ON QUANTIZABLE ODD LIE BIALGEBRAS

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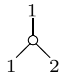
Abstract. The notion of a quantizable odd Lie bialgebra is introduced. A minimal resolution of the properad governing such Lie bialgebras is constructed.

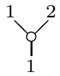
1. Introduction

1.1. Even and odd Lie bialgebras. A Lie (c,d)-bialgebra is a graded vector space V which carries both a degree c Lie algebra structure



$$\simeq [\ ,\ ] : V\otimes V \to V[c]$$

and a degree d Lie coalgebra structure,

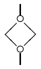


$$\simeq \Delta : V \to V\otimes V[d]$$

satisfying a certain compatibility condition. If the $\mathbb{Z}_2$-parities of both structures are the same, i.e. if $c+d \in 2\mathbb{Z}$, the Lie bialgebra is called *even*, if the $\mathbb{Z}_2$-parities are opposite, $c+d\in 2\mathbb{Z}+1$, it is called *odd*.

In the even case the most interesting for applications Lie bialgebras have $c=d=0$. Such Lie bialgebras were introduced by Drinfeld in [D] in the context of the theory of Yang-Baxter equations, and they have since found numerous applications, most prominently in the theory of Hopf algebra deformations of universal enveloping algebras (see the book [ES] and references cited there). If the composition of the cobracket and bracket of a Lie bialgebra is zero, that is

(1)



$$= 0,$$

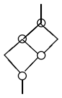
then the Lie bialgebra is called *involutive*. This additional constraint is satisfied in many interesting examples studied in homological algebra, string topology, symplectic field theory, Lagrangian Floer theory of higher genus, and the theory of cohomology groups $H(\mathcal{M}_{g,n})$ of moduli spaces of algebraic curves with labelings of punctures skewsymmetrized [D, ES, C, CS, CFL, Sc, CMW, MW1].

In the odd case the most interesting for applications Lie bialgebras have $c=1$, $d=0$. They have been introduced in [M1] and have seen applications in Poisson geometry, deformation quantization of Poisson structures [M2] and in the theory of cohomology groups $H(\mathcal{M}_{g,n})$ of moduli spaces of algebraic curves with labelings of punctures symmetrized [MW1].

The homotopy and deformation theories of even/odd Lie bialgebras and also of involutive Lie bialgebras have been studied in [CMW, MW2]. A key tool in those studies is a minimal resolution of the properad governing the algebraic structure under consideration.

The minimal resolutions of properads $\mathcal{L}ieb$ and $\mathcal{L}ieb_{odd}$ governing even and, respectively, odd Lie bialgebras were constructed in [Ko, MaVo] and, respectively, in [M1, M2]. Constructing a minimal resolution $\mathcal{H}olieb^{\diamond}$ of the properad $\mathcal{L}ieb^{\diamond}$ governing *involutive* Lie bialgebras turned out to be a more difficult problem, and that goal was achieved only very recently in [CMW].

1.2. Quantizable odd Lie bialgebras. For odd Lie bialgebras the involutivity condition (1) is trivial, i.e. it is satisfied automatically for any odd Lie bialgebra V. There is, however, a higher genus analogue of that condition,

(2)  $= 0,$

which is highly non-trivial, and which can be considered as an odd analogue of (1). We prefer, however, to call odd Lie bialgebras satisfying the extra constraint (2) *quantizable* ones rather than involutive. The reason for this terminology is explained in §2.

Our main result is an explicit construction in §3 of a (highly non-obvious) minimal resolution $\mathcal{H}olieb^{\diamond}_{odd}$ of the properad $\mathcal{L}ieb^{\diamond}_{odd}$ governing quantizable Lie bialgebras. One of the key tricks in [CMW] used to solve an analogous problem for the properad $\mathcal{L}ieb^{\diamond}$ of even involutive Lie bialgebras reduced the "hard" problem of computing the cohomology of some dg properad to the "easy" computation of the minimal resolutions of a family of some auxiliary *quadratic* algebras. Remarkable enough, this approach works for the constraint (2) as well, but it leads instead to a certain family of *cubic* homogeneous algebras which are studied in the Appendix.

