An extension result of CR functions by a general Schwarz Reflection Principle

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We study a general Schwarz Reflection Principle in one complex variable and obtain a holomorphic extension result of continuous CR functions defined on a real analytic, generic, CR submanifold in $\mathbb{C}^N$.

1 A general Schwarz Reflection Principle

The classical Schwarz Reflection Principle can be stated as follows:

**Theorem 1.1.** Suppose that $\Omega$ is a connected domain, symmetric with respect to the real axis, and that $L = \Omega \cap \mathbb{R}$ is an interval. Let $\Omega^+ = \{ z \in \Omega : \text{Im}(z) > 0 \}$. Suppose that $f \in A(\Omega^+)$, a function holomorphic on $\Omega^+$ and that $\text{Im}(f)$ has a continuous extension to $\Omega^+ \cup L$ that vanishes on $L$. Then there is a $F \in A(\Omega)$ such that $F = f$ in $\Omega^+$ and $F(z) = \overline{f(\overline{z})}$ in $\Omega - \Omega^+$.

Our generalized version replaces the vanishing of $\text{Im}(f)$ on $L$ by the real analyticity of $\text{Im}(f)$ on $L$ which is a necessary condition for the holomorphic extension. We simplify our version as follows:

**Theorem 1.2.** Let $D = D(0,1)$ be the open disc centered at the origin with radius $1$, $D^+ = \{ z \in D : \text{Im}(z) > 0 \}$, $L = D \cap \mathbb{R}$. Suppose that $f \in A(D^+)$ and $\text{Im}(f)$ has a continuous extension to $D^+ \cup L$ such that $v(x,0) = \text{Im}(f)|_L$ is real analytic at $0$ with radius of convergence $r$. Denote $D_r = D(0,r)$, then there exists $F \in A(D^+ \cup D_r)$ such that $F = f$ in $D^+$ and $F(z) = f(\overline{z}) + 2iv(z,0)$ for all $z \in D_r - D_r^+$ where $v(z,0)$ denotes a holomorphic function on $D_r$.

**Proof:** Since $v(x,0) = \text{Im}(f)|_L$ is real analytic at $0$ with radius of convergence $r$, we can write $v(x,0) = \sum a_n x^n \forall |x| < r$. Thus we get a holomorphic function $v(z,0) = \sum a_n z^n$ on $D_r$ by the complexification of the variable $x$. Now $\text{Im}(f(z) - iv(z,0))|_{(-r,r)} \equiv 0$, by the Schwarz Reflection Principle, the function

$$g(z) := \begin{cases} f(z) - iv(z,0) & \text{for } z \in D_r^+ \\ f(\overline{z}) - iv(\overline{z},0) & \text{for } z \in D_r - D_r^+ \end{cases}$$

is holomorphic on $D_r$. Thus the function

$$F(z) := g(z) + iv(z,0) = \begin{cases} f(z) & \text{for } z \in D^+ \\ \frac{f(z)}{f(\overline{z}) - iv(\overline{z},0)} + iv(z,0) & \text{for } z \in D_r - D_r^+ \end{cases}$$

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is holomorphic on \( D^+ \cup D_\rho \) and is the desire extension. Since \( a_n \) are real, 
\[
F(z) = \overline{f(z)} + 2iv(z, 0) \quad \forall z \in D_\rho - D^+.
\]

Theorem 1.2 leads to a reflection principle of harmonic functions if we impose 
some real analyticity condition on the boundary.

**Proposition 1.3.** Every regular real analytic curve \( S \subset \mathbb{C} \) is locally biholomorphic equivalent to the real line, i.e. \( \forall p_0 \in S, \exists \) a local biholomorphic map \( \phi \) from a neighborhood \( U \) of 0 in \( \mathbb{C} \) that takes 0 to \( p_0 \) and \( U \cap \mathbb{R} \) to \( \phi(U) \cap S \).

