ON 1-DIMENSIONAL SHEAVES ON PROJECTIVE PLANE
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Abstract. Let $M$ be the Simpson moduli space of semistable sheaves on the projective plane with fixed linear Hilbert polynomial $P(m) = dm + c$. A generic sheaf in $M$ is a vector bundle on its Fitting support, which is a planar projective curve of degree $d$. The sheaves that are not vector bundles on their support constitute a closed subvariety $M'$ in $M$.

We study the geometry of $M'$ in the case of Hilbert polynomials $dm - 1$, $d > 3$, and demonstrate that $M'$ is a singular variety of codimension 2 in $M$.

We speculate on how the question we study is related to recompactifying of the Simpson moduli spaces by vector bundles.

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0. Introduction

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero, say $\mathbb{C}$. Let $V$ be a vector space over $\mathbb{k}$ of dimension 3, and let $\mathbb{P}^2 = \mathbb{P}V$ be the corresponding projective plane.

**Definition 0.1.** A 1-dimensional sheaf $\mathcal{F}$ on $\mathbb{P}^2$ is a pure coherent with $\dim \text{Supp} \mathcal{F} = 1$, i.e., $C = \text{Supp} \mathcal{F} \subseteq \mathbb{P}^2$ is a curve.

Since purity implies torsion-freeness on support and since torsion free sheaves on smooth curves are locally free, a generic 1-dimensional sheaf is a vector bundle on an algebraic curve.

One can see (cf. [14]) that 1-dimensional sheaves on $\mathbb{P}^2$ are in one-to-one correspondence with the pairs $(E, f)$, where $E = \oplus \mathcal{O}_{\mathbb{P}^2}(a_i)$ is a direct sum of line bundles on $\mathbb{P}^2$ and $E \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{f} E$ is an injective morphism of sheaves.

For a sheaf $\mathcal{F}$, let $P(m) = P_\mathcal{F}(m)$ be its Hilbert polynomial. Its degree equals the dimension of the support of $\mathcal{F}$, so 1-dimensional sheaves have linear Hilbert polynomials $dm + c$, $d, m \in \mathbb{Z}$.

Let $M = M_{dm+c}(\mathbb{P}^2)$ be the Simpson moduli space of semi-stable sheaves on $\mathbb{P}^2$ with Hilbert polynomial $dm + c$.

Recall that $\mathcal{F}$ is called semistable resp. stable if it is pure and for every proper subsheaf $\mathcal{E} \subseteq \mathcal{F}$ with $P_\mathcal{E}(m) = d'm + c'$ it holds $c'/d' \leq c/d$ resp. $c'/d' < c/d$.

**Properties of $M$.**

- $M$ is projective, irreducible, locally factorial, dim $M = d^2 + 1$ (Le Potier [7]). If gcd$(d, c) = 1$, $M$ is a fine moduli space, there are only stable sheaves, $M$ is smooth (Le Potier [7]).

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• Canonical singularities for $\gcd(d, c) \neq 1$ (Woolf [13]).
• For small $d$ one has
  – the Betti numbers (Choi, Chung, Maican [1], [2], [3]);
  – description of $M$ in terms of locally closed strata (Drezet, Maican [4], [10], [11]).
• Isomorphisms:
  – an obvious one $M_{dm+c} \cong M_{dm+c+d}$, $F \mapsto F \otimes \mathcal{O}_{\mathbb{P}_2}(1)$;
  – a non-obvious one $M_{dm+c} \cong M_{dm-c}$, $F \mapsto \mathcal{E}xt^1(F, \omega_{\mathbb{P}_2})$ (Maican [9])
• $M_{dm+c} \cong M_{dm+c'}$ iff $d = d'$ and $c = \pm c'$ mod $d$ (Woolf [13]).

**Definition 0.2.** A 1-dimensional sheaf $F$ is called singular if it is locally free on its support.

Let $M' \subseteq M$ be the closed subvariety of singular sheaves. If $M'$ is non-empty, then $M \setminus M'$ is a space of vector bundles on support and $M$ is its compactification. Since $\text{codim}_M M' > 1$ in general, this compactification is not maximal.