Another important technical ingredient in our construction of $\mathcal{H}olieb^{\diamond}_{odd}$ comes from the paper [MW2], in which the cohomologies of the deformation complexes of the properads $\mathcal{L}ieb$, $\mathcal{L}ieb_{odd}$ and $\mathcal{L}ieb^{\diamond}$ have been computed, and it was proven in particular that the properad $\mathcal{L}ieb_{odd}$ admits precisely one non-trivial deformation; in fact it is that unique non-trivial deformation which leads us to the dg properad $\mathcal{H}olieb^{\diamond}_{odd}$. We explain this link in §3.

1.3. Some notation. The set $\{1, 2, \ldots, n\}$ is abbreviated to $[n]$; its group of automorphisms is denoted by $\mathbb{S}_n$; the trivial one-dimensional representation of $\mathbb{S}_n$ is denoted by $\mathbf{1}_n$, while its one dimensional sign representation is denoted by sgn_n. The cardinality of a finite set A is denoted by $\#A$.

We work throughout in the category of $\mathbb{Z}$-graded vector spaces over a field $\mathbb{K}$ of characteristic zero. If $V = \oplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ stands for the graded vector space with $V[k]^i := V^{i+k}$. For a prop(erad) $\mathcal{P}$ we denote by $\mathcal{P}\{k\}$ a prop(erad) which is uniquely defined by the following property: for any graded vector space V a representation of $\mathcal{P}\{k\}$ in V is identical to a representation of $\mathcal{P}$ in $V[k]$.

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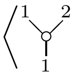
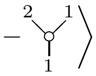
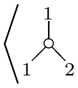
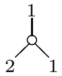
2. Quantizable odd Lie bialgebras

2.1. Odd lie bialgebras. By definition [M1], the properad, $\mathcal{L}ieb^{\diamond}_{odd}$, of odd Lie bialgebras is a quadratic properad given as the quotient,

$$\mathcal{L}ieb^{\diamond}_{odd} := \mathcal{F}ree\langle E\rangle/\langle\mathcal{R}\rangle,$$

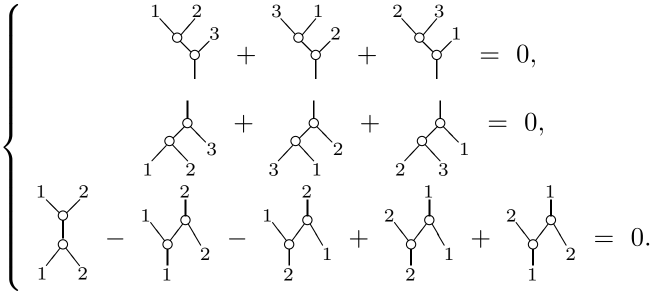
of the free properad generated by an $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n\geq 1}$ with all $E(m,n) = 0$ except

$E(2,1) := sgn_2 \otimes \mathbf{1}_1 = \text{span}\Big\langle$  $= -$  $\Big\rangle$

$E(1,2) := \mathbf{1}_1 \otimes \mathbf{1}_2[-1] = \text{span}\Big\langle$  $=$  $\Big\rangle$

modulo the ideal generated by the following relations

(3) $\mathcal{R}:$

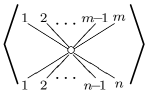


A minimal resolution $\mathcal{H}olieb_{odd}$ of $\mathcal{L}ieb_{odd}$ was constructed in [M1, M2]. It is a free properad,

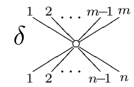
$$\mathcal{H}olieb_{odd} = \mathcal{F}ree\langle \hat{E}\rangle$$

generated by an $\mathbb{S}$–bimodule $\hat{E} = \{\hat{E}(m,n)\}_{m,n\geq 1, m+n\geq 3}$,

