Weak Poincaré Inequalities for Convolution Probability Measures

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Abstract. By using Lyapunov conditions, weak Poincaré inequalities are established for some probability measures on a manifold (M, g). These results are further applied to the convolution of two probability measures on $\mathbb{R}^d$. Along with explicit results we study concrete examples.

1 Introduction

During the last decades, a lot of attention has been devoted to the study of ergodic theory for Markov processes. Specifically a lot of effort has been made on the stability speed for the corresponding Markov processes (see e.g. [1, 8, 9, 10, 12]). From this former work, functional inequalities of Dirichlet forms play important roles in characterizing the convergence speed of ergodic Markov processes. For instance, Poincaré inequalities imply the exponential ergodic speed of Markov processes; super Poincaré inequalities imply the strong ergodicity of the corresponding processes; weak Poincaré inequalities are used to characterize the non-exponential convergence rate for semigroup (see [12] for details).

However, to establish a functional inequality, we always need the coefficients of the generator to satisfy some regularity conditions. To deal with generators with less regular or less explicit coefficients, an efficient way is to regard the measures as perturbations from better ones, which satisfy the underlying functional inequalities. The convolution probability measure, in the sense of an independent sum of random variables, can be regarded as a kind of perturbation; see e.g. [5, 14] and references therein. Moveover, the study of functional inequalities for convolution probability measures is helpful in describing some behaviors of random variables under independent perturbations, see e.g. [14, Section 3] for an application to the study of random matrices.

Recently, F.-Y. Wang and J. Wang [13] gave some sufficient conditions for log-Sobolev/ Poincaré/ super Poincaré inequalities for convolution probability measures. The present article is thus a continuation of [13] to study weak Poincaré inequalities for the convolution probability measures.

Before moving on, let us briefly review some background about the weak Poincaré inequality. The weak Poincaré inequality was first introduced in [11] to characterize the non-exponential

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convergence rate of Markov processes and the concentration of measure phenomenon for sub-exponential laws (see [2]). Let (M, g) be a d-dimensional complete connected Riemannian manifold and $\mathrm{d}x$ be the volume measure. For a probability measure $\mu(\mathrm{d}x) := \mathrm{e}^{-V(x)}\,\mathrm{d}x$ with some locally bounded function V on M, we say that μ satisfies the *weak Poincaré inequality* if

$$\|f\|^2 \leq \alpha(r)\mu(|\nabla f|^2) + r\mathrm{Osc}^2(f), \quad r > 0, f \in C_b^2(M) \tag{1.1}$$

holds for some decreasing function $\alpha : [0, \infty) \to (0, \infty)$, where $\|\cdot\|$ denotes the $L^2(\mu)$-norm and $\mathrm{Osc}(f) := \sup_{x,y\in M} |f(x) - f(y)|$. Indeed, the function α can be estimated by using the growth of $|V|$ (see [11, 12]). However, in general, the resulting estimate of the rate function is less sharp. Therefore, in Section 2, we will revisit this problem on Riemannian manifolds by using some Lyapunov conditions.

As an application of the results in Section 2, we consider the weak Poincaré inequality for convolution probability measures on $\mathbb{R}^d$. Let μ and ν be two probability measures on $\mathbb{R}^d$. The perturbation of μ by the probability measure ν is given by their convolution

$$(\nu * \mu)(A) := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathbb{1}_A(x + y)\mu(\mathrm{d}x)\nu(\mathrm{d}y),$$

where $A \in \mathcal{B}(\mathbb{R}^d)$. In particular, let $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ be a probability measure on $\mathbb{R}^d$ such that $V \in C^1(\mathbb{R}^d)$ and ν be a probability measure on $\mathbb{R}^d$ such that

$$p_\nu(\cdot) := \int \mathrm{e}^{-V(\cdot-z)}\,\nu(\mathrm{d}z) \in C^1(\mathbb{R}^d). \tag{1.2}$$

