Asset Pricing Models with Underlying Time-varying Lévy Processes

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Time-varying Jump Diffusion Framework

- Time-varying volatility: Empirical studies on the statistical properties of realized and/or implied volatilities have given rise to various stochastic volatility models in the literature, such as the Heston model, CEV models and also stochastic volatility models with jumps etc.
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- **Existence of jumps is empirically supported**: Carr and Wu (2003), Pan (2002).
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- Theoretical part: We assume a time-varying Lévy process, with time-varying drift, volatility and jump intensity parameters, to model the jump diffusion economy. We study an equilibrium asset and option pricing model in this economy.
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- Theoretical part: We assume a time-varying Lévy process, with time-varying drift, volatility and jump intensity parameters, to model the jump diffusion economy. We study an equilibrium asset and option pricing model in this economy.
- Empirical part: Under this general framework, we decompose S&P500 index into time-varying processes of drift, volatility and jump, using the Hodrick-Prescott filter and a particle filter.
Stock Market

An investment of $S_t$ in the stock market is governed by:

$$\frac{dS_t}{S_{t^-}} = \mu(t)dt + \sigma(t)dB_t + (e^x - 1)dN_t - \lambda(t)E(e^x - 1)dt, \quad (1)$$

where $S_{t^-}$ is the value of $S_t$ before a possible jump occurs; $\mu(t)$ and $\sigma(t)$ are the rate of return and the volatility of the investment.
Model Build-up

Stock Market

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where $S_{t^-}$ is the value of $S_t$ before a possible jump occurs; $\mu(t)$ and $\sigma(t)$ are the rate of return and the volatility of the investment.

The jump part is assumed to be a Poisson process, with jump intensity $\lambda(t)$ and jump size $x$ which follows an arbitrary distribution.
Money Market Account

- We further assume that there is a market for instantaneous borrowing and lending at a risk-free rate \( r(t) \). The money market account, \( M_t \), follows

\[
\frac{dM_t}{M_t} = r(t)dt. \tag{2}
\]

The risk-free rate, \( r(t) \), will be derived from the general equilibrium later, as a part of the solution.
Representative Investor

- Maximize the expected utility function of lifetime consumption

\[
\max E_t \int_t^T p(t)U(c_t) dt,
\]

where \( c_t \) is the rate of consumption at time \( t \), \( U(c) \) the utility function with \( U' > 0, U'' < 0 \), and \( p(t) \geq 0, 0 \leq t \leq T \) the time preference function.

- Assume constant relative risk aversion (CRRA) utility function.
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- Assume constant relative risk aversion (CRRA) utility function.

\[
U(c) = \begin{cases} 
\frac{c^{1-\gamma}}{1-\gamma} & \gamma > 0, \gamma \neq 1, \\
\ln c & \gamma = 1,
\end{cases}
\]

where the constant \(\gamma\) is the relative risk aversion coefficient, \(\gamma = -cU''/U'\).
Total Wealth

- The total wealth of the representative investor at time $t$:

$$W_t = W_{1t} + W_{2t}$$

where $W_{1t} = \omega W_t$ is invested in the stock market, and $W_{2t} = (1 - \omega)W_t$ is invested in the money market.
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$\omega$ is called the *wealth ratio*. 
Representative Investor’s Optimal Control Problem:

\[
\max_{c_t, \omega} E_t \int_t^T p(t) U(c_t) dt, 
\]

subject to

\[
\frac{dW_t}{W_t} = \omega \frac{dS_t}{S_t} + (1-\omega) \frac{dM_t}{M_t} - \frac{c_t}{W_t} dt \\
= [r(t) + \omega \mu(t) - \omega r(t) - \omega \lambda(t) E(e^x - 1) - \frac{c_t}{W_t}] dt + \omega \sigma(t) dB_t + \omega (e^x - 1) dN_t,
\]

where \( \phi(t) = \mu(t) - r(t) \) is the equity premium.
■ Representative Investor’s Optimal Control Problem:

$$\max_{c_t, \omega} E_t \int_t^T p(t) U(c_t) dt,$$

subject to

$$\frac{dW_t}{W_t} = \omega \frac{dS_t}{S_t} + (1 - \omega) \frac{dM_t}{M_t} - \frac{c_t}{W_t} dt$$

$$= [r(t) + \omega \mu(t) - \omega r(t) - \omega \lambda(t) E(e^x - 1) - \frac{c_t}{W_t}] dt + \omega \sigma(t) d B_t$$

$$+ \omega (e^x - 1) d N_t,$$

where $\phi(t) = \mu(t) - r(t)$ is the equity premium.