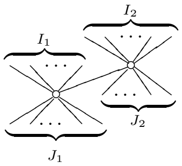
$$\hat{E}(m,n) := sgn_m \otimes \mathbf{1}_n[m-2] = \text{span}\left\langle \quad \right\rangle,$$



and comes equipped with the differential

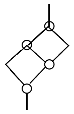


$$= \sum_{\substack{[1,\ldots,m]=I_1\sqcup I_2 \\ |I_1|\geq 0, |I_2|\geq 1}} \sum_{\substack{[1,\ldots,n]=J_1\sqcup J_2 \\ |J_1|\geq 1, |J_2|\geq 1}} \pm$$



It was shown in [M1, M2] that representations $\mathcal{H}olieb_{odd} \to \mathcal{E}nd_V$ of the minimal resolution of $\mathcal{L}ieb_{odd}$ in a graded vector space V are in 1-1 correspondence with formal graded Poisson structures $\pi \in \mathcal{T}_{poly}^{\geq 1}(V^*)$ on the dual vector space V^* (viewed as a linear manifold) which vanish at the zero point in V, $\pi|_0 = 0$.

2.2. **Quantizable odd Lie bialgebras.** We define the properad $\mathcal{L}ieb_{odd}^{\diamond}$ of *quantizable* odd Lie bialgebras as the quotient of the properad $\mathcal{L}ieb_{odd}$ by the ideal generated by the following element



$$\in \mathcal{L}ieb_{odd}.$$

The associated relation on Lie and coLie brackets looks like a higher genus odd analogue of the involutivity condition (1) in the case of even Lie bialgebras. However, we prefer to use the adjective *quantizable* rather than *involutive* for odd Lie bialgebras satisfying (2) because that condition has a clear interpretation within the framework of the theory of deformation quantization, and its quantizability property becomes even more clear when one raises it to the level of representations of its minimal resolution $\mathcal{H}olieb_{odd}^{\diamond}$.

An odd Lie bialgebra structure in a vector space V can be understood as a pair

$$\left(\xi \in \mathcal{T}_{V^*}, \Phi \in \wedge^2\mathcal{T}_{V^*}\right)$$

consisting of a degree 1 quadratic vector field ξ (corresponding to the Lie brackets [,] in V) and a linear Poisson structure Φ in V^* (corresponding to the Lie cobracket Δ in V. All the (compatibility) equations for the algebraic operations [,] and Δ get encoded into a single equation,

$$\{\xi + \Phi, \xi + \Phi\} = 0,$$

where { , } stand for the standard Schouten bracket in the algebra $\mathcal{T}_{poly}(V^*)$ of polyvector fields on V^* (viewed as an affine manifold). Therefore, the sum $\xi + \Phi$ gives us a graded Poisson structure on V^* and one can talk about its deformation quantization, that is, about an associated Maurer-Cartan element Γ in the Hochschild dg Lie algebra,

$$C^\bullet(\mathcal{O}_V, \mathcal{O}_V) := \bigoplus_{n\geq 0} \mathrm{Hom}(\mathcal{O}_V^{\otimes n}, \mathcal{O}_V)$$

where $\mathrm{Hom}(\mathcal{O}_V^{\otimes n}, \mathcal{O}_V)$ stands for vector space polydifferential operators on the graded commutative algebra $\mathcal{O}_V := \odot^\bullet V$ of polynomial functions on V^*. As the graded Poisson structure $\xi + \Phi$ is non-negatively graded, its deformation quantization must satisfy the condition

$$\Gamma \in \mathrm{Hom}(\mathbb{K}, \mathcal{O}_V) \oplus \mathrm{Hom}(\mathcal{O}_V, \mathcal{O}_V) \oplus \mathrm{Hom}(\mathcal{O}_V^{\otimes 2}, \mathcal{O}_V)$$

with the corresponding splitting of Γ into a sum of three terms,

$$\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2$$

of degrees 2, 1, and 0 respectively. The term $\Gamma_2 = \Gamma_2(\Phi)$ has degree zero and hence can depend only on the Lie cobracket Φ. It makes $(\mathcal{O}_V, \star := \Gamma_2)$ into an associative non-commutative algebra, and up to gauge equivalence, the algebra $(\mathcal{O}_V, \star)$ can always be identified with the universal enveloping algebra of the Lie algebra $(V, [\ ,\ ])$. The standard compatibility condition (the third one in (3)) says that the homological vector field ξ defines a derivation of the algebra $(\mathcal{O}_V, \star := \Gamma_2)$. As a derivation of the graded commutative algebra $\mathcal{O}_V = \odot^\bullet V$ ξ obviously satisfies the condition $\xi^2 = 0$ (this follows from the second Jacobi identity in the list (3)). As a derivation of the $\star$-product it satisfies the same condition