Then

$$(\mu * \nu)(\mathrm{d}x) = p_\nu(x)\,\mathrm{d}x = \mathrm{e}^{-V_\nu(x)}\,\mathrm{d}x, \tag{1.3}$$

where $V_\nu(x) := -\log p_\nu(x)$. Let $L_\nu = \Delta - \nabla V_\nu$, which is the generator associated with some independent sum of two Markov diffusion processes with invariant measures μ and ν, respectively. This article aims to prove that the measure $\mu * \nu$ satisfies (1.1) for some explicit function α, which characterizes the explicit L^2-ergodic speed of some diffusion generated by L_ν. Actually, the existence of weak Poincaré inequalities for $\mu * \nu$ holds automatically due to the positivity of the density $\mathrm{e}^{-V_\nu(x)}$ (see [11]). So the main topic of this article is to find an explicit function α in the weak Poincaré inequality.

Our method is based on the use of Lyapunov type conditions. These conditions are well known to furnish some results on the long time behavior of the laws of Markov processes (see e.g. [1, 4, 8, 9, 10, 13] and references therein). In the recent work [13], the authors partly use Lyapunov conditions to study ordinary or super Poincaré inequality for convolution probability measures. As announced, the present paper is thus a complement of [13] for the study of the weak Poincaré inequality. The main idea of the use of a Lyapunov function is similar to [13] and in the present work, however we have to face some technical difficulties when choosing suitable Lyapunov functions and handling the "local term" in the proof of Theorem 2.1 below. It is worthy to mentioning that a new and reasonable Lyapunov function, constructed for establishing weak Poincaré inequalities, can also be applied to improving some results obtained in [13] for the super Poincaré inequality. In addition we will use a comparison method to simplify the assumptions in general results, and then give some concrete examples as applications.

The parts of the paper are organized as follows. In the following section, we study the weak Poincaré inequality by Lyapunov conditions and the comparison theorem for some probability measures on Riemannian manifolds. In Section 3, we apply results in Section 2 to convolution probability measures on $\mathbb{R}^d$. Some explicit examples are studied in Section 4.

2 Weak Poincaré inequality on manifolds via Lyapunov conditions

We organize this section by first introducing main results and then giving proofs.

2.1 Main results

Let (M,g) be a d-dimensional complete connected Riemannian manifold. Let ∇ and Δ be the Levi-Civita connection and the Laplacian associated with g, respectively. Consider the elliptic operator $L=\Delta-\nabla V$ for $V\in C^1(M)$ such that $\mu(\mathrm{d}x):=\mathrm{e}^{-V(x)}\,\mathrm{d}x$ is a probability measure, where $\mathrm{d}x$ is the Riemannian volume measure.

Given $o\in M$. For any $x\in M$, let $\rho_o(x)$ be the Riemannian distance on M between x and o and Cut_o be the set of cut-locus points of o which is closed and has volume zero. Define $\varphi(s)$ to be the continuous version of

$$\inf_{\rho_o(x)=s,\ x\notin \mathrm{Cut}_o}\left(\langle \nabla V(x),\nabla\rho_o(x)\rangle-\Delta\rho_o(x)\right)$$

for $s\geq 0$. We now introduce the main results about weak Poincaré inequalities for μ via Lyapunov conditions.

Theorem 2.1. *Let $\mu(\mathrm{d}x)=\mathrm{e}^{-V(x)}\,\mathrm{d}x$ be a probability measure on M for some $V\in C^1(M)$.*

(a) *Assume that for some constant R_0 and any $\sigma\in(0,1)$, one has*

$$\theta(r):=\frac{(1-\sigma)\varphi(r)\exp\left[\sigma\int_{R_0}^r\varphi(u)\,\mathrm{d}u\right]}{\int_{R_0}^r\exp\left[\sigma\int_{R_0}^s\varphi(u)\,\mathrm{d}u\right]\mathrm{d}s+1}>0,\quad r\geq R_0.$$

Let

$$\phi(x)=\theta(\rho_o(x)\vee R_0),\quad x\in M. \tag{2.1}$$

Then μ satisfies the weak Poincaré inequality with $\alpha(r):=cF_\phi^{-1}(r)$ for some positive constant c, where $F_\phi(r):=\mu(\phi\leq\frac{1}{r})$ and $F_\phi^{-1}(r)=\inf\{s:\ F_\phi(s)\leq r\}$.