**Proof:** WLOG, assume \( p_0 = 0 \), since \( S \) is regular real analytic, then \( S \) can be locally parametrized by \((\sum_{n=1}^{\infty} a_n t^n, \sum_{n=1}^{\infty} b_n t^n)\), \( t \in (-r, r) \) with \((a_1, b_1) \neq (0, 0)\). Then the local biholomorphic mapping \( \phi(z) = \sum_{n=1}^{\infty} (a_n + ib_n) z^n \) defined on \( U = D(0, \frac{1}{\sqrt{2}}) \) maps \( U \cap \mathbb{R} \) to \( \phi(U) \cap S \) and takes 0 to 0.

By Proposition 1.3, Theorem 1.2 can be generalized as follows: if a function \( f \) holomorphic on one side of \( S \), \( \text{Im}(f) \) has a continuous extension to a neighborhood \( U \) of \( p_0 \) in \( S \) such that the restriction to \( U \) is real analytic, then \( f \) can be holomorphically extended to a neighborhood of \( p_0 \) in \( \mathbb{C} \). A generalized Corollary 1.1 will follow easily.

## 2 Holomorphic extension of CR functions on real analytic, generic CR submanifold in \( \mathbb{C}^N \)

We first introduce the necessary notations and definitions needed in the sequel. We mainly follow [BER]. A smooth (real analytic) real submanifold of \( \mathbb{C}^N \) of codimension \( d \) is a subset \( M \subset \mathbb{C}^N \) such that \( \forall p_0 \in M \), there is a neighborhood \( U \) of \( p_0 \) and a smooth (real analytic) real vector-valued function \( \rho = (\rho_1, \ldots, \rho_d) \) defined in \( U \) such that 
\[
M \cap U = \{ Z \in U : \rho(Z, \overline{Z}) = 0 \},
\]
with differentials \( d\rho_1, \ldots, d\rho_d \) linearly independent in \( U \).

**Definition 2.1.** A real submanifold \( M \subset \mathbb{C}^N \) is CR if the complex rank of the complex differentials \( \partial \rho_1, \ldots, \partial \rho_d \) is constant for \( p \in M \). It is generic if \( \partial \rho_1, \ldots, \partial \rho_d \) are \( \mathbb{C} \) linearly independent for \( p \in M \).
It should be noted that the above definitions are independent of choice of local defining functions $\rho$. Let $M \subset \mathbb{C}^N$ be a germ of real analytic, generic, CR submanifold of codimension $d$ at $p_0$, write $N = n + d$, $n \geq 1$. After a local holomorphic change of coordinates, we can assume that there exists $\Omega$, a sufficiently small open neighborhood of 0 in $\mathbb{C}^{n+d}$, such that $M$ is given in $\Omega$ by

$$Im(w) = \phi(z, \overline{z}, Re(w)),$$

with $z \in \mathbb{C}^n$, $w \in \mathbb{C}^d$, $\phi$ a real-valued analytic function (power series) of $z, \overline{z}, Re(w)$ such that $\phi(0,0,0) = 0$ and $d\phi(0,0,0) = 0$. Such a choice of coordinates is called regular coordinates.

Now suppose $M$ is a real analytic, generic submanifold, locally parametrized near $0 \in M$. Since the function $\phi(z, \overline{z}, s)$ is real analytic near $0 \in \mathbb{R}^2n$, the variable $s$ can be complexified. Thus, we extend the parametrization $\Psi(z, \overline{z}, s) = (z, s + i\phi(z, \overline{z}, s)) \in \mathbb{C}^{n+d}$ to a local real analytic diffeomorphism $\tilde{\Psi}(z, \overline{z}, s + it) = (z, s + it + i\phi(z, \overline{z}, s + it)) \in \mathbb{C}^{n+d}$, defined in a neighborhood of $0 \in \mathbb{R}^{2n+2d}$.

By the above notation, we have the following holomorphic extension result, see [BER].

**Proposition 2.1.** Let $M$ be a real analytic generic submanifold of $\mathbb{C}^{n+d}$ of codimension $d$ in regular coordinates near $0 \in M$. If $h$ be a CR function on $M$, then $h$ extends holomorphically to a full neighborhood of $0$ if and only if there exists $\epsilon > 0$ such that for every $z \in \mathbb{C}^n$, $|z| < \epsilon$, the function $s \mapsto h \circ \Psi(z, \overline{z}, s)$ extends holomorphically to the open set $\{s + it \in \mathbb{C}^d, |s| < \epsilon, |t| < \epsilon\}$ in such a way that the extension $H := h \circ \Psi(z, \overline{z}, s + it)$ is a bounded, measurable function of all its variables.