$$\xi^2 = 0$$

if and only if $\xi \simeq$  and $\Phi \simeq$  satisfy the extra compatibility condition (2) (see [M3]). Therefore, if ξ and Φ come from a representation of $\mathcal{L}ieb^\diamond_{odd}$ in V, then they admit a very simple deformation quantization in the form

$$\Gamma = \xi + \Gamma_2(\Phi)$$

and this quantization makes sense even in the case when V is *infinite*-dimensional. If ξ and Φ dot not satisfy the extra compatibility condition (2), then their deformation quantization is possible only in *finite* dimensions, and involve a non-zero "curvature" term Γ_0 which in turn involves graphs with closed paths of directed edges and is given explicitly in [M3] (in fact this argument proves non-existence of Kontsevich formality maps for *infinite* dimensional manifolds). We shall construct below a minimal resolution $\mathcal{H}olieb^\diamond_{odd}$ of the properad $\mathcal{L}ieb^\diamond_{odd}$; its representations in a graded vector space V give us so called *quantizable* Poisson structures on V which can be deformation quantized via a trivial (i.e. without using Drinfeld associators) perturbation even if $\dim V = \infty$ (see [W2, B]); in finite dimensions there is a 1-1 correspondence between ordinary Poisson structure on V and quantizable ones, but this correspondence is highly non-trivial — it depends on the choice of a Drinfeld associator [MW3].

3. A minimal resolution of $\mathcal{L}ieb^\diamond_{odd}$

3.1. Oriented graph complexes and a Kontsevich-Shoikhet MC element. Let $G^{or}_{n,l}$ be a set of connected graphs Γ with n vertices and l directed edges such that (i) Γ has no *closed* directed paths of edges, and (ii) some bijection from the set of edges $E(\Gamma)$ to the set $[l]$ is fixed. There is a natural right action of the group $\mathbb{S}_l$ on the set $G^{or}_{n,l}$ by relabeling the edges.

Consider a graded vector space

$$\mathsf{fGC}^{or}_2 := \prod_{n\geq 1, l\geq 0} \mathbb{K}\langle G^{or}_{n,l}\rangle \otimes_{\mathbb{S}_l} sgn_l[l + 2(1-n)]$$

It was shown in [W2] that this vector space comes equipped with a Lie bracket $[\ ,\ ]$ (given, as often in the theory of graph complexes, by substituting graphs into vertices of another graphs), and that the degree +1 graph

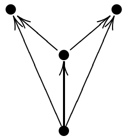
$$\bullet\!\!\longrightarrow\!\!\bullet \in \mathsf{fGC}^{or}_2$$

is a Maurer-Cartan element making fGC^{or}_2 into a *differential* Lie algebra with the differential given by

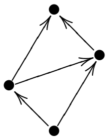
$$\delta := [\bullet\!\!\longrightarrow\!\!\bullet,\ ].$$

It was proven in [W2] that the cohomology group $H^1(\mathsf{fGC}_2^{or})$ is one-dimensional and is spanned by the following graph

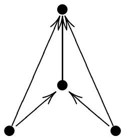
$$\Upsilon_4 :=$$



$$+\,2$$



$$+$$



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while the cohomology group $H^2(\mathsf{fGC}_2^{or},\delta_0)$ is also one-dimensional and is generated by a linear combination of graphs with four vertices (whose explicit form plays no role in this paper). This means that one can construct by induction a new Maurer-Cartan element (the integer subscript stand for the number of vertices)

$$\Upsilon_{KS} = \bullet\!\!\longrightarrow\!\!\bullet + \Upsilon_4 + +\Upsilon_6 + \Upsilon_8 + \ldots$$