■ Market Clearing: Because there is only one investor in the economy, he has to put all the wealth into the stock market. The general equilibrium occurs at $\omega = 1$, under which the market is cleared.
Equity Premium

Introduction

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Decomposing S&P500 index

Summary

Proposition

In the production economy with jump diffusion and one representative investor with CRRA utility function, the equilibrium equity premium is given by

\[ \phi(t) = \phi_{\sigma}(t) + \phi_J(t), \]

where

\[ \phi_{\sigma}(t) = \gamma \sigma(t)^2 \]

- diffusion risk premium

\[ \phi_J(t) = \lambda(t) \mathbb{E}[(1 - e^{-\gamma x})(e^x - 1)] \]

- jump risk premium

The risk-free rate is a time-varying function:

\[ r(t) = \mu(t) - \phi(t) = \mu(t) - \phi_{\sigma}(t) - \phi_J(t). \]
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\phi_J(t) = \lambda(t) E[(1 - e^{-\gamma x})(e^x - 1)] -jump risk premium
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Equity Premium

Proposition

- **In the production economy with jump diffusion and one representative investor with CRRA utility function, the equilibrium equity premium is given by**

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  where \( \phi_\sigma(t) = \gamma \sigma(t)^2 \) -diffusion risk premium

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- **The risk-free rate is a time-varying function:**

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General Pricing Kernel

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General Pricing Kernel

The pricing kernel is given by

\[ d \pi_t = -r(t) \, dt - \gamma \sigma(t) \, dB_t + (e^y - 1) \, dN_t - \lambda(t) \, E(e^y - 1) \, dt, \]

or equivalently, after integration

\[ \pi_T = \exp \left\{ -\int_T^t \gamma \sigma(s) \, dB_s - \int_T^t \left[ r(s) + \frac{1}{2} \gamma^2 \sigma^2(s) \right] ds - \lambda(t) \, E(e^y - 1) \, ds + N_t \right\}. \]

The random variable \( y \) modeling the jump size in the logarithm of the pricing kernel, satisfies

\[ E[(e^y - e^{-\gamma x})(e^x - 1)] = 0. \]
The pricing kernel is given by

$$\frac{d\pi_t}{\pi_t} = -r(t)dt - \gamma\sigma(t)dB_t + (e^y - 1)dN_t - \lambda(t)E(e^y - 1)dt,$$

where $y$ modeling the jump size in the logarithm of the pricing kernel, satisfies $E[(e^y - e^{-\gamma x})(e^x - 1)] = 0$. 

**Proposition**
General Pricing Kernel

Proposition

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- or equivalently, after integration
  \[
  \frac{\pi_T}{\pi_t} = \exp\{ - \int_t^T \gamma \sigma(s) dB_s - \int_t^T [r(s) + \frac{1}{2} \gamma^2 \sigma^2(s)] ds - E(e^y - 1) \int_t^T \lambda(s) ds + \sum_{i=1}^{N_{t,T}} y_i \}.\]
General Pricing Kernel

**Proposition**

- The pricing kernel is given by
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  \]

- The random variable \( y \) modeling the jump size in the logarithm of the pricing kernel, satisfies \( E[(e^y - e^{-\gamma x})(e^x - 1)] = 0 \).
European Call