(b) *Let $V\in C^2(M)$ such that for some positive constants R_0 and $\delta\in(0,1)$, there exists some positive function ϕ on M such that*

$$\phi(x)=(1-\delta)(\delta|\nabla V|^2(x)-\Delta V(x))>0,\quad \rho_o(x)\geq R_0.$$

Then μ satisfies the weak Poincaré inequality with $\alpha(r) := cF_\phi^{-1}(r)$ for some positive constant c.

Remark 2.2. (i) In Theorem 2.1 (a), it is easy to see that a different $\sigma \in (0,1)$ does not affect the sign of θ. But suitable choosing of σ seems to get the best α in the weak Poincaré inequality; see the proof of Example 4.3 for more explanations.

(ii) In the proof of this theorem, we use two ways to construct Lyapunov functions, the first is new and the second is due to [13]. Our new Lyapunov function can improve the result in [13, Theroem 4.1(a)] for the super Poincaré inequality of convolution probability measures on $\mathbb{R}^d$, see Remark 3.2 for details.

We now assume that (M,g) satisfies the following curvature condition:

Assumption (A) : Ric$\geq -(d-1)k$ for some constant k, where Ric is the Ricci curvature tensor with respect to g.

Let

$$h_k(r) = \begin{cases} \sin(\sqrt{-k}r)/\sqrt{-k}, & \text{if } k < 0; \\ r, & \text{if } k = 0; \\ \sinh(\sqrt{k}r)/\sqrt{k}, & \text{if } k > 0. \end{cases}$$

Under assumption **(A)**, we can use the following comparison theorem to handle $\Delta\rho_o$ (see [5, Section 1]):

$$\Delta\rho_o \leq \frac{(d-1)h_k'(\rho_o)}{h_k(\rho_o)}$$

outside the cut-locus. Then, the following corollary can be proved by a similar discussion as in Theorem 2.1.

Corollary 2.3. *Let $\mu(\mathrm{d}x) = \mathrm{e}^{-V(\rho_o(x))}\,\mathrm{d}x$ be a probability measure on M for some function $V \in C^1(\mathbb{R}^+)$. Suppose that assumption* **(A)** *holds, then we have the following two assertions.*

(a) *Assume that for some positive constant R_0 and any $\sigma \in (0,1)$, one has*

$$\theta(r) := \frac{(1-\sigma)h_k(r)^{1-d}V'(r)\,\mathrm{e}^{\sigma V(r)}}{\int_{R_0}^r h_k(s)^{1-d}\,\mathrm{e}^{\sigma V(s)}\,\mathrm{d}s + 1} > 0, \quad r \geq R_0.$$

Then μ satisfies the weak Poincaré inequality with $\alpha(r) := cF_\phi^{-1}(r)$ for some positive constant c, where $\phi(x) := \theta(\rho_o(x))$ for $\rho_o(x) \geq R_0$.

(b) *Assume that for some positive constants R_0 and $\delta \in (0,1)$, one has $V \in C^2([R_0,\infty))$ and*

$$\theta(r) = (1-\delta)\left[\left(\delta - \frac{(d-1)h_k'(r)}{h_k(r)}\right)V'(r) - V''(r)\right] > 0, \quad r \geq R_0. \tag{2.2}$$

Then μ satisfies a weak Poincaré inequality with $\alpha(r) := cF_\phi^{-1}(r)$ for some positive constant c, where $\phi(x) := \theta(\rho_o(x))$ for $\rho_o(x) \geq R_0$.

