On strategically equivalent contests

Jang SCHILTZ (University of Luxembourg)

joint work with

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June 2, 2016
Outline

1. Basic Framework - Deterministic rent

2. General Framework - Risky rent

Strategically equivalent contests

Situations in which the proportional contest dominates
Outline

1. Basic Framework - Deterministic rent

2. General Framework - Risky rent
   - Strategically equivalent contests
   - Situations in which the proportional contest dominates
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1. Basic Framework - Deterministic rent

2. General Framework - Risky rent
   - Strategically equivalent contests
   - Situations in which the proportional contest dominates
General description of the contest

Two players compete for a prize. Each player chooses an effort level $x_i$ given the expected effort level $x_j$ of his rival. The vector of efforts $(x_i, x_j)$ determines what each player will finally get.
General description of the contest

Two players compete for a prize $V$. 

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June 2, 2016 4 / 22
General description of the contest

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Two players compete for a prize $V$.

Each player chooses an effort level $x_i$ given the expected effort level $x_j$ of his rival.

The vector of efforts $(x_i, x_j)$ determines what each player will finally get.
The assumptions

Assumption 1. The contest success function \( p_i \), describing the effort of player \( i \) relative to the total effort is given by
\[
\frac{x_i}{x_i + x_j}.
\]

There are two possible interpretations for \( p_i \).
- In a Proportional contest, it is interpreted as a share of the prize.
- In a Lottery contest, as a win probability.

Assumption 2. Players are expected utility maximizers.
The assumptions

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Strategically equivalent contests

Definition (Chowdhury and Sheremeta (2014))
Contests are effort equivalent if they result in the same Nash equilibrium level of effort.
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Contests are effort equivalent if they result in the same Nash equilibrium level of effort.
The case of a deterministic rent

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- Lottery contest (L)

\[ \pi_L^i = (V - x_i, -x_i; p_i, 1 - p_i) \]

Expected utility

\[ EU_{\pi_L}^i(x) = x_i x_i + x_j V - x_i \]

Proportional contest (P)

\[ \pi_P^i = (p_i V - x_i; 1) \]

Expected utility

\[ EU_{\pi_P}^i(x) = U(x_i x_i + x_j V - x_i) \]
The case of a deterministic rent

- **Lottery contest (L)**

Payoff

\[ \pi^L_i = (V - x_i, -x_i; p_i, 1 - p_i). \]
The case of a deterministic rent

- **Lottery contest (L)**

Payoff

\[ \pi_i^L = (V - x_i, -x_i; p_i, 1 - p_i). \]

Expected utility

\[ EU\pi_i^L(x) = \frac{x_i}{x_i + x_j} U_i(V - x_i) + \frac{x_j}{x_i + x_j} U_i(-x_i). \]
The case of a deterministic rent

- **Lottery contest (L)**

  Payoff
  \[
  \pi^L_i = (V - x_i, -x_i; p_i, 1 - p_i).
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- **Proportional contest (P)**
The case of a deterministic rent

- **Lottery contest (L)**

Payoff

\[ \pi_i^L = (V - x_i, -x_i; p_i, 1 - p_i). \]

Expected utility

\[ EU_{\pi_i^L}(x) = \frac{x_i}{x_i + x_j} U_i(V - x_i) + \frac{x_j}{x_i + x_j} U_i(-x_i). \]

- **Proportional contest (P)**

Payoff

\[ \pi_i^P = (p_iV - x_i; 1). \]
The case of a deterministic rent

- **Lottery contest (L)**

Payoff

$$\pi^L_i = (V - x_i, -x_i; p_i, 1 - p_i).$$

Expected utility

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- **Proportional contest (P)**

Payoff

$$\pi^P_i = (p_i V - x_i; 1).$$

Expected utility

$$EU_{\pi^P_i}(x) = U(\frac{x_i}{x_i + x_j} V - x_i).$$
The case of risk-neutral players

If players are risk-neutral expected utility maximizers ($U(z) = z$), then their expected utility $EU_{\pi_L}(x) = x_i + x_j - x_i \equiv UE_{\pi_L}(x) \equiv U_{\pi_P}(x)$. 

