An extension of the concept of distance to functions of several variables

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A pair \((X, d)\) is called a **metric space**, if \(X\) is a nonempty set and \(d\) is a distance on \(X\), that is a function \(d: X^2 \to \mathbb{R}_+\) such that:

(i) \(d(x_1, x_2) = 0\) if and only if \(x_1 = x_2\),

(ii) \(d(x_1, x_2) = d(x_2, x_1)\) for all \(x_1, x_2 \in X\),

(iii) \(d(x_1, x_2) \leq d(x_1, z) + d(z, x_2)\) for all \(x_1, x_2, z \in X\).
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**Multidistance:** A generalization of a distance by Martín and Mayor.

We say that \(d : \bigcup_{n \geq 1} X^n \to \mathbb{R}_+\) is a *multidistance* if:

(i) \(d(x_1, \ldots, x_n) = 0\) if and only if \(x_1 = \cdots = x_n\),
(ii) \(d(x_1, \ldots, x_n) = d(x_{\pi(1)}, \ldots, x_{\pi(n)})\) for all \(x_1, \ldots, x_n \in X\) and all \(\pi \in S_n\),
(iii) \(d(x_1, \ldots, x_n) \leq \sum_{i=1}^n d(x_i, z)\) for all \(x_1, \ldots, x_n, z \in X\).
Definition

We say that $d : X^n \to \mathbb{R}_+ \ (n \geq 2)$ is an $n$-distance if:

1. $d(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n$,
2. $d(x_1, \ldots, x_n) = d(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for all $x_1, \ldots, x_n \in X$ and all $\pi \in S_n$,
3. There is a $0 \leq K \leq 1$ such that $d(x_1, \ldots, x_n) \leq K \sum_{i=1}^{n} d(x_1, \ldots, x_n) \mid x_i = z$ for all $x_1, \ldots, x_n, z \in X$.

We denote by $K^*$ the smallest constant $K$ for which (iii) holds.

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We say that \( d : X^n \to \mathbb{R}_+ \) \((n \geq 2)\) is an \( n \)-distance if:

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We denote by $K^*$ the smallest constant $K$ for which (iii) holds. For $n = 2$, we assume that $K^* = 1$. 
Example (Drastic $n$-distance)

The function $d: X^n \rightarrow \mathbb{R}_+$ defined by $d(x_1, \ldots, x_n) = 0$, if $x_1 = \cdots = x_n$, and $d(x_1, \ldots, x_n) = 1$, otherwise.
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$K^* = \frac{1}{n-1}$ for every $n \geq 2$. 

Proposition

Let $d$ and $d'$ be $n$-distances on $X$ and let $\lambda > 0$. The following assertions hold.

(1) $d + d'$ and $\lambda d$ are $n$-distance on $X$.

(2) $d_1 + d_2$ is an $n$-distance on $X$, with value in $[0, 1]$. 

Lemma

Let $a, a_1, \ldots, a_n$ be nonnegative real numbers such that $\sum_{i=1}^n a_i \geq a_i$. Then $a_1 + a_2 \leq a_1 + a_2 + \cdots + a_n$. 

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Lemma

Let $a, a_1, \ldots, a_n$ be nonnegative real numbers such that $\sum_{i=1}^n a_i \geq a$. Then

$$\frac{a}{1+a} \leq \frac{a_1}{1+a_1} + \cdots + \frac{a_n}{1+a_n}.$$
A generalization of $n$-distance

Condition (iii) in Definition 1 can be generalized as follows.

**Definition**

Let $g : \mathbb{R}_+^n \to \mathbb{R}_+$ be a symmetric function. We say that a function $d : X^n \to \mathbb{R}_+$ is a $g$-distance if it satisfies conditions (i), (ii) and

$$d(x_1, \ldots, x_n) \leq g(d(x_1, \ldots, x_n)|_{x_1 = z}, \ldots, d(x_1, \ldots, x_n)|_{x_n = z})$$

for all $x_1, \ldots, x_n, z \in X$. 
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for all $x_1, \ldots, x_n, z \in X$.