**Questions.**

• Study $M'$.
• Find a maximal compactification with a geometric meaning.
• Find a maximal compactification by vector bundles (on support).

We restrict ourselves to the case $\gcd(d, c) = 1$, i.e., to the case of the moduli spaces of isomorphism classes.

1. **FIRST EXAMPLES**

**Trivial examples.**

• A sheaf $F$ belongs to $M_{m+1}(\mathbb{P}_2)$ if and only $F \cong \mathcal{O}_L$ for a line $L \subseteq \mathbb{P}_2$.
• A sheaf $F$ belongs to $M_{m+1}(\mathbb{P}_2)$ if and only $F \cong \mathcal{O}_C$ for a conic $C \subseteq \mathbb{P}_2$.

In these cases $M' = \emptyset$.

**A non-trivial example.** A sheaf $F$ belongs to $M = M_{3m-1}(\mathbb{P}_2)$ if and only if it is isomorphic to the ideal sheaf of a point $p$ on a cubic planar curve $C \subseteq \mathbb{P}_2$, i.e., there is an exact sequence

$$0 \rightarrow F \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{\{p\}} \rightarrow 0.$$ 

Then $M$ is isomorphic to the universal cubic curve

$$\{(p, C) \mid p \in C, \ C \text{ is a cubic curve in } \mathbb{P}_2\}.$$ 

$F$ is singular iff $F_p \not\cong \mathcal{O}_{C,p}$ iff $p \in \text{Sing } C$, hence $M'$ is the universal singular locus

$$\{(p, C) \mid p \in \text{Sing } C, \ C \text{ is a cubic curve in } \mathbb{P}_2\},$$

which is a smooth subvariety of codimension 2 in $M$.

A construction that interprets $\text{Bl}_{M'} M$ as a compactification of $M \setminus M'$ by vector bundles on curves in reducible surfaces $D(p)$ was given in [9]. A singular sheaf $\tilde{F}$ given by $p \in C$ is substituted by sheaves on curves $C_0 \cup C_1 \subseteq D(p)$.

The surface $D(p)$ consists of two irreducible components $D_0(p) \cup D_1(p)$, where $D_0(p) = \text{Bl}_p \mathbb{P}_2$ is the blow up of $\mathbb{P}_2$ at $p$ and $D_1(p)$ is a projective plane attached to $D_0(p)$ along the exceptional line.
This construction is very explicit and uses heavily the properties of $M'$, which motivated us to study the subvarieties of singular sheaves in the Simpson moduli spaces for Hilbert polynomials $dm + c$, $d \geq 4$.

2. MODULI SPACES $M_{dm-1}(\mathbb{P}_2)$

We consider the moduli spaces $M = M_{dm-1}(\mathbb{P}_2)$, $d \geq 4$. Their description in terms of locally closed strata, each of which is described as a quotient, is given in [4], [10], [11]. For an arbitrary $d$ one has a good understanding of the open Brill-Noether locus

$M_0 = \{ F \in M \mid h^0(F) = 0 \}$.

$\mathcal{F} \in M_0$ if and only if $\mathcal{F}$ has a locally free resolution

$0 \to \mathcal{E}_1 \xrightarrow{A} \mathcal{E}_0 \to \mathcal{F} \to 0, \quad \mathcal{E}_1 = \mathcal{O}_{\mathbb{P}_2}(-3) \oplus (d-2)\mathcal{O}_{\mathbb{P}_2}(-2), \quad \mathcal{E}_0 = (d-1)\mathcal{O}_{\mathbb{P}_2}(-1)$

with $A = \begin{pmatrix} Q \end{pmatrix}$ such that $Q$ is a $1 \times (d-1)$ of quadratic forms and $\Phi$ is a stable Kronecker module, i.e., a $(d-2) \times (d-1)$ matrix of linear forms that is not equivalent under the action of $\text{GL}_{d-2}(k) \times \text{GL}_{d-1}(k)$ to a matrix with a zero block of size $j \times (d-1-j)$, $j = 1, \ldots, d-2$.