Proposition

The price of a European call, \( c(S_t, t) \), in the jump diffusion economy satisfies

\[
\frac{\partial c(S_t, t)}{\partial t} + \frac{1}{2} \sigma^2(t) S_t^2 \frac{\partial^2 c(S_t, t)}{\partial S^2} + [r(t) - \lambda^Q(t) E^Q(e^x - 1)] S_t \frac{\partial c(S_t, t)}{\partial S} - r(t)c(S_t, t) + \lambda^Q(t) \{ E^Q[c(S_te^x, t)] - c(S_t, t) \} = 0,
\]

with final condition

\[
c(S_T, T) = max(S_T - K, 0),
\]

where \( \lambda^Q(t) \equiv \lambda(t)E(e^y) \): jump intensity in the risk-neutral measure \( Q \), defined by \( E^Q[f(x)] := \frac{E[e^y f(x)]}{E(e^y)} \), for any function \( f(x) \).
European Call

Proposition

Pricing formula of a European call option:

\[
c(S_t, t) = \sum_{n=0}^{+\infty} e^{-\int_t^T \lambda^Q(s)ds} \frac{\left(\int_t^T \lambda^Q(s)ds\right)^n}{n!} E^n_Q \left[c^{BS}(Se^X e^{-E^n_Q(e^x-1)\int_t^T \lambda^Q(s)ds}, t)\right],
\]

where \(c^{BS}(S, t)\) is the Black-Scholes formula price for the European call option and

\[
X = \sum_{i=1}^n x_i.
\]
Decompose the S&P500 Index into time-varying components, using

- the Hodrick-Prescott Filter
- a particle filter (Sequential Monte Carlo Method)
The Hodrick-Prescott filters was first proposed in Whittaker (1923), then popularized in economics by Hodrick and Prescott (1997).
Hodrick-Prescott Filter

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- The method serves to decompose the time series $y_t = \ln(S_t)$ into a trend component $\tau_t$, and a cyclical component $c_t$:

$$y_t = \tau_t + c_t, \text{ for } t = 1, \ldots, T.$$  

---

1Ravn and Uhlig (2002)
Hodrick-Prescott Filter

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  $$y_t = \tau_t + c_t, \text{ for } t = 1, \ldots, T.$$ 

- Condition: For a given $a$, $\tau_t$ satisfies

  $$\min_{\tau} \left( \sum_{t=1}^{T} (y_t - \tau_t)^2 + a \sum_{t=2}^{T-1} \left[ (\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}) \right]^2 \right),$$

  where $a = 129600$ for monthly data\(^1\).

\(^1\)Ravn and Uhlig (2002)
Extract Drift

Data:

- In each month, we use the 5% to 95% quantile of $ln(S_t)$, compute the mean as a monthly data input for HP filter.
Extract Drift

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As a result, we decompose the stock index into a time-varying trend component $T$ and a component $C$:

$$ln(S_t) = T + C.$$  

$T$ is a monthly drift, $C$ the remaining process of volatility plus jumps.
**Time-varying Drift**

**Figure 1.**

<table>
<thead>
<tr>
<th></th>
<th>mean ($\times 10^{-5}$)</th>
<th>volatility</th>
<th>skewness</th>
<th>kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \ln(S)$</td>
<td>31.9</td>
<td>0.0115</td>
<td>-1.3044</td>
<td>31.8</td>
</tr>
<tr>
<td>$\Delta C$</td>
<td>1.91</td>
<td>0.0117</td>
<td>-1.2229</td>
<td>30.1</td>
</tr>
</tbody>
</table>

**Table 1.**
By taking the difference of the time-varying trend ($\Delta T$), we can observe that:

- **Regime Switching**: Before 2000, stock return was positive. However, after 2000 we can observe that it fluctuates around zero.

- **Volatility/Jump Clustering**: In negative return periods, there exists jumps and volatility clustering. By contrast, in positive return period, volatility/jump process is much less volatile.