It is natural to ask that $d + d', \lambda d$, and $\frac{d}{1+d}$ be $g$-distances whenever so are $d$ and $d'$. 
Proposition

Let $g : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ be a (symmetric) function, $d$ and $d'$ be $g$-distances. The following assertions hold.

(1) If $g$ is positively homogeneous, i.e., $g(\lambda r) = \lambda g(r)$ for all $r \in \mathbb{R}_+^n$ and all $\lambda > 0$, then for every $\lambda > 0$, $\lambda d$ is a $g$-distance.
Proposition

Let \( g : \mathbb{R}^n_+ \to \mathbb{R}_+ \) be a (symmetric) function, \( d \) and \( d' \) be \( g \)-distances. The following assertions hold.

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(2) If \( g \) is superadditive, i.e., \( g(\mathbf{r} + \mathbf{s}) \geq g(\mathbf{r}) + g(\mathbf{s}) \) for all \( \mathbf{r}, \mathbf{s} \in \mathbb{R}^n_+ \), then \( d + d' \) is a \( g \)-distance.
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2. If \( g \) is superadditive, i.e., \( g(r + s) \geq g(r) + g(s) \) for all \( r, s \in \mathbb{R}^n_+ \), then \( d + d' \) is a \( g \)-distance.

3. If \( g \) is both positively homogeneous and superadditive, then it is concave.
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Let $g : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ be a (symmetric) function, $d$ and $d'$ be $g$-distances. The following assertions hold.

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(3) If $g$ is both positively homogeneous and superadditive, then it is concave.

(4) If $g$ is bounded below (at least on a measurable set) and additive, that is, $g(\mathbf{r} + \mathbf{s}) = g(\mathbf{r}) + g(\mathbf{s})$ for all $\mathbf{r}, \mathbf{s} \in \mathbb{R}_+^n$, then and only then there exist $\lambda_1, \ldots, \lambda_n \geq 0$ such that

$$g(\mathbf{r}) = \sum_{i=1}^{n} \lambda_i \mathbf{r}_i$$

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d: $X^n \to \mathbb{R}_+$ ($n \geq 2$) is an $n$-distance if satisfies (i), (ii) and (iii) There is a $0 \leq K \leq 1$ such that

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Example I.

What would be $K*$?
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**Example (Basic examples)**

Given a metric space $(X, d)$ and $n \geq 2$, the maps $d_{\text{max}} : X^n \to \mathbb{R}_+$ and $d_{\Sigma} : X^n \to \mathbb{R}_+$ defined by

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d_{\text{max}}(x_1, \ldots, x_n) = \max_{1 \leq i < j \leq n} d(x_i, x_j)
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are $n$-distances for which the best constants are given by $K^* = \frac{1}{n-1}$. 
Generalization

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Let $\mathcal{P}$ be a class of graphs over $x_1, \ldots, x_n$.

**Theorem**

Let $(X, d)$ be a metric space and $n \geq 2$. Then for any nonempty class $\mathcal{P}$ the map $d_{Gr}: X^n \to \mathbb{R}_+$ defined by

$$d_{Gr}(x_1, \ldots, x_n) = \max_{G \in \mathcal{P}} \sum_{(x_i, x_j) \in E(G)} d(x_i, x_j)$$

are $n$-distances for which the best constants are given by $K^* = \frac{1}{n-1}$.
Example

1. If \( \mathcal{P} = \{ G \cong K_2 \} \), then \( d_{Gr} = d_{\max}(x_1, \ldots, x_n) \).
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2. If $\mathcal{P} = \{ G \cong K_n \}$, then $d_{Gr} = d_{\Sigma}(x_1, \ldots, x_n)$. 