in the Lie algebra fGC_2^{or}. Indeed, the Lie brackets in fGC_2^{or} has the property that a commutator $[A,B]$ of a graph A with p vertices and a graph B with q vertices has $p+q-1$ vertices. Therefore, all the obstructions to extending the sum $\bullet\!\!\longrightarrow\!\!\bullet + \Upsilon_4$ to a Maurer-Cartan element have 7 or more vertices and hence do not hit the unique cohomology class in $H^2(\mathsf{fGC}_2^{or},\delta)$. Up to gauge equivalence, this new MC element Υ_{KS} is the *only* non-trivial deformation of the standard MC element $\bullet\!\!\longrightarrow\!\!\bullet$. We call it the *Kontsevich-Shoikhet* element as it was introduced (via a different line of thought) by Boris Shoikhet in [Sh] with a reference to an important contribution by Maxim Kontsevich via an informal communication.

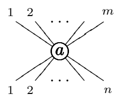
3.1.1. A formal power series extension of fGC_2^{or}. Let $\hbar$ be a formal parameter of degree 0 and let $\mathsf{fGC}_2^{or}[[\hbar]]$ be a topological vector space of formal power series in $\hbar$ with coefficients in fGC_2^{or}. This is naturally a topological Lie algebra in which the formal power series

$$\Upsilon_{KS}^{\hbar} = \bullet\!\!\longrightarrow\!\!\bullet + \hbar^2\Upsilon_4 + +\hbar^4\Upsilon_6 + \hbar^6\Upsilon_8 + \ldots$$

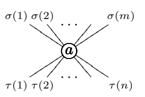
is a Maurer-Cartan element.

3.2. From the Kontsevich-Shoikhet element to a minimal resolution of $\mathcal{L}ieb^{\diamond}_{odd}$. Consider a (non-differential) free properad $\mathcal{H}olieb^{\diamond}_{odd}$ generated by the following (skewsymmetric in outputs and symmetric in inputs) corollas of degree $2-m$,

(4)



$$= (-1)^{\sigma}$$



$$\forall\sigma\in\mathbb{S}_m, \forall\tau\in\mathbb{S}_n,$$

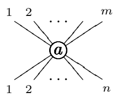
where $m+n+a\geq 3$, $m\geq 1$, $n\geq 1$, $a\geq 0$. Let $\widehat{\mathcal{H}olieb}^{\diamond}_{odd}$ be the genus completion of $\mathcal{H}olieb^{\diamond}_{odd}$.

3.2.1. Lemma. *The Lie algebra* $\mathsf{fGC}_2^{or}[[\hbar]]$ *acts (from the right) on the properad* $\widehat{\mathcal{H}olieb}^{\diamond}_{odd}$ *by continuous derivations, that is, there is a morphism of Lie algebras*

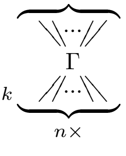
$$\begin{array}{rccc} F: & \mathsf{fGC}_2^{or}[[\hbar]] & \longrightarrow & \mathrm{Der}(\widehat{\mathcal{H}olieb}^{\diamond}_{odd}) \\ & \hbar^k\Gamma & \longrightarrow & F(\hbar^k\Gamma) \end{array}$$

where the derivation $F(\hbar^k\Gamma)$ is given on the generators as follows

$$F(\hbar^k\Gamma)\cdot$$



$$:= \left\{\begin{array}{l} \displaystyle\sum_{m,n}\ \sum_{a=a_1+\ldots+a_{\#V(\Gamma)}+k} \end{array}\right.$$



$$\forall\ \Gamma\in\mathsf{fGC}_2^{or},\qquad \forall\ k\in[0,1,\ldots,a]$$

$$0 \qquad\qquad \forall\ \Gamma\in\mathsf{fGC}_2^{or},\qquad \forall\ k>a,$$

where the first sum is taking over to attach m output legs and n input legs to the vertices of the graph Γ, and the second sum is taken over all possible ways to decorate the vertices of Γ with non-negative integers $a_1,\ldots,a_{\#V(\Gamma)}$ such they sum to $a-k$.

Proof is identical to the proofs of similar statements (Theorems 1.2.1 and 1.2.2) in [MW2].