![Figure 2.](image-url)
Filtering Problems

For filtering problem, the data is generated by the state space model, which consists of the observation and state evolution equations,

- Observation equation: \( y_t = f(x_t, \epsilon^y_t) \)
- State evolution: \( x_{t+1} = g(x_t, \epsilon^x_{t+1}) \)

where \( \epsilon^y_{t+1} \) is the observation error or “noise”, and \( \epsilon^x_{t+1} \) are state shocks.
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Particle filters belong to statistical filtering methods, which usually refer to an algorithm for extracting a latent state variable (e.g. volatility) from noisy observations (e.g. stock price/return) using a statistical model.
Particle filters use a sampling approach with a set of particles to represent the posterior density of a latent state space (Johannes, Polson and Stroud 2009, RFS).
Particle Filters

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- They are simulation-based estimation methods, which include a set of algorithms that estimate the posterior density by directly implementing the Bayesian recursion equations.
Particle Filters

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- They are simulation-based estimation methods, which include a set of algorithms that estimate the posterior density by directly implementing the Bayesian recursion equations.
- The state space model used in particle filters can be non-linear and the initial state and noise distributions can take any form required.
The Sampling Importance Resampling (SIR) algorithm is a classical particle filtering algorithm developed by Gordon, Salmond, and Smith (1993).
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SIR includes two steps: given samples from $p^N(x_t|y^t)$,

- **S1. Propagation:** for $i = 1, \ldots, N$, draw $x_{t+1}^{(i)} \sim p(x_{t+1}|x_t^{(i)})$.
- **S2. Resampling:** for $i = 1, \ldots, N$, draw $z^{(i)} \sim \text{Mult}_N(w_{t+1}^{(1)}, \ldots, w_{t+1}^{(N)})$,

with importance sampling weights $w_{t+1}^{(i)} = \frac{p(y_{t+1}|x_{t+1}^{(i)})}{\sum_{l=1}^{N} p(y_{t+1}|x_{t+1}^{(l)})}$,

and set $x_{t+1}^{(i)} = x_{t+1}^{z(i)}$. 

![SIR Algorithm Diagram]
As a state space model (for volatility) is necessary to implement particle filters, we assume that following dynamics for the stochastic variance:

$$d\nu_t = k(\theta - \nu_t)dt + \sigma\nu\sqrt{\nu_t}dB^\nu_t,$$

where $\nu_t$ is a mean-reverting stochastic process. $B^\nu_t$ is a Brownian motion correlated with $B_t$, with correlation coefficient $\rho$. 
Based on the result from HP filter, we further apply SIR to decompose the time-varying volatility and jump (component $C$).
Filter out Volatility

- Based on the result from HP filter, we further apply SIR to decompose the time-varying volatility and jump (component C).
- We start with particle filters under stochastic volatility (SV) model without jumps, then we apply particle filters under stochastic volatility and jump (SVJ) model.
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- We start with particle filters under stochastic volatility (SV) model without jumps, then we apply particle filters under stochastic volatility and jump (SVJ) model.
- The parameters used for the particle filters are taken from Eraker, Johannes and Polson (2003).
Filtered Volatility Processes I - SV model

- We run particle filters under SV model three times. Estimated volatilities stay around 0.34-0.35. However, the pattern of the volatility processes varies each time.

- Note that the hump shape on the left sides are caused by an adaptation period (around 200 initial data points) needed by the algorithm.
Following Eraker, Johannes and Polson (2003), we assume a jump intensity of $\lambda = 0.006$, meaning 1 to 2 jumps per year; the jump size follows a normal distribution.
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With the SVJ model, filtered volatilities decrease to 0.2-0.25, as jumps account for some of the excess variance.

We detect a high possibility of jumps around 1987-1988, and some other infrequent jumps. Overall jumps are rare in this model. We observe high level of volatilities when the probability of a jump occurring is high.
Figure 4. Filtered volatility and jump processes under SVJ model
Future Research

- The decomposition of the time-varying component of drift, volatility and jumps from S&P500 index using HP filter and particle filter is still a preliminary result.
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- Here we studied the SVJ model only with fixed jump intensity; another possibility is to consider the jump intensity as a time-varying function, for example a function of time-varying drift or volatility, or some other possible exogenous determinant.
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It will be interesting to use option data jointly with return data in the filtering methods.
Main References


Thank you!