5. $\mathcal{P}$ is a class of circles of given size, or the class of spanning trees, etc.
Example

1. If $\mathcal{P} = \{ G \simeq K_2 \}$, then $d_{Gr} = d_{\text{max}}(x_1, \ldots, x_n)$.
2. If $\mathcal{P} = \{ G \simeq K_n \}$, then $d_{Gr} = d_\Sigma(x_1, \ldots, x_n)$.
3. For any $1 \leq s \leq n$ let $\mathcal{P} = \{ G \simeq K_s \}$. Then

$$d_{K_s}(x_1, \ldots, x_n) = \max_{G \in \mathcal{P}} \sum_{(x_i, x_j) \in E(G)} d(x_i, x_j)$$

is an $n$-metric with $K^* = \frac{1}{n-1}$.
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4. If $\mathcal{P}$ is the class of Hamiltonian cycles of $K_n$. Then
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Examples II.

Example (Geometric constructions)
Let $x_1, \ldots, x_n$ be $n \geq 2$ arbitrary points in $\mathbb{R}^k$ ($k \geq 2$) and denote by $B(x_1, \ldots, x_n)$ the smallest closed ball containing $x_1, \ldots, x_n$. It can be shown that this ball always exist, is unique, and can be determined in linear time.
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(1) The radius of $B(x_1, \ldots, x_n)$ is an $n$-distance whose best constant $K^* = \frac{1}{n-1}$.
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2. If $k = 2$, then the area of $B(x_1, \ldots, x_n)$ is an $n$-distance whose best constant $K^* = \frac{1}{n-3/2}$.
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1. The radius of $B(x_1, \ldots, x_n)$ is an $n$-distance whose best constant $K^* = \frac{1}{n-1}$.

2. If $k = 2$, then the area of $B(x_1, \ldots, x_n)$ is an $n$-distance whose best constant $K^* = \frac{1}{n-3/2}$.

3. The $k$-dimensional volume of $B(x_1, \ldots, x_n)$ is an $n$-distance and we conjecture that the best constant $K^*$ is given by $K^* = \frac{1}{n-2+(1/2)^{k-1}}$. This is correct for $k = 1$ or 2.
Examples III.

Example (Fermat point based $n$-distances)

Given a metric space $(X, d)$, and an integer $n \geq 2$, the Fermat set $F_Y$ of any element subset $Y = \{x_1, \ldots, x_n\}$ of $X$, is defined as

$$F_Y = \left\{ x \in X : \sum_{i=1}^{n} d(x_i, x) \leq \sum_{i=1}^{n} d(x_i, z) \text{ for all } z \in X \right\}.$$
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Since $h(x) = \sum_{i=1}^{n} d(x_i, x)$ is continuous and bounded from below by $0$, $F_Y$ is non-empty but usually not a singleton.

We can define $d_F : X^n \to \mathbb{R}_+$ by

$$d_F(x_1, \ldots, x_n) = \min \left\{ \sum_{i=1}^{n} d(x_i, x) : x \in X \right\}.$$ 

**Proposition**

$d_F$ is an $n$-distance and $K^* \leq \frac{1}{\lceil \frac{n-1}{2} \rceil}$. 
Median graphs

Let $G = (V, E)$ be an undirected graph.
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Let $G = (V, E)$ be an undirected graph. $G$ is called median graph if for every $u, v, w \in V$ there is a unique $z := m(u, v, w)$ such that $z$ is in the intersection of shortest paths between any two elements among $u, v, w$. 

Examples: Hypercubes and trees. We can define $d_m: V^3 \to \mathbb{R}^+$ by $d_m(u, v, w) = \min_{s \in V} \{d(u, s) + d(v, s) + d(w, s)\}$.