Theorem: The two contests have Nash equilibrium $x^* = V/4$ and are thus strategically equivalent.
The case of risk-neutral players

If players are **risk-neutral** expected utility maximizer \((U(z) = z)\)

\[
EU_{\pi_i}^L(x) = \frac{x_i}{x_i + x_j} V - x_i \equiv UE_{\pi_i}^L(x) \equiv U_{\pi_i}^P(x).
\]

\[
D_{x_i}(\frac{x_i}{x_i + x_j} V - x_i) = V \frac{x_j}{(x_i + x_j)^2} - 1.
\]

**Theorem**

*The two contests have Nash equilibrium \(x^* = V/4\) and are thus strategically equivalent.*
Assumption 3. The players have constant absolute risk-averse (CARA) preferences represented by the negative exponential utility function
\[ u(\pi_i) = -\exp(-r\pi_i) , \]
where \( r > 0 \) represents the coefficient of absolute risk aversion. We can then show that
\[ x^*_P = \sqrt{V}, \]
\[ x^*_L = \frac{e^{Vr} - 1}{2r(1 + e^{Vr})} = \frac{1}{2} r \tanh(\frac{Vr}{2}). \]
The case of risk averse players

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- \( x_P^* = \frac{V}{4} \).
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where \( r > 0 \) represents the coefficient of absolute risk aversion.

We can then show that

- \( x_P^* = \frac{V}{4} \).
- \( x_L^* = \frac{e^{Vr} - 1}{2r(e^{Vr} + 1)} = \frac{1}{2r} \tanh\left(\frac{Vr}{2}\right). \)
The case of risk-neutral players

Hence,
The case of risk-neutral players

Hence,

\[ 2r(x_P^* - x_L^*) = 2r \left( \frac{V}{4} - \frac{1}{2r} \tanh\left( \frac{V_r}{2} \right) \right) \]

\[ = \frac{V_r}{2} - \tanh\left( \frac{V_r}{2} \right) > 0, \]

since \( x \geq \tanh(x), \forall x \geq 0 \)
The case of risk-neutral players

Hence,

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1. Basic Framework - Deterministic rent

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   - Strategically equivalent contests
   - Situations in which the proportional contest dominates
In this section we suppose that the rent $V$ is a random variable with density $f$. 

It's Laplace transform $f^\ast(t)$ is defined by $f^\ast(t) = \mathbb{E}\left[\exp\left(tV\right)\right]$. 

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Strategically equivalent contests 

June 2, 2016 12 / 22
In this section we suppose that the rent $V$ is a random variable with density $f$.

It’s Laplace transform $f^*$ is defined by

$$f^*(t) = E[\exp(tV)].$$
The proportional contest

Expected utility:

$$\text{Eu}(\pi_i) = E(-e^{-r\pi_i}) = \int_{-\infty}^{+\infty}(-e^{-r(p_iu-x_i)})f(u)du = -e^{rx_i}\int_{-\infty}^{+\infty}e^{-rp_iu}f(u)du = -e^{rx_i}f^*(x_i + x_j).$$

This gives us an equilibrium effort

$$x^p = \frac{1}{4}(f^*(r^2))'.$$
The proportional contest

Expected utility:

\[ Eu(\pi_i) = E \left( -e^{-r\pi_i} \right) = \int_{-\infty}^{+\infty} \left( -e^{-r(p_iu-x_i)} \right) f(u) du \]

\[ = -e^{rx_i} \int_{-\infty}^{+\infty} e^{-rp_iu} f(u) du \]

\[ = -e^{rx_i} f^* \left( \frac{-rx_i}{x_i + x_j} \right). \]
The proportional contest

Expected utility:

\[ Eu(\pi_i) = E(-e^{-r\pi_i}) = \int_{-\infty}^{+\infty} (-e^{-r(p_i u - x_i)}) f(u) du \]

\[ = -e^{rx_i} \int_{-\infty}^{+\infty} e^{-rp_i u} f(u) du \]

\[ = -e^{rx_i} f^* \left( \frac{-rx_i}{x_i + x_j} \right). \]

