Median algebra with a retraction: an example of variety closed under natural extension

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(joint work with Georges Hansoul)
The Big Picture
Claims about natural extensions

Natural extension provides

‘canonical extension’ for non lattice-based algebras;
insight about the construction of canonical extension.
Back to the roots: canonical extension

\[ L = \langle L, \lor, \land, 0, 1 \rangle \] is a DL

Canonical extension \( L^\delta \) of \( L \) comes with topologies \( \iota \) and \( \delta \):

- \( L^\delta \) is doubly algebraic.
- \( L \hookrightarrow L^\delta \).
- \( L \) is dense in \( L^\delta \).
- \( L \) is dense and discrete in \( L^\delta \).
**Problem.** Given \( f : L \rightarrow E \), define \( f^\delta : L^\delta \rightarrow E^\delta \).

**Solution.**

- \( L \) is made of the isolated points of \( L^\delta \),
- \( L \) is dense in \( L^\delta \),
- \( f^\delta := \liminf_\delta f \) and \( f^\pi := \limsup_\delta f \).

Leads to canonical extension of ordered algebras:

Why canonical extension?

It provides completeness results for modal logics with respects to classes of Kripke frames:


Is it possible to generalize canonical extension to non lattice-based algebras?

**Problem 1.** Define the natural extension $A^\delta$ of $A$:


**Problem 2.** Extend $f : A \rightarrow B$ to $f^\delta : A^\delta \rightarrow B^\delta$.

We give a partial solution.
Natural extension of algebras
The framework of natural extension

A$^\delta$ can be defined if A belongs to some

$$\text{ISP}(\mathcal{M})$$

where $\mathcal{M}$ is class of finite algebras of the same type.

A$^\delta$ can more easily be computed if $\text{ISP}(\mathcal{M})$ is dualisable (in the sense of natural dualities).
We adopt the setting of natural dualities

\[ \mathbf{M} \equiv \text{a finite algebra} \]

A discrete alter-ego topological structure \( \overset{\sim}{\mathbf{M}} \)

\[ \mathbf{A} \in \text{ISP}(\mathbf{M}) \]

<table>
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<th>Algebra</th>
<th>Topology</th>
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<td>( \mathbf{M} )</td>
<td>( \overset{\sim}{\mathbf{M}} )</td>
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<td>( \mathcal{A} = \text{ISP}(\mathbf{M}) )</td>
<td>( \mathcal{X} = \text{ISP}_{c}^{+}(\overset{\sim}{\mathbf{M}}) )</td>
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<td>( \mathbf{A} )</td>
<td>( \mathbf{A}^{*} = \mathcal{A}(\mathbf{A}, \mathbf{M}) \leq_{c} \mathbf{M}^{\mathbf{A}} )</td>
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<td>( \mathcal{X} \ast = \mathcal{X}(\mathcal{X}, \mathbf{M}) \leq \mathbf{M}^{\mathcal{X}} )</td>
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**Definition.** \( \overset{\sim}{\mathbf{M}} \) **yields a natural duality** for \( \text{ISP}(\mathbf{M}) \) if

\[ (\mathbf{A}^{*})_{\ast} \simeq \mathbf{A}, \quad \mathbf{A} \in \text{ISP}(\mathbf{M}). \]
Natural extension of an algebra can be constructed from its dual

Priestley duality is a natural duality: \( L \simeq (L^*)^* \).

**Proposition** (Gehrke and Jónsson)
If \( L \in DL \) then \( L^\delta \) is the algebra of order-preserving maps from \( L^* \) to \( \tilde{2} \).

Assume that \( M \) yields a duality for \( ISP(M) \).

**Proposition** (Davey and al.)
If \( A \in ISP(M) \), then \( A^\delta \) is the algebra of structure-preserving maps from \( A^* \) to \( \tilde{M} \).
Natural extension of median algebras
The variety of median algebras is an old friend.

The expression

$$(x \land y) \lor (x \land z) \lor (y \land z)$$

defines an operation $m_{\leq}(x, y, z)$ on a distributive lattice $(L, \leq)$.

**Definition.** (Avann, 1948)

*median algebra* $A = (A, m) \iff$ subalgebra of some $(L, m_{\leq})$

**Example.** Set $2 := \langle \{0, 1\}, m \rangle$ where $m$ is the majority function.

**Theorem.** The variety $\mathcal{A}_m$ of median algebras is $\text{ISP}(2)$. 
\[(x \land y) \lor (x \land z) \lor (y \land z)\]

Examples.

\[m(\bullet, \bullet, \bullet) = \bullet\]

Median graphs

Some metric spaces
For every $a \in A$, the relation $\leq_a$ defined on $A$ by

$$b \leq_a c \quad \text{if} \quad m(a, b, c) = b.$$ 

is a $\land$-semilattice order on $A$ with $b \land_a c = m(a, b, c)$.

Semilattices obtained in this way are the *median semilattices*.

**Proposition.** In a median semilattice, principal ideals are distributive lattices.

There is a natural duality for median algebras

\[ \mathcal{A}_m = \ISP(2) \]

\[ \mathcal{Z} := \langle \{0, 1\}, \leq, \cdot, 0, 1, \iota \rangle. \]

**Theorem** (Isbell (1980), Werner (1981)). The structure \( \mathcal{Z} \) yields a logarithmic duality for \( \mathcal{A}_m \).