The following important corollary is an easy consequence of the above proposition.

**Corollary 2.1.** Let $M \subset \mathbb{C}^N$ be a real analytic generic submanifold and $f$ a CR function in a neighborhood of $p \in M$. Then $f$ extends as a holomorphic function in a neighborhood of $p$ in $\mathbb{C}^N$ if and only if $f$ is real analytic in a neighborhood of $p$ in $M$.

We shall describe an open wedge with a generic edge in $\mathbb{C}^{n+d}$. Let $M$ be a generic submanifold of $\mathbb{C}^{n+d}$ of codimension $d$ and $p_0 \in M$. Let $\rho = (\rho_1, ..., \rho_d)$ be a defining functions of $M$ near $p_0$ and $\Omega$ a small neighborhood of $p_0$ in $\mathbb{C}^{n+d}$ in which $\rho$ is defined. If $\Gamma$ is an open convex cone with vertex at the origin in $\mathbb{R}^d$, we set

$$W(\Omega, \rho, \Gamma) := \{Z \in \Omega : \rho(Z, \overline{Z}) \in \Gamma\}.$$

The above set is an open subset of $\mathbb{C}^{n+d}$ whose boundary contains $M \cap \Omega$. Such a set is called a wedge of edge $M$ in the direction $\Gamma$ centered at $p_0$. 

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The following shows that $W(\Omega, \rho, \Gamma)$ is in a sense independent of the choice of $\rho$, which will allow us to change defining functions freely. [BER]

**Proposition 2.2.** Let $\rho$ and $\rho'$ be two defining functions for $M$ near $p_0$, where $M$ and $p_0$ are as above. Then there is a $d \times d$ real invertible matrix $B$ such that for every $\Omega$ and $\Gamma$ as above the following holds. For any open convex cone $\Gamma_1 \subset \mathbb{R}^d$ with $B\Gamma_1 \cap S^{d-1}$ relatively compact in $\Gamma \cap S^{d-1}$ (where $S^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$), there exists $\Omega_1$, an open neighborhood of $p_0$ in $\mathbb{C}^{n+d}$, such that

$$W(\Omega_1, \rho', \Gamma_1) \subset W(\Omega, \rho, \Gamma).$$

Below is an Edge-of-the-Wedge Theorem, see [Kr], [Ru]. This theorem together with Proposition 2.1 are essential to the proof of the main theorem.

**Theorem 2.3.** If $\Gamma \subset \mathbb{R}^d$ is an open convex cone, $R > 0$, $d \geq 2$. Let $V = \Gamma \cap B(0, R)$. Let $E \subset \mathbb{R}^d$ be a nonempty neighborhood of 0. Define $W^+ \subset \mathbb{C}^d$, $W^- \subset \mathbb{C}^d$ by

$$W^+ = E + iV, \quad W^- = E - iV$$

Then there exists a fixed neighborhood $U$ of $0 \in \mathbb{C}^d$ such that the following property holds: For any continuous function $g : W^+ \cup W^- \cup E \rightarrow \mathbb{C}$ that is holomorphic on $W^+ \cup W^-$, there is a holomorphic $G$ on $U$ such that $G|_{U \cap (W^+ \cup W^- \cup E)} = g$.

**Proof:** Let $s \in E$. We may assume that $s = 0$. After composition with a linear isomorphism $A$ in $\mathbb{C}^d$ with real coefficients, we may assume that $i\{(t_1, ..., t_d) \in \mathbb{R}^d : t_j > 0, j = 1, ..., d\} \subset A^{-1}(i\Gamma)$, there exists $B(0, R') \subset \mathbb{R}^d$ with $R' > 6\sqrt{d}$ and $E' = \{s \in \mathbb{R}^d : |s_j| < 6, j = 1, ..., d\}$ such that $E' + iB(0, R') \subset A^{-1}(E + iB(0, R))$. So we reduce the case to the following.

**Claim:** Let

- $E = \{s \in \mathbb{R}^d : |s_j| < 6, j = 1, ..., d\}$,
- $V = \{t \in \mathbb{R}^d : 0 < t_j < 6, j = 1, ..., d\}$,
- $W^+ = E + iV, \quad W^- = E - iV$.