This gives us an equilibrium effort

\[ x_P^* = \frac{1}{4} \left( \frac{f^* \left( \frac{-r}{2} \right)'}{f^* \left( \frac{-r}{2} \right)} \right). \]
The lottery contest

\[ E_u = x_1 x_2 \left( \int_{-\infty}^{\infty} e^{-u x_1} f(u) \, du \right) + x_2 x_1 + x_2 \left( e^{r x_1} \right) = -x_1 e^{r x_1} x_1 + x_2 f^{*} (-r) - x_2 e^{r x_1} x_1 + x_2. \]

This gives us an equilibrium effort

\[ x^* L = 1 - f^{*} (-r) \frac{2}{r (1 + f^{*} (-r))}. \]

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The lottery contest

Expected utility:

\[ Eu(\pi_1) = \frac{x_1}{x_1 + x_2} \left( \int_{-\infty}^{+\infty} (- \exp(-ru) f(u) du) \right) + \frac{x_2}{x_1 + x_2} \left( -e^{rx_1} \right) \]

\[ = -\frac{x_1 \exp(rx_1)}{x_1 + x_2} \left( \int_{-\infty}^{+\infty} \exp(-ru) f(u) du \right) - \frac{x_2}{x_1 + x_2} \left( e^{rx_1} \right) \]

\[ = -\frac{x_1 \exp(rx_1)}{x_1 + x_2} f^*(-r) - \frac{x_2 \exp(rx_1)}{x_1 + x_2} . \]
The lottery contest

Expected utility:

$$Eu(\pi_1) = \frac{x_1}{x_1 + x_2} \left( \int_{-\infty}^{+\infty} (-\exp(-ru)f(u)du) + \frac{x_2}{x_1 + x_2} (-e^{rx_1}) \right)$$

$$= -\frac{x_1 \exp(rx_1)}{x_1 + x_2} \left( \int_{-\infty}^{+\infty} \exp(-ru)f(u)du \right) - \frac{x_2}{x_1 + x_2} (e^{rx_1})$$

$$= -\frac{x_1 \exp(rx_1)}{x_1 + x_2} f^*(-r) - \frac{x_2 \exp(rx_1)}{x_1 + x_2}.$$

This gives us an equilibrium effort

$$x_L^* = \frac{1 - f^*(-r)}{2r(1 + f^*(-r))}.$$
Strategically equivalent contests thus exist if

\[
\frac{1}{4} \frac{(f^*)'(-\frac{r}{2})}{f^*(-\frac{r}{2})} = \frac{1 - f^*(-r)}{2r(1 + f^*(-r))},
\]

where \( f^* \) is the Laplace transform of the density of the rent.
Strategically equivalent contests thus exist if

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\frac{1}{4} \frac{(f^*)'(-\frac{r}{2})}{f^*(-\frac{r}{2})} = \frac{1 - f^*(-r)}{2r(1 + f^*(-r))},
\]

where \( f^* \) is the Laplace transform of the density of the rent.

This is a nonlinear, non local differential equation.
First set of solutions: The exponential law

Let $V$ be exponentially distributed with parameter $\lambda > 0$. Then,

$$f(x) = \lambda e^{-\lambda x},$$

for $t$ such that $0 < t < \lambda$ and

$$f'(t) = \lambda \left(t - \frac{1}{\lambda}ight)^2.$$

Hence,

$$x^*P = \frac{1}{4} \left( f^*(-r^2) f^*(-r^2) \right),$$

and

$$x^*L = 1 - f^*(-r) \frac{1}{1 + f^*(-r)} = \frac{1}{2} r + \frac{4}{\lambda}.$$
First set of solutions: The exponential law

Let $V$ be exponentially distributed with parameter $\lambda > 0$. 

\[ f(x) = \lambda e^{-\lambda x}, \quad f^*(t) = \lambda \frac{\lambda}{\lambda - t} \text{ for } t < \lambda \]

\[ f^*(t)' = \lambda \left( t - \frac{\lambda}{\lambda - t} \right)^2. \]

Hence,

\[ x^*P = \frac{1}{4} \left( f^*(-r)^2 \right)' f^*(-r)^2 = \frac{1}{2} r + 4 \lambda. \]

\[ x^*L = 1 - f^*(-r)^2 r \left( 1 + f^*(-r) \right) = \frac{1}{2} r + 4 \lambda. \]

\[ \Rightarrow x^*P = x^*L = \frac{1}{2} r + 4 \lambda. \]
First set of solutions: The exponential law

Let $V$ be exponentially distributed with parameter $\lambda > 0$.