2.2 Proofs

Let L be a second order elliptic operator. To prove these results above, let us first introduce the following *general Lyapunov condition* with respect to L (see [7, Subsection 3.3]).

Hypothesis (L) There exist some positive constants b, r_0, some positive function ϕ on M and function $W \in \mathcal{D}(L)$ with $W \geq 1$ such that

$$\frac{LW}{W} \leq -\phi + b\mathbb{1}_{B_{r_0}}, \tag{2.3}$$

where $\mathcal{D}(L)$ is the weak domain of L and $B_{r_0} := \{x \in M : \rho_o(x) \leq r_0\}$ is the ball with center o and radius r_0.

Our first step is to prove that if hypothesis **(L)** holds for $L = \Delta - \nabla V$, then there exists some function α such that the weak Poincaré inequality holds for μ.

Lemma 2.4. *Let $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ be a probability measure on M. Assume the Lyapunov condition* **(L)** *holds for $L = \Delta - \nabla V$. Then the following weak Poincaré inequality*

$$\mu(f^2) \leq c_0 F_\phi^{-1}(r)\mu(|\nabla f|^2) + r\mathrm{Osc}(f)^2$$

holds for some positive constant c_0 and $F_\phi(r) := \mu(\phi \leq \frac{1}{r})$.

Proof. The proof is given by combining [3, Theorem 4.6] with [3, Theorem 2.18]. For the sake of completeness, we include it here. For any $r > 0$ and $f \in C_b^1(M)$ with $\mu(f) = 0$, we have

$$\begin{aligned}
\mu(f^2) = \inf_{c\in\mathbb{R}} \mu(f-c)^2 &\leq \int_{\{\phi>1/r\}} (f - f(x_0))^2\,\mathrm{d}\mu + \int_{\{\phi\leq 1/r\}} (f - f(x_0))^2\,\mathrm{d}\mu \\
&\leq \int_{\{\phi>1/r\}} (f - f(x_0))^2\,\mathrm{d}\mu + \mu(\phi \leq 1/r)\mathrm{Osc}(f)^2 \\
&\leq r\int \phi(f - f(x_0))^2\,\mathrm{d}\mu + \mu(\phi \leq 1/r)\mathrm{Osc}(f)^2 \\
&\leq -r\int \frac{LW}{W}(f - f(x_0))^2\,\mathrm{d}\mu + rb\int_{B_{r_0}} (f - f(x_0))^2\,\mathrm{d}\mu \\
&\quad + \mu(\phi \leq 1/r)\mathrm{Osc}(f)^2,
\end{aligned} \tag{2.4}$$

where $x_0 \in M$ will be specified later. Now we need to estimate the first two terms on the right hand side of the latter inequality: a global term and a local term. For the global term, by [4, Lemma 2.12], we have

$$-\int \frac{LW}{W}(f - f(x_0))^2\,\mathrm{d}\mu \leq \int |\nabla f|^2\,\mathrm{d}\mu. \tag{2.5}$$

For the local one, choose $x_0 \in M$ such that $f(x_0) = \frac{1}{\mu(B_{r_0})}\big(\int_{B_{r_0}} f\,\mathrm{d}\mu\big)$ and define $g = f - f(x_0)$. Then we obtain

$$\int_{B_{r_0}} (f - f(x_0))^2\,\mathrm{d}\mu = \int_{B_{r_0}} g^2\,\mathrm{d}\mu \leq \lambda_{r_0}^{-1}\int |\nabla g|^2\,\mathrm{d}\mu + \frac{1}{\mu(B_{r_0})}\left(\int_{B_{r_0}} g\,\mathrm{d}\mu\right)^2$$

$$= \lambda_{r_0}^{-1}\mu(|\nabla g|^2) = \lambda_{r_0}^{-1}\mu(|\nabla f|^2), \tag{2.6}$$

where by [12, (4.3.5)],

$$\lambda_{r_0}^{-1} \le \frac{4r_0^2}{\pi^2}\exp\left\{\sup_{x,y\in B_{r_0}}(V(x)-V(y))\right\} < \infty.$$