Let $U = D^d(0, 1)$. If $g : W^+ \cup W^- \cup E \rightarrow \mathbb{C}$ is continuous and $g$ is holomorphic on $W^+ \cup W^-$, then there is a holomorphic $G$ on $U$ such that $G|_{U \cap (W^+ \cup W^- \cup E)} = g$.

Let $c = \sqrt{2} - 1$ and define

$$\varphi : \overline{D}^2(0, 1) \rightarrow \mathbb{C}$$

$$(w, \lambda) \mapsto \frac{w + \lambda/c}{1 + c\lambda w}$$

Then

$$\varphi(w, \lambda) = \frac{w + \lambda/c + |\lambda|^2 w + c\lambda w}{|1 + c\lambda w|^2};$$
hence

\[\text{Im}\varphi(w, \lambda) = \frac{(1-|\lambda|^2)\text{Im}(cw) + (1-|cw|^2)\text{Im}(\lambda)}{c|1+c\lambda w|^2}\]

Notice that

1. \(\text{sgn}(\text{Im}\varphi) = \text{sgn}(\text{Im}\lambda)\) if \(|\lambda| = 1\);
2. \(\text{sgn}(\text{Im}\varphi) = \text{sgn}(\text{Im}\lambda)\) if \(w \in \mathbb{R}\);
3. \(\varphi(w, 0) = w\);
4. \(|\varphi(w, \lambda)| \leq (1 + 1/c)(1 - c) < 6\). It follows that the function

\[\Phi : D^d(0, 1) \times \mathbb{T} \rightarrow \mathbb{C}^d \quad (w, \lambda) \mapsto (\varphi(w_1, \lambda), \ldots, \varphi(w_d, \lambda))\]

satisfies

5. \(\Phi(w, e^{i\theta}) \in W^+\) if \(0 < \theta < \pi\);
6. \(\Phi(w, e^{i\theta}) \in W^-\) if \(\pi < \theta < 2\pi\);
7. \(\Phi(w, e^{i\theta}) \in E\) if \(\theta = 0\) or \(\theta = \pi\);

In short, \(\Phi(z, e^{i\theta}) \in \text{Dom}(g)\) for all \(0 \leq \theta < 2\pi\), all \(w \in D^d(0, 1)\). So we may define

\[G(w) = \frac{1}{2\pi} \int_0^{2\pi} g(\Phi(w, e^{i\theta}))d\theta, \quad w \in D^d(0, 1).\]

We claim that this is what we seek. First, \(G\) is holomorphic by an application of Morera’s theorem. Next, we note that for fixed \(s \in E \cap D^d(0, 1)\), the function \(g(\Phi(s, \cdot))\) is continuous on \(\mathbb{T}\) and holomorphic on \(\{\lambda \in D : \text{Im}\lambda > 0\text{ or } < 0\}\).

Again, by Morera, \(g(\Phi(s, \cdot))\) is holomorphic on all of \(D\). It follows by Mean Value Property that

\[G(s) = \frac{1}{2\pi} \int_0^{2\pi} g(\Phi(s, e^{i\theta}))d\theta = g(\Phi(s, 0)) = g(s);\]

hence \(G = g\) on \(E \cap D^d(0, 1)\). If \(s + it \in \mathbb{C}^d\) is fixed, \(|s + it| < 1/2, t > 0\), then the function

\[\xi \mapsto G(s + \xi t) - g(s + \xi t)\]

is holomorphic for \(\xi \in \mathbb{C}\) small and \(\equiv 0\) when \(\xi\) is real. If follows that \(G\) and \(g\) are identical on \(W^+ \cup W^-\). If follows that \(G \equiv g\) on \(W^+ \cup W^- \cup E\).

If we return to our original case, the desire extension of \(g\) is

\[G(w) = \frac{1}{2\pi} \int_0^{2\pi} g \circ A(\Phi(A^{-1}(w), e^{i\theta}))d\theta, \quad w \in A^{-1}(D^d(0, 1)).\]
since it agrees with our $g$ in some open set. Our desire $U$ is $A^{-1}(D^d(0,1))$. 