Then, $f(x) = \lambda e^{-\lambda x}$, $f^*(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$ and $(f^*(t))' = \frac{\lambda}{(t-\lambda)^2}$. 
First set of solutions: The exponential law

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Hence,

$$x^*_P = \frac{1}{4} \left( \frac{f^* \left( -\frac{r}{2} \right)'}{f^* \left( -\frac{r}{2} \right)} \right) = \frac{1}{4} \left( \frac{\lambda}{\left( -\frac{r}{2} - \lambda \right)^2} \right) = \frac{1}{2r + 4\lambda}$$
First set of solutions: The exponential law

Let \( V \) be exponentially distributed with parameter \( \lambda > 0 \).

Then, \( f(x) = \lambda e^{-\lambda x} \), \( f^*(t) = \frac{\lambda}{\lambda - t} \) for \( t < \lambda \) and \((f^*(t))' = \frac{\lambda}{(t-\lambda)^2}\).

Hence,

\[
x_P^* = \frac{1}{4} \left( \frac{f^* \left( -\frac{r}{2} \right)'}{f^* \left( -\frac{r}{2} \right)} \right) = \frac{1}{4} \left( \frac{\lambda}{\left( -\frac{r}{2} - \lambda \right)^2} \right) = \frac{1}{2r + 4\lambda}
\]

and

\[
x_L^* = \frac{1 - f^*(-r)}{2r(1 + f^*(-r))} = \frac{1 - \frac{\lambda}{\lambda - r}}{2r(1 + \frac{\lambda}{\lambda - r})} = \frac{1}{2r + 4\lambda}
\]
First set of solutions: The exponential law

Let $V$ be exponentially distributed with parameter $\lambda > 0$.

Then, $f(x) = \lambda e^{-\lambda x}$, $f^*(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$ and $(f^*(t))' = \frac{\lambda}{(t - \lambda)^2}$.

Hence,

$$x_P^* = \frac{1}{4} \left( \frac{f^*\left(-\frac{r}{2}\right)}{f^*\left(-\frac{r}{2}\right)} \right) = \frac{1}{4} \left( \frac{\lambda}{\left(\frac{r}{2} - \frac{\lambda}{\lambda - r}\right)^2} \right) = \frac{1}{2r + 4\lambda}$$

and

$$x_L^* = \frac{1 - f^*(-r)}{2r(1 + f^*(-r))} = \frac{1 - \frac{\lambda}{\lambda - r}}{2r(1 + \frac{\lambda}{\lambda - r})} = \frac{1}{2r + 4\lambda}$$

$$\Rightarrow \quad x_P^* = x_L^* = \frac{1}{2r + 4\lambda}.$$
Second set of solutions: A polynomial function

\[ f^*(t) = at + bt + ct^2. \]

Then, \( (f^*)'(t) = b + 2ct \) and we can compute the equilibrium efforts as

\[ x^*P = \frac{1}{4}(b - cr) \quad \text{and} \quad x^*L = \frac{1}{4}a + b - cr^2 - r(1 + a - br + cr^2). \]
Second set of solutions : A polynomial function

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and we can compute the equilibrium efforts as

$$x_P^* = \frac{1}{4} \left( \frac{b - cr}{a - b^2 + c} \right)$$
Second set of solutions: A polynomial function

Let $f^*(t) = a + bt + ct^2$.

Then, $(f^*(t))' = b + 2ct$

and we can compute the equilibrium efforts as

$$x_P^* = \frac{1}{4} \left( \frac{b-cr}{a-br^2+c\frac{r^2}{4}} \right)$$

$$x_L^* = \frac{1-a+br-cr^2}{2r(1+a-br+cr^2)}.$$
Second set of solutions: A polynomial function

Both contests are strategically equivalent and $f$ is a density function of a random variable iff $a = 1$, $c = 0$ and $0 < b < 2/r$. 

$\therefore \quad x^* P = x^* S = b^2 (2 - b r)$. 