\( \mathcal{A}^\delta \) is the algebra of structure-preserving maps \( x : \mathcal{A}^* \to \mathcal{Z} \).
Natural extension completes everything it can complete

**Theorem.** Let \( a \in A \).

- \( \langle A^\delta, \leq_a \rangle \) a bounded-complete extension of \( \langle A, \leq_a \rangle \).
- If \( I \) is a distributive lattice in \( A \) then \( c_{I_{A^\delta}}(I) = I^\delta \)
Natural extension of maps
**A can be defined topologically in** $A^\delta$

$\mathcal{X}_p(A^*, M) \equiv$ set of morphisms defined on a closed substructure of $A^*$.

**Definition.**

$$O_f := \{ x \in \mathcal{X}(A^*, M) \mid x \supseteq f \}, \quad f \in \mathcal{X}_p(A^*, M)$$

$$\Delta := \{ O_f \mid f \in \mathcal{X}_p(A^*, M) \}$$

**Working assumption.** $M$ yields a full logarithmic duality for $\text{ISP}(M)$ and $M$ is injective in $\text{ISP}^+(M)$.

**Proposition.**

- $\Delta$ is a basis of topology $\delta$
- $A$ is dense and discrete in $A^\delta$.
- In the settings of DL, we get the known topology.
We canonically extends maps to multi-maps

Input:

\[ f : A \rightarrow B \]
We canonically extend maps to multi-maps

Input:

\[ f : A \rightarrow B \]

Output:

\[ f^+ : A^\delta \rightarrow \Gamma(B^\delta_\iota) \]
The multi-extension of $f : A \to B$

Intermediate step: Consider

$$\bar{f} : A \to \Gamma(B^\delta_\iota) : a \mapsto \{f(a)\}.$$ 

Recall that $A$ is dense in $A^\delta_\iota$ and $\Gamma(B^\delta_\iota)$ is a complete lattice.

**Definition.** The *multi-extension* $f^+$ of $f$ is defined by

$$f^+ : A^\delta_\iota \to \Gamma(B^\delta_\iota) : x \mapsto \limsup_\delta \bar{f}(x),$$

In other words,

$$f^+(x) = \bigcap \{\cl_{B^\delta_\iota}(f(A \cap V)) \mid V \in \delta_x\},$$

$$f^+(x)|_F = \bigcap \{f(A \cap V)|_F \mid V \in \delta_x\}, \quad F \in B^\delta.$$
The multi-extension is a continuous map

**Definition.**

We say that $f$ is **smooth** if $\#f^+(x) = 1$ for all $x \in A^\delta$.

Let $\sigma \downarrow$ be the co-Scott topology on $\Gamma(B^\delta \iota)$.

**Proposition.**

- $f^+$ is the smallest $(\delta, \sigma \downarrow)$-continuous extension from $A^\delta$ to $\Gamma(B^\delta \iota)$.

- $f$ is smooth if and only if it admits an $(\delta, \iota)$-continuous extension $f^\delta : A^\delta \to B^\delta$ satisfying $f^\delta(x) \in f^+(x)$.
This construction sheds light on canonical extension

**Proposition.** If \( f : A \to B \) is a map between DLs with lower extension \( f^\delta \) and upper extension \( f^\pi \), then for any \( x \in L^\delta \)

\[
f^\delta(x) = \bigwedge f^+(x),
\]

\[
f^\pi(x) = \bigvee f^+(x).
\]
Natural extension of median algebras with a retraction.
Natural extension of expansions of median algebras

General framework.

Let

\[ A = \langle A, m, r, a \rangle \]

where \( \langle A, m \rangle \in \mathcal{A}_m \), \( a \in A \) and \( r : A \to A \)

Set

\[ r^\delta(x) = \bigwedge_a r^+(x), \quad x \in \langle A, m \rangle^\delta. \]

\[ A^\delta := \langle \langle A, m \rangle^\delta, r^\delta, a \rangle \]
Natural properties

**Definition.** A property $P$ of algebras in $\mathcal{A}$ is *natural* if

$$\mathbf{A} \models P \implies \mathbf{A}^\delta \models P,$$

$\mathbf{A} \in \mathcal{A}$

**Example.** The property ‘being a median algebra of a Boolean algebra’ is natural.
Median algebras with a retraction

**Definition.** An idempotent homomorphism $r : A \to A$ such that $u(A)$ is convex is called a *retraction*.

**Proposition.** A map $r : A \to A$ is a retraction if and only if

$$r(m(x, y, z)) = m(x, r(y), r(z)), \quad x, y, z \in A.$$

**Definition.** An algebra $\langle A, m, r, a \rangle$ is a *pointed retract algebra* if $r$ is a retraction of the median algebra $\langle A, m \rangle$ and $a \in A$. 
The variety of pointed retract algebras is natural

**Theorem.** If $A$ is a pointed retract algebra then $A^\delta$ is a pointed retract algebra.

**Sketch of the proof.**
Proves equalities of the type

$$(r \circ m)^\delta = r^\delta \circ m$$

using continuity properties of the extensions.
The variety of pointed algebras with operator is natural.

**Definition.** An algebra $A = \langle A, m, f, a \rangle$ is a **pointed median algebra with operator** if $\langle A, m \rangle$ is a median algebra, $a \in A$ and

$$f(m(a, x, y)) = m(a, f(x), f(y)), \quad x, y \in A.$$ 

**Theorem** Let $\langle A, m, f, a \rangle$ be a pointed median algebra with operator.

- $f$ is smooth.
- $A^\delta$ is a pointed median algebra with operator.

**Sketch of the proof.**

$f$ can be dualized as a relation $R$ on $A^*$ and $f^\delta$ can be explicitly computed with $R$. 

□
Questions/Problems.

- Interesting instances of natural extensions of maps (in non-ordered based algebras).
- Successful applications of the whole theory.
- Find canonical (continuous) way to pick-up some element in $f^+(x)$.
- Intrinsic definition of $\delta$ in the non-dualizable setting.