According to Corollary 2.1, a CR function $f = u + iv$ on a real analytic, generic submanifold can be holomorphically extended to a full neighborhood if and only if $u, v$ are real analytic on $M$. Our main theorem, to some extent, reduces the case to $u$ continuous and $v$ real analytic on $M$.

**Theorem 2.4.** Let $M \subset \mathbb{C}^{n+d}$ be a real analytic, generic submanifold.

1. $0 \in M$, in a neighborhood $\Omega$ of $0$ in $\mathbb{C}^{n+d}$, $M$ is given by regular coordinates, $\rho := \text{Im}(w) - \phi(z, \bar{z}, \text{Re}(w)) = 0$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$. Let $f = u + iv$ be a holomorphic function defined in the wedge $W = W(\Omega, \rho, \Gamma)$, where $W(\Omega, \rho, \Gamma)$ is as above defined with $\Gamma$ an open convex cone in $\mathbb{R}^d$. If $f$ extends continuously up to the edge $M \cap \Omega$ with $v = \text{Im}(f)$ is real analytic on the edge $M \cap \Omega$. Then there exists a holomorphic function $F$ in a neighborhood $\tilde{\Omega}$ of $0$ in $\mathbb{C}^{n+d}$ such that $F|_{W \cap \tilde{\Omega}} = f$.

Before proving the theorem, we need a lemma which ensures a wedge with edge as an open subset of $\mathbb{R}^{2n+d}$ contained in $\tilde{\Psi}^{-1}(\Omega \cap W)$.

**Lemma 2.5.** There exists an neighborhood $U_1$ of $0$ in $\mathbb{R}^{2n+2d}$, an neighborhood $Q_1$ of $0$ in $\mathbb{R}^{2n+d}$, an open convex cone $\Gamma'$ with vertex at $0$ in $\mathbb{R}^d$ such that $\tilde{\Psi}$ is a real analytic diffeomorphism from $U_1$ to $\tilde{\Psi}(U_1) \subset \Omega$ and $\tilde{\Psi}(Q_1 \times \Gamma') \cap U_1 \subset \Omega \cap W$.

**Proof of the lemma:** Take an neighborhood $U_1$ of $0$ in $\mathbb{R}^{2n+2d}$ such that $\tilde{\Psi}$ is a real analytic diffeomorphism from $U_1$ to $\tilde{\Psi}(U_1) \subset \Omega$. $\tilde{\Psi}(z, s + it) = (z, s + it + i\phi(z, \bar{z}, s + it)) = (\tilde{z}, \tilde{w})$, we can write $t = \Theta(\tilde{z}, \tilde{w})$, where $\Theta$ is $\mathbb{R}^d$ valued, real analytic in all its variables. Note that $t := \Theta(\tilde{z}, \tilde{w})$ can be taken as the defining functions for $M$ in $\tilde{\Psi}(U)$. By Proposition 2.2, there exists a neighborhood of $0$ in $\mathbb{C}^{n+d}$ (we may take it as a subset of $U_1$ and still denote it as $U_1$), a convex open cone $\Gamma'$ such that

$$W(U_1, t, \Gamma') \subset W(\Omega, \rho, \Gamma).$$

Take $Q_1 = U_1 \cap \mathbb{R}^{2n+d}$, we then have $\tilde{\Psi}((Q_1 \times \Gamma') \cap U_1) \subset \Omega \cap W$.

**Proof of the theorem:** $f|_{M \cap \Omega}$ is a continuous CR function on $M \cap \Omega$. By the lemma, the function $f \circ \tilde{\Psi}(z, \bar{z}, s + it) = f(z, \bar{z}, s + it + i\phi(z, \bar{z}, s + it))$ is defined on $(z, s + it) \in (Q_1 \times \Gamma') \cap U_1$, note that it is holomorphic in the variables $w = s + it$. To use Proposition 2.1, it suffices to show that $f \circ \tilde{\Psi}$ can be continuously extended to a full neighborhood of $0$ in $\mathbb{R}^{2n+2d}$ such that the extension is holomorphic in variables $w = s + it$.