Moreover, $f(t) = \text{dirac}(t) + b \text{dirac}(1, t)$, where $\text{dirac}(1, t)$ denotes the derivative of $\text{dirac}(t)$. 

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In that case, $x_P^* = x_S^* = \frac{b}{2(2-br)}$. 
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Both contests are strategically equivalent and \( f \) is a density function of a random variable iff \( a = 1, c = 0 \) and \( 0 < b < 2/r \).

In that case, \( x^*_P = x^*_S = \frac{b}{2(2-br)} \).

Moreover,

\[
f(t) = \text{dirac}(t) + b \text{dirac}(1, t),
\]

where \( \text{dirac}(1, t) \) denotes the derivative of \( \text{dirac}(t) \).
Open question 1

Are these all possible solution?
The normal distribution

If the rent is normally distributed with mean $V$ and variance $\sigma^2$ then

$$f^*(r) = \exp(rV + \frac{1}{2}r^2\sigma^2)$$

and

$$(f^*(r))^\prime = (r\sigma^2 + V)e^{\frac{1}{2}r^2\sigma^2 + Vr}.$$ 

The contest equilibrium levels are then given by

$$x^*P = \frac{(f^*(r))^\prime}{4f^*(r)} = 1 - \frac{V}{8\sigma^2}r^2$$

and

$$x^*L = 1 - \frac{2r(1 + f^*(r))^2}{r(1 + e^{\frac{1}{2}r^2\sigma^2 + Vr})} = \frac{1}{2}r \tanh(2r(\frac{1}{4}V - \frac{1}{8}r^2\sigma^2)).$$

Hence,

$$x^*P - x^*L = 2r(1 - \frac{V}{8\sigma^2}r^2) - \tanh(2r(\frac{1}{4}V - \frac{1}{8}r^2\sigma^2)) > 0$$

since $x^* \geq \tanh(x^*) \forall x^* \geq 0$. 

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and

$$(f^*(r))' = (r\sigma^2 + V) e^{\frac{1}{2}r^2\sigma^2 + Vr}.$$
The normal distribution

If the rent is normally distributed with mean $V$ and variance $\sigma^2$ then

$$f^*(r) = \exp(rV + \frac{1}{2} r^2 \sigma^2) \quad \text{and} \quad (f^*(r))' = (r\sigma^2 + V) e^{\frac{1}{2} r^2 \sigma^2 + Vr}.$$ 

The contest equilibrium levels are then given by
The normal distribution

If the rent is normally distributed with mean $V$ and variance $\sigma^2$ then $f^*(r) = \exp(rV + \frac{1}{2} r^2 \sigma^2)$ and $(f^*(r))' = (r\sigma^2 + V) e^{\frac{1}{2} r^2 \sigma^2 + Vr}$.

The contest equilibrium levels are then given by

$$x_P^* = \frac{(f^*)'(-\frac{r}{2})}{4f^*(-\frac{r}{2})} = \frac{1}{4} V - \frac{1}{8} r\sigma^2$$
The normal distribution

If the rent is normally distributed with mean $V$ and variance $\sigma^2$ then

$$f^*(r) = \exp(rV + \frac{1}{2} r^2 \sigma^2)$$

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The contest equilibrium levels are then given by

$$x_P^* = \frac{(f^*)'(-\frac{r}{2})}{4f^*(-\frac{r}{2})} = \frac{1}{4} V - \frac{1}{8} r\sigma^2$$

and

$$x_L^* = \frac{1 - f^*(-r)}{2r(1 + f^*(-r))} = \frac{1 - e^{\frac{1}{2} r^2 \sigma^2 - Vr}}{2r(1 + e^{\frac{1}{2} r^2 \sigma^2 - Vr})} = \frac{1}{2r} \tanh\left(\frac{r}{2}(V - \frac{1}{2} r\sigma^2)\right).$$
The normal distribution