Proposition $d_m$ is a 3-distance, $d_m(u, v, w)$ is realized by $s = m(u, v, w)$ and $K^* = \frac{1}{2}$. 
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Every median graph can be embedded into a hypercube $H_m = \{0, 1\}^m$ for some $m$ (with respect to the Hamming-distance). For a given $m$, we can define $d_{gm}$ by

$$d_{gm}(x_1, \ldots, x_n) = \min_{z \in V(H_m)} \sum_{i=1}^{n} d(z, x_i).$$

Let $m = \text{Maj}(x_1, \ldots, x_n)$ denote the majority of $x_1, \ldots, x_n$. Theorem $d_{gm}$ is a $n$-distance, $d_{gm}(x_1, \ldots, x_n)$ is realized by (any) $m = \text{Maj}(x_1, \ldots, x_n)$ and $K^* = \frac{1}{n-1}$. 
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**Theorem**

$d_{gm}$ is a $n$-distance, $d_{gm}(x_1, \ldots, x_n)$ is realized by (any) $m = Maj(x_1, \ldots, x_n)$ and $K^* = \frac{1}{n-1}$.
\(K^* = 1, \text{ Example IV.}\)

For all of the previous examples \(\frac{1}{n-1} \leq K^* \leq \frac{1}{n-2}\) (when we know the exact value).

Question

Are there any \(n\)-distance \(d\) such that the \(K^* = 1\) for any \(n\)?
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*Are there any \( n \)-distance \( d \) such that the \( K^* = 1 \) for any \( n \)?*

Yes. In \( \mathbb{R} \) we can define

\[
A_n(x) = \frac{x_1 + \cdots + x_n}{n}, \quad \min_n(x) = \min\{x_1, \ldots, x_n\}
\]

and \( d_n(x) = A_n(x) - \min_n(x) \), where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \).
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$d_n$ is an $n$-distance for every $n \geq 2$ and $K^* = 1$. 
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$d_n$ is an $n$-distance for every $n \geq 2$ and $K^* = 1$.

*But it is not realized. (For every $\varepsilon > 0$ it can be shown that $K^* > 1 - \varepsilon$.)*
### Summary

#### Table: Critical values

<table>
<thead>
<tr>
<th>$n$-distance</th>
<th>space $X$</th>
<th>$K^*$</th>
<th>nb. of var.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{Gr}$, $d_{max}$, $d_\sum$</td>
<td>arbitrary metric</td>
<td>$\frac{1}{n-1}$</td>
<td>$n &gt; 1$</td>
</tr>
<tr>
<td>$d_{diameter}$</td>
<td>$\mathbb{R}^m$ ($m \geq 1$)</td>
<td>$\frac{1}{n-1}$</td>
<td>$n &gt; 1$</td>
</tr>
<tr>
<td>$d_{area}$</td>
<td>$\mathbb{R}^m$ ($m \geq 2$)</td>
<td>$\frac{1}{n-3/2}$</td>
<td>$n &gt; 1$</td>
</tr>
<tr>
<td>$d_{volume(k)}$</td>
<td>$\mathbb{R}^m$ ($m \geq k$)</td>
<td>$\frac{1}{n-1-(1/2)^{k-1}}$</td>
<td>$n &gt; 1$</td>
</tr>
<tr>
<td>$d_{Fermat}$</td>
<td>arbitrary metric</td>
<td>$\frac{1}{\lceil n-1/2 \rceil}$</td>
<td>$n &gt; 1$</td>
</tr>
<tr>
<td>$d_{median}$</td>
<td>median graph $G$</td>
<td>$\frac{1}{2}$</td>
<td>$n = 3$</td>
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<tr>
<td>$d_{hypercube}$</td>
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<tr>
<td>$d_n$</td>
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### Conjecture

\[
\frac{1}{n-1} \leq K^* \leq 1.
\]
Question

1. Are there any $n$-distance such that $K^* < \frac{1}{n-1}$?
2. Can we characterize the $n$-distances for which $K^* = \frac{1}{n-1}$?
3. Can we characterize the $n$-distances for which $K^* = 1$?
4. Can we show an example where $K^* = 1$ is realized?
Thank you for your kind attention!