Now, denote the restriction of the function $v \circ \tilde{\Psi}(z, s + it)$ on $Q_1$ to be $v(z, \bar{z}, s)$. By the real analyticity of $v$ and $\Psi$, we can treat $v(z, \bar{z}, s)$ as a power series expansion at $0 \in \mathbb{R}^{2n+d}$ and complexify the variable $s$ to $w$. Denote the new function to be $v(z, \bar{z}, w)$ defined in a product neighborhood $U_2 = U_3 \times U_4 \subset \mathbb{C}^n \times \mathbb{C}^d$ of $0$. Note that $v(z, \bar{z}, w)$ is continuous in $U_2$ and
holomorphic in variables $w$.

Following the idea of Theorem 1.2, the function

$$g(z,\overline{z}, s + it) :=
\begin{cases}
    f \circ \tilde{\Psi}(z,\overline{z}, s + it) - iv(z,\overline{z}, s + it) & \text{for} \ (z, s + it) \in (Q_1 \times (\Gamma' \cup \{0\})) \cap U_2 \\
    f \circ \tilde{\Psi}(z,\overline{z}, s - it) - iv(z,\overline{z}, s - it) & \text{for} \ (z, s + it) \in (Q_1 \times -\Gamma') \cap U_2
\end{cases}
$$

is holomorphic in variables $w = s + it$ for fixed $z$ and continuously up to $Q_1 \cap U_2$.

We shall apply the construction of holomorphic extension in the proof of Theorem 2.3 to $g(z,\overline{z}, s + it)$ for every fixed $z$. According to the proof, we have a linear isomorphism $A$ in $\mathbb{C}^d$ with real coefficients and $\Phi : D^d(0, 1) \times \overline{D} \longrightarrow \mathbb{C}^d$. Now we extend $A$ to a linear isomorphism in $\mathbb{C}^{n+d}$ which maps $(z, w) \in \mathbb{C}^{n+d}$ to $(z, A(w))$, still denote it as $A$. We also extend $\Phi$ to a mapping which takes $(z, w, \lambda) \in U_3 \times D^d(0, 1) \times \overline{D}$ to $(z, \Phi(w, \lambda)) \in \mathbb{C}^n \times \mathbb{C}^d$, still denote it as $\Phi$. Since the choice of $A$ depends only on $U_4$ and the cone $\Gamma'$, so we have the extension

$$G(z,\overline{z}, w) = \frac{1}{2\pi i} \int_0^{2\pi} g \circ A(\Phi(A^{-1}(z, w), e^{i\theta}))d\theta, \ \forall (z, w) \in A^{-1}(U_3 \times D^d(0, 1))$$

by Theorem 2.3 and Lemma 2.5.

From the construction above, the extension $G(z,\overline{z}, s + it)$ is continuous on $U_3 \times U_5 = A^{-1}(U_3 \times D^d(0, 1))$ and holomorphic in variables $w$. Thus, the function $F(z,\overline{z}, s + it) := G + iv(z,\overline{z}, s + it)$ is a continuous extension of $f \circ \tilde{\Psi}(z,\overline{z}, s + it)$ in $U_3 \times U_5$, holomorphic in $w = s + it$. Thus by Proposition 2.1, $f$ can be holomorphically extended to a full neighborhood $\tilde{\Omega}$ of $0$ in $\mathbb{C}^{n+d}$. $lacksquare$

The following theorem states that minimality is a sufficient condition for holomorphic extension of all CR functions from a generic submanifold $M$ in $\mathbb{C}^N$ into an open wedge. See [BER], [T].

**Theorem 2.6.** Let $M$ be a generic submanifold of $\mathbb{C}^{n+d}$ of codimension $d$ and $p_0 \in M$. If $M$ is minimal at $p_0$, then for every open neighborhood $U$ of $p_0$ in $M$ there exists a wedge $W$ with edge $M$ centered at $p_0$ such that every continuous CR function in $U$ extends holomorphically to the wedge $W$.

Since a real analytic CR submanifold $M$ in $\mathbb{C}^N$ is finite type at $p_0 \in M$ if and only if it is minimal at $p_0$. Thus, together Theorem 2.6, we have the following corollary.