If the rent is normally distributed with mean $V$ and variance $\sigma^2$ then

$$f^*(r) = \exp(rV + \frac{1}{2}r^2\sigma^2)$$

and

$$(f^*(r))' = (r\sigma^2 + V) \exp\left(\frac{1}{2}r^2\sigma^2 + Vr\right).$$

The contest equilibrium levels are then given by

$$x^*_P = \frac{(f^*)'(-\frac{r}{2})}{4f^*(-\frac{r}{2})} = \frac{1}{4}V - \frac{1}{8}r\sigma^2$$

and

$$x^*_L = \frac{1 - f^*(-r)}{2r(1 + f^*(-r))} = \frac{1 - \exp\left(\frac{1}{2}r^2\sigma^2 - Vr\right)}{2r(1 + \exp\left(\frac{1}{2}r^2\sigma^2 - Vr\right))} = \frac{1}{2r} \tanh\left(\frac{r}{2}(V - \frac{1}{2}r\sigma^2)\right).$$

Hence,

$$x^*_P - x^*_L = 2r\left(\frac{1}{4}V - \frac{1}{8}r\sigma^2\right) - \tanh\left(2r\left(\frac{1}{4}V - \frac{1}{8}r\sigma^2\right)\right) > 0$$

since $x \geq \tanh(x) \ \forall x \geq 0.$
The uniform distribution

If the rent is uniformly distributed on an interval \([a, b]\) then

\[ f^*(r) = e^{rb} - e^{ra} \]

and

\[ (f^*)' = -\frac{1}{r^2} (a - b) (e^{ar} - e^{br} - ar + bre) \]
The uniform distribution

If the rent is uniformly distributed on an interval \([a, b]\) then
\[
f^\ast(r) = \frac{e^{rb} - e^{ra}}{r(b-a)} \quad \text{and} \quad (f^\ast)'(r) = -\frac{1}{r^2(a-b)} (e^{ar} - e^{br} - a e^{ar} + b e^{br}).
\]
The uniform distribution

If the rent is uniformly distributed on an interval \([a, b]\) then
\[f^*(r) = \frac{e^{rb} - e^{ra}}{r(b-a)}\]
and
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The contest equilibrium levels are then given by
The uniform distribution

If the rent is uniformly distributed on an interval \([a, b]\) then 
\[
 f^*(r) = \frac{e^{rb} - e^{ra}}{r(b-a)} \quad \text{and} \quad (f^*)'(r) = -\frac{1}{r^2(a-b)} \left( e^{ar} - e^{br} - are^{ar} + bre^{br} \right).
\]

The contest equilibrium levels are then given by

\[
x_P^* = \frac{1}{4r} \frac{1}{e^{-\frac{1}{2}ar} - e^{-\frac{1}{2}br}} \left( 2e^{-\frac{1}{2}ar} - 2e^{-\frac{1}{2}br} + are^{-\frac{1}{2}ar} - bre^{-\frac{1}{2}br} \right)
\]
The uniform distribution

If the rent is uniformly distributed on an interval \([a, b]\) then
\[ f^*(r) = \frac{e^{rb} - e^{ra}}{r(b-a)} \quad \text{and} \quad (f^*)'(r) = -\frac{1}{r^2(a-b)} \left( e^{ar} - e^{br} - ar e^{ar} + br e^{br} \right). \]

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and

\[ x^*_L = \frac{e^{-br} - e^{-ar} + br - ar}{2re^{-ar} - 2re^{-br} - 2ar^2 + 2br^2}. \]
The uniform distribution

If the rent is uniformly distributed on an interval \([a, b]\) then
\[
f^*(r) = \frac{e^{rb} - e^{ra}}{r(b-a)} \quad \text{and} \quad (f^*)'(r) = -\frac{1}{r^2(a-b)} \left( e^{ar} - e^{br} - are^{ar} + bre^{br} \right).
\]

The contest equilibrium levels are then given by

\[
x_P^* = \frac{1}{4r} \frac{1}{e^{\frac{1}{2}ar} - e^{\frac{1}{2}br}} \left( 2e^{-\frac{1}{2}ar} - 2e^{-\frac{1}{2}br} + are^{-\frac{1}{2}ar} - bre^{-\frac{1}{2}br} \right)
\]

and

\[
x_L^* = \frac{e^{-br} - e^{-ar} + br - ar}{2re^{-ar} - 2re^{-br} - 2ar^2 + 2br^2}.
\]

Again, one can show that \(X_P^* > X_L^*\).
Open question 2

Does the proportional contest always dominate the lottery contest?