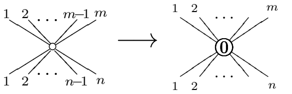
3.2.2. Corollary. *The completed free properad* $\widehat{\mathcal{H}olieb}^{\diamond}_{odd}$ *comes equipped with a differential* $\delta_\diamond := F(\Upsilon^{\hbar}_{KS})$. *The differential* δ *restricts to a differential in the free properad* $\mathcal{H}olieb^{\diamond}_{odd}$.

Proof. When applied to any generator of $\widehat{\mathcal{H}olieb}^{\diamond}_{odd}$ the differential δ gives always a *finite* sum of graphs. It follows that it is well defined in $\mathcal{H}olieb^{\diamond}_{odd}$ as well. □

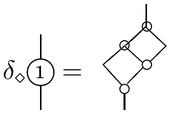
There is an injection of dg free properads

$$(\mathcal{H}olieb_{odd}, \delta) \longrightarrow (\mathcal{H}olieb^{\diamond}_{odd}, \delta_\diamond)$$

given on generators by



Identifying from now on weight zero generators of $\mathcal{H}olieb^{\diamond}_{odd}$ with generators of $\mathcal{H}olieb_{odd}$, we may write

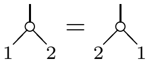


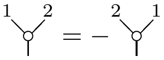
and hence conclude that there is a natural morphism of dg properads

$$\pi : (\mathcal{H}olieb^{\diamond}_{odd}, \delta_\diamond) \longrightarrow (\mathcal{L}ieb^{\diamond}_{odd}, 0)$$

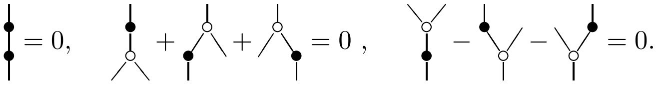
Our main result in this paper is the following theorem.

3.3. Main Theorem. *The map* π *is a quasi-isomorphism, i.e.* $\mathcal{L}ieb^{\diamond}_{odd\infty}$ *is a minimal resolution of* $\mathcal{L}ieb^{\diamond}_{odd}$.

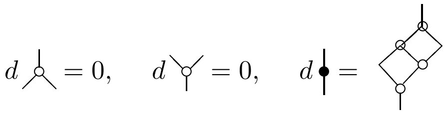
Proof. Let $\mathcal{P}$ be a dg properad generated by a degree 1 corollas  and 

, and a degree zero corolla, 

and modulo relations

(5) 

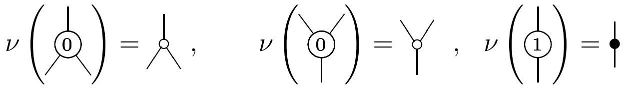
and the three relations in (3). The differential in $\mathcal{P}$ is given on the generators by

(6) 

Claim I: *The surjective morphism of dg properads,*

(7) $$\nu : \mathcal{L}ieb^{\diamond}_{odd\infty} \longrightarrow \mathcal{P},$$

which sends all generators to zero except for the following ones

(8) 

is a quasi-isomorphism.

The proof of this claim is identical to the proof of Theorem 2.7.1 in [CMW] so that we can omit the details.

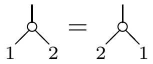
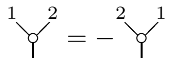
The proof of the Main Theorem will be completed once we show the following

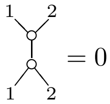
Claim II: *The natural map*

(9) $$\mu : (\mathcal{P}, d) \longrightarrow (\mathcal{L}ieb^{\diamond}_{odd}, 0)$$

is a quasi-isomorphism.

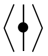
Let us define a new homological grading in the properad $\mathcal{P}$ by assigning to the generator  degree -1 and to the remaining generators the degree zero; to avoid confusion with the original grading let us call this new grading *s-grading.* Then Claim II is proven once we show that the cohomology $H(\mathcal{P})$ of $\mathcal{P}$ is concentrated in s-degree zero.