**Corollary 2.2.** Let $M \subset \mathbb{C}^N$ be a real analytic, generic CR submanifold, finite type at $p_0 \in M$. If $f = u + iv$ is a continuous CR function defined in a neighborhood of $p_0$ in $M$ with $v$ real analytic. Then $f$ can be holomorphically extended to a full neighborhood of $p_0$ in $\mathbb{C}^N$. (Or $u$ is also real analytic in a neighborhood of $p_0$ in $M$ by Corollary 2.1)

**Definition 2.2.** A CR submanifold of the form
\[ M = \{(z, s + it) \in \mathbb{C}^n \times \mathbb{C}^d; \ t = \phi(z, \overline{z})\} \]

where \( \phi : \mathbb{C}^n \rightarrow \mathbb{R}^d \) is smooth with \( \phi(0) = 0 \) and \( d\phi(0) = 0 \) is called rigid.

**Corollary 2.3.** Let \( M \) be a real analytic, generic submanifold, \( 0 \in M \), given by \( \{(z, s + it) \in U \subset \mathbb{C}^n \times \mathbb{C}^d : t = \phi(z, \overline{z}, s)\} \). Let \( M' \) be a real analytic, generic, rigid submanifold, \( 0 \in M' \), given by \( \{(z', s' + it') \in U' \subset \mathbb{C}^{n'} \times \mathbb{C}^{d'} : t' = \phi'(z', \overline{z'})\} \). If \( H = (f_1, ..., f_{n'}, g_1, ..., g_{d'}) \) is a holomorphic mapping defined in the wedge \( W = W(U, \rho, \Gamma) \) with \( \Gamma \) an open convex cone. If \( H \) extends continuously to the edge \( M \cap U \) with \( H(0) = 0 \), then \( H(M) \subset M' \). Then \( (f_1, ..., f_{n'}) \) can be holomorphically extended near \( 0 \) implies \( H \) can be holomorphically extended near \( 0 \).

**Proof:** Denote \( \text{Im}(g_i)(z, \overline{z}, s) := \text{Im}(g_i) \circ \Psi(z, \overline{z}, s), \ 1 \leq i \leq d' \), similarly for \( f_j(z, \overline{z}, s), \ 1 \leq j \leq n' \). Now \( \text{Im}(g_i)(z, \overline{z}, s) = \phi'(f(z, \overline{z}, s), \overline{f}(z, \overline{z}, s)) \) and \( f_j \) real analytic. Write \( f_j(z, \overline{z}, s) = \sum a_{\alpha \beta \gamma} z^\alpha \overline{z}^\beta s^\gamma \) for all \( j \) in a neighborhood of \( 0 \) in \( \mathbb{R}^{2n+d} \), similarly \( \overline{f_j}(z, \overline{z}, s) = \sum a_{\alpha \beta \gamma} z^\alpha \overline{z}^\beta s^\gamma \) for all \( j \) in the same neighborhood. We extend \( f_j(z, \overline{z}, s) \) to \( \overline{f_j}(z, \chi, w) = \sum a_{\alpha \beta \gamma} z^\alpha \chi^\beta w^\gamma, \overline{f_j}(z, \overline{z}, s) \) to \( \overline{f_j}(z, \chi, w) = \sum a_{\alpha \beta \gamma} \chi^\alpha z^\beta w^\gamma \) in a neighborhood of \( 0 \) of \( \mathbb{C}^{2n+d} \).

Note that the function \( \phi'_i(f_1(z, \chi, w), ..., f_{n'}(z, \chi, w), \overline{f_1}(z, \chi, w), ..., \overline{f_{n'}}(z, \chi, w)) \) is holomorphic in a neighborhood of \( 0 \) in \( \mathbb{C}^{2n+d} \) by the real analyticity of \( \phi' \), it is therefore real analytic in that neighborhood. The restriction of such function to the plane \( \{(z, \chi, w) \in \mathbb{C}^{2n+d} : \chi = \overline{z}, w \in \mathbb{R}^d\} \) is also real analytic and equals \( \text{Im}(g_i)(z, \overline{z}, s) \), thus \( \text{Im}(g_i) \) are real analytic for all \( i \), by Theorem 2.4, \( g_i \) can be holomorphically extended to a neighborhood, we get the desire result. \( \blacksquare \)