This describes $M_0$ as a quotient

$\{ A = \begin{pmatrix} Q \end{pmatrix} \text{ as above} \} / \text{Aut}(\mathcal{E}_1) \times \text{Aut}(\mathcal{E}_0)$

with the induced map to the quotient space of the stable Kronecker modules $\Phi$

$M_0 \to N, \quad N = N(3; d-2, d-1) = \{ \Phi \} / \text{GL}_{d-2}(k) \times \text{GL}_{d-1}(k)$.

Taking all matrices $A$ as above (not necessarily injective), one gets a projective quotient $\mathbb{B} = \{ A \}^{ss} / \text{Aut}(\mathcal{E}_1) \times \text{Aut}(\mathcal{E}_0)$, with a map $\mathbb{B} \to N$, which is a projective bundle associated to a vector bundle of rank $3d$ over $N$ (cf. [8]).

Consider an open subvariety $N_0 \subseteq N$ corresponding to $\Phi$ with coprime maximal minors. Then $N_0$ is isomorphic to an open subvariety $H_0 \subseteq \mathbb{P}_2[l]$, $l = (d-1)(d-2)/2$, in the Hilbert scheme of $l$ points on $\mathbb{P}_2$ that do not lie on a curve of degree $d - 3$. The class of $[\Phi]$ is sent to the zero scheme of its maximal minors.

Put $M_{00} = \mathbb{B}|_{N_0}$, then codim$_M M \setminus M_0 \geq 2$ as shown in [14].

$\mathcal{F} \in M_{00}$ if and only if $\mathcal{F}$ is a twisted ideal sheaf of $Z \in H_0$ on $C = \text{Supp} \mathcal{F}$:

$0 \to \mathcal{F} \to \mathcal{O}_C(d-3) \to \mathcal{O}_Z \to 0$.

The fibre over $[\Phi] \in N_0$ can be interpreted as curves of degree $d$ through $Z \in H_0$ that corresponds to $[\Phi]$ under the isomorphism $N_0 \cong H_0$. Thus $M_{00}$ can be seen as an open subvariety of the Hilbert flag scheme $H(l, d)$ of $l$ points on a curve of degree $d$.

3. SINGULAR SHEAVES IN $M_{00}$

The results in this section are obtained in [8] together with Alain Leytem, a PhD student of Martin Schlichenmaier at the University of Luxembourg.

Let us mention some necessary conditions for $\mathcal{F} \in M_{00}$ to be singular.

- $C = \text{Supp} \mathcal{F}$ must be singular as torsion free sheaves on a smooth curves are locally free.
As $\mathcal{F}$ is a twisted ideal sheaf of $Z \subseteq C$, $\mathcal{F}$ can only be singular at points from $Z$, thus $Z \cap \text{Sing } C \neq \emptyset$ if $\mathcal{F}$ is singular.

**Claim.** If $Z$ consists of $l$ different points, then $\mathcal{F}$ is singular if and only if $Z \cap \text{Sing } C \neq \emptyset$.

Indeed, $\mathcal{F}$ is singular if and only if there exists $p \in Z$ with $\mathcal{F}_p \not\cong \mathcal{O}_{C,p}$, which, in turn, holds if and only if there is a point $p \in Z \cap \text{Sing } C$.

Let now $Z = \bigsqcup Z_i$, where $Z_i$ is a (fat) point at $p_i \in \mathbb{P}_2$. Assume that for a given $i$, $Z_i$ is a curvilinear point that in some local coordinates $x, y$ at $p_i$ is given as $Z(x - h(y), y^n)$. Let $(C_i, p_i)$ be the germ of the smooth curve $C_i = Z(x - h(y))$.

**Claim.** $\mathcal{F}$ is non-singular at $p_i \in \text{Sing } C$ if and only if $(C_i, p_i) \cap (C, p) = (Z_i, p_i)$ (intersection of germs of curves).

**Proof.** Straightforward: write $C = Z(\det \begin{pmatrix} x - h(y) \\ w(y) \\ v(x,y) \end{pmatrix})$ and study when the ideal of $Z_i \subseteq C$ is 1-generated. 