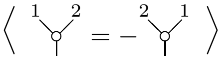
Consider a path filtration [Ko, MaVo] of the dg properad $\mathcal{P}$. The associated graded gr$\mathcal{P}$ can be identified with dg properad generated by the same corollas ,  and  which are subject to the relations (5), the first two relation in (3) and the following one

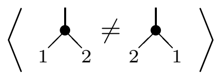


The differential in gr$\mathcal{P}$ is given by the original formula (6). The Claim II is proven once it is shown that the cohomology of the properad gr$\mathcal{P}$ is concentrated in s-degree zero, or equivalently, the cohomology of dg prop $\mathcal{U}(\text{gr}\mathcal{P})$ generated by this properad is concentrated in s-degree zero (as the universal enveloping functor $\mathcal{U}$ from the category of properads to the category of props is exact).

Consider a free prop $\mathcal{F}ree\langle E\rangle$ generated by an $\mathbb{S}$-bimodule $E = \{E(m,n)\}$ with all $E(m,n) = 0$ except the following ones,

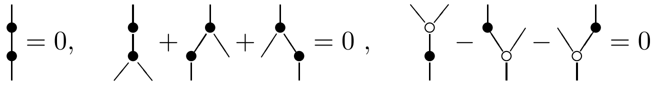
$E(1,1) = \mathbb{K}[-1] = \text{span}$ 

$E(2,1) = sgn_2 = \text{span}$ 

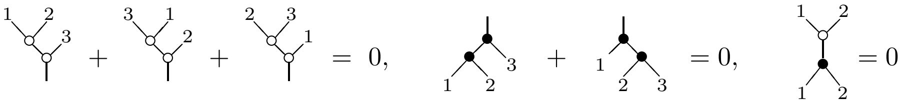
$E(1,2) = \mathbb{K}[\mathbb{S}_2][-1] = \text{span}$ 

We assign to the above generators s-degrees -1, 0 and 0 respectively.

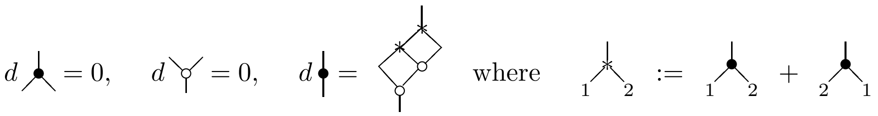
Define next a dg prop $\mathcal{A}$ as the quotient of the above free prop $\mathcal{F}ree\langle E\rangle$ by the ideal generated by the relations

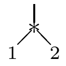


and



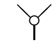
A differential in $\mathcal{A}$ is defined by

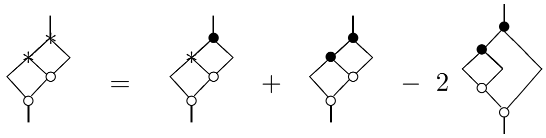


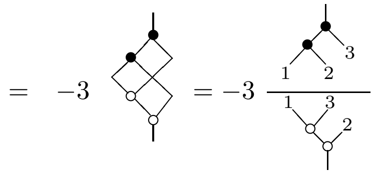
Note that the generator  satisfies the second relation in (3) so that we have a canonical injection of dg props

$$i : \mathcal{U}(\text{gr}\mathcal{P}) \longrightarrow \mathcal{A}$$

It is easy to see that image of $\mathcal{U}(\text{gr}\mathcal{P})$ under this injection is a *direct* summand in the complex $(\mathcal{A}, d)$. Hence Claim 2 is proven once we show that the cohomology of the prop $\mathcal{A}$ is concentrated in s-degree zero.

Using the associativity relation for the generator  and the Jacobi relation for the generator  one obtains an equality





where the horizontal line stands for the properadic composition in accordance with the labels shown. This result propmts us to consider a dg associative non-commutative algebra $\mathbf{A}_n$ generated by degree zero variable $\{x_i\}_{1\leq i\leq n}$ and degree -1 generators $\{u_{i,i+1,u+2}\}_{1\leq i\leq n-2}$ with the differential

$$du_{i,i+1,i+2} = -3[[x_i, x_{i+2}], x_{i+1}]$$

or, equivalently (after rescaling the generators $u_\bullet$), with the differential

$$du_{i,i+1,i+2} = [[x_i, x_{i+2}], x_{i+1}]$$

Arguing in exactly the same way as in [CMW] one concludes that the cohomology of the dg operad operad $\mathcal{A}$ is concentrated in s-degree zero if and only if the collections of algebras $\mathbf{A}_n$, $n \geq 3$, has cohomology concentrated in ordinary degree zero. The latter fact is established in the appendix. The proof is completed. □

APPENDIX A

3.3.1. Lemma. *Let $\{c_\sigma | \sigma \in S_3\}$ be a collection of 6 numbers such that*

$$\text{for each pair } (c_{ijk}, c_{ikj}) \text{ with the same first index } i \atop \text{at least one of these elements is different from zero} \tag{10}$$

Then the associative algebra $A := \Bbbk\left\langle x_1,\dots,x_n \;\middle|\; \begin{array}{c}\sum_{\sigma\in S_3} c_\sigma x_{i+\sigma(1)}x_{i+\sigma(2)}x_{i+\sigma(3)},\\ i=0,\dots,n-2\end{array}\right\rangle$ *has global dimension 2.*

Proof. Let us consider any linear ordering of the set of generators, such that

$$\forall k,l,m \quad x_{3k} > x_{3l+2}, x_{3m+1}$$

We extend this ordering to a degree-lexicographical ordering of the set of monomials in the free associative algebra $\Bbbk\langle x_1,\dots,x_n\rangle$. The leading monomials of relation $\sum_{\sigma\in S_3} c_\sigma x_{i+\sigma(1)}x_{i+\sigma(2)}x_{i+\sigma(3)}$ are different for all i because they contain different letters $\{x_{i+1}, x_{i+2}, x_{i+3}\}$. Moreover, there is exactly one number divisible by 3 in each subsequent triple of integer numbers, thus after reordering we have $\{i+1, i+2, i+3\} = \{3s, r, t\}$ for appropriate r, s and t, such that r and t are not divisible by 3. Recall that by property (10) at least one of the two monomials $c_{3srt}x_{3s}x_r x_t$ and $c_{3str}x_{3s}x_t x_r$ is different from zero. Hence, the first letters in the leading monomials of the relation $\sum_{\sigma\in S_3} c_\sigma x_{i+\sigma(1)}x_{i+\sigma(2)}x_{i+\sigma(3)}$ has index divisible by 3 and two remaining letters is not divisible by 3. Concequently, the leading monomials of generating relations has no compositions and the set of generating relations form a *strongly free* Gröbner bases following that the algebra A has global dimension 2. (See [U] §4.3 for details on strongly free relations.) □

3.3.2. Corollary. *The minimal free resolution $\mathbf{A}_n$ of the algebra*

$$A_n := \Bbbk\left\langle x_1,\dots,x_n \;\middle|\; \begin{array}{c}[[x_i, x_{i+2}], x_{i+1}]\\ i=1,\dots,n-2\end{array}\right\rangle$$

is generated by $x_1,\dots,x_n$ and $u_{1,2,3}\dots,u_{n-2,n-1,n}$ such that

$$\begin{array}{ll}\deg(x_i) = 0, & \deg(u_{i,i+1,i+2}) = -1;\\ d(x_i) = 0; & d(u_{i,i+1,i+2}) = [[x_i, x_{i+2}], x_{i+1}].\end{array}$$

Proof. Let us expand the commutators in the relations we are working with:

$$[[x_1, x_3], x_2] = (x_1x_3x_2 - x_3x_1x_2 - x_2x_1x_3 + x_2x_3x_1)$$

As we can see they satisfy the condition (10) of Lemma **3.3.1** and algebra A_n has global dimension 2, meaning that the following complex

$$0 \to \mathrm{span}\langle\mathrm{relations}\rangle \otimes A_n \to \mathrm{span}\langle\mathrm{generators}\rangle \otimes A_n \to A_n \to \Bbbk \to 0$$

is acyclic in the leftmost term and, consequently, acyclic everywhere. Therefore, the minimal resolution of A_n is generated by generators and generating relations of A_n. ☐

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