If $Z_i$ is a double point on a line $L$, then $\mathcal{F}$ is non-singular at $p_i \in \text{Sing } C$ iff the tangent cone of $C$ at $p_i$ consists of two lines different from $L$. This shows that the sheaves singular at double points are limits of the sheaves singular at simple points.

Now fix a basis $(x_0, x_1, x_2) \in V^*$, assume $p_i = (1, 0, 0) = Z(x_1, x_2)$, and assume that $Z$ contains at most 1 fat point and this fat point can only be a double point. Then the requirement for $\mathcal{F}$ to be singular at $p_i$ imposes 2 linear independent conditions on $C$ (independent from the conditions imposed by the condition $Z \subseteq C$):

- vanishing of the coefficients of the monomials $x_0^{d-1}x_1$, $x_0^{d-1}x_2$ in the equation of $C$ if $Z_1 = p_1$ is a simple point;
- vanishing of the coefficients of the monomials $x_0^{d-1}x_1$, $x_0^{d-2}x_2^2$ in the equation of $C$ if $Z_1 = Z(x_1, x_2^2)$ is a double point.

The imposed conditions are independent because $Z$ does not lie on a curve of degree $d - 3$.

We conclude that for $M'_{00} = M_{00} \cap M' \subseteq M_{00} \rightarrow N_0 \cong H_0$, the fibre over the locus $H_c$ of the configurations ($l$ different points) is a union of $l$ linear subspaces of codimension 2 in $\mathbb{P}_{3d-1}$ (the fibre of $M_{00} \rightarrow N_0$). The fibre over

$$H_1 = \{Z \mid \text{with exactly 1 fat point}\}$$

is a union of $l - 1$ linear subspaces of codimension 2.

**Conclusion.** Fibres of $M'_{00}$ over $N_c \subseteq N_1 \cong H_c \cap H_1$ are singular of codimension 2. Therefore, $M'_{00}$ is singular of codimension 2 in $M_{00}$. Since $\text{codim}_M M \setminus M_{00} \geq 2$ (Yuan, [14]), we obtain the following theorem.

**Theorem.** $M'$ is singular of codimension 2 in $M$.

The restriction $M'_{00}|_{N_c}$ is a family of arrangements of linear subspaces of codimension 2 in $\mathbb{P}_{3d-1}$.

4. Speculations on recompactifying the Simpson moduli spaces

**Aim.** Interpret the blow-up

$$\text{Bl}_{M'} M \rightarrow M$$

as a process that substitutes a singular sheaf $\mathcal{F} \in M_{00}|_{H_c}$ given by $Z \subseteq C$ with non-empty $Z \cap \text{Sing } C = \{q_1, \ldots, q_r\}$ by sheaves $\tilde{\mathcal{F}}$ on a curve $C' = C_0 \cup C_1 \cup \cdots \cup C_r$ in a reduced surface $D(q_1, \ldots, q_r)$ obtained by blowing up the points $q_1, \ldots, q_r$ and attaching to the exceptional lines $L_1, \ldots, L_r$ surfaces $D_1(q_i) \cong \mathbb{P}_2$ such that $C_0$ is the proper transform of $C$ in $D_0(q_1, \ldots, q_r) = \text{Bl}_{\{q_1, \ldots, q_r\}} \mathbb{P}_2$ and $C_i \subseteq D_1(q_i)$. $\tilde{\mathcal{F}}$ is a twisted ideal sheaf of $l$ points $\{\tilde{q}_1, \ldots, \tilde{q}_r, \tilde{p}_{r+1}, \ldots, \tilde{p}_l\}$ in $C'$ with $\tilde{q}_i \in C_i \subseteq D_1(q_i)$ and the points $\tilde{p}_{r+1}, \ldots, \tilde{p}_l$ being preimages of $p_{r+1}, \ldots, p_l$ in $D_0(q_1, \ldots, q_r)$. The sheaf $\tilde{\mathcal{F}}$ is locally free on $C'$ or “less singular”.
Iterating this (to be) construction we want to get a recompatification of the Simpson moduli spaces by vector bundles (on 1-dimensional support).

Remark. It should be mentioned that the construction indicated here resembles the construction from [12, Theorem 4.3]. I was happy to learn this from the talk of Szilárd Szabó given at this conference.

References


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