Now, taking (2.5) and (2.6) into (2.4), we arrive at

$$\mu(f^2) \le r(b\lambda_{r_0}^{-1}+1)\int |\nabla f|^2\,\mathrm{d}\mu + \mu\left(\phi \le \frac{1}{r}\right)\mathrm{Osc}(f)^2. \tag{2.7}$$

Let

$$F_\phi(r) = \mu\left(\phi \le \frac{1}{r}\right).$$

Then

$$\lim_{r\to+\infty} F_\phi(r) = 0$$

due to the fact that ϕ is positive and μ is a probability measure on M. From this, we derive that $F_\phi : (0,+\infty) \to (0,1)$ is a decreasing function. Then

$$\alpha(r) := (b\lambda_{r_0}^{-1}+1)F_\phi^{-1}(r)$$

is a function from $(0,+\infty)$ to $(0,+\infty)$ and the weak Poincaré inequality holds for such α. □

Proof of Theorem 2.1. In case (a), let $0<\sigma<1$, and define the Lyapunov function by

$$W_\sigma(r) = \int_{R_0}^r \exp\left[\sigma\int_{R_0}^s \varphi(u)\,\mathrm{d}u\right]\mathrm{d}s + 1, \quad \text{for all} \quad r \ge R_0.$$

By an approximation argument, we may consider $\rho_o \in C^2(M)$ for the sake of conciseness. Then for all $\rho_o(x) \ge R_0$, we have

$$\begin{aligned}
\frac{LW_\sigma(\rho_o(x))}{W_\sigma(\rho_o(x))} &= \frac{1}{W_\sigma(\rho_o(x))}\left[W_\sigma'(\rho_o(x))\Delta\rho_o(x) + W_\sigma''(\rho_o(x))|\nabla\rho_o(x)|^2 - W_\sigma'(\rho_o(x))\langle\nabla V, \nabla\rho_o(x)\rangle\right] \\
&= \frac{1}{W_\sigma(\rho_o(x))}\left\{W_\sigma'(\rho_o(x))[\Delta\rho_o(x) - \langle\nabla V,\nabla\rho_o(x)\rangle] + W_\sigma''(\rho_o(x))\right\} \\
&= \frac{1}{W_\sigma(\rho_o(x))}\exp\left[\sigma\int_{R_0}^{\rho_o(x)}\varphi(u)\,\mathrm{d}u\right][\Delta\rho_o(x) - \langle\nabla V,\nabla\rho_o(x)\rangle + \sigma\varphi(\rho_o(x))] \\
&\le -\frac{1-\sigma}{W_\sigma(\rho_o(x))}\exp\left[\sigma\int_{R_0}^{\rho_o(x)}\varphi(u)\,\mathrm{d}u\right]\varphi(\rho_o(x)).
\end{aligned}$$

Thus, there exists a constant $b>0$ such that

$$\frac{LW_\sigma(\rho_o(x))}{W_\sigma(\rho_o(x))} \le -\,\theta(\rho_o(x))\mathbb{1}_{\{\rho_o(x)\ge R_0\}} + b\mathbb{1}_{\{\rho_o(x)<R_0\}},$$

which combining with Lemma 2.4 implies Theorem 2.1(a).

In case (b), we consider a function W in $C^2(M)$ such that $W(x) = \mathrm{e}^{(1-\delta)V(x)}$ for all $\rho_o(x) \geq R_0$. It is easy to see that $W(x) \geq 1$ for all $x \in M$ and

$$\begin{aligned}\frac{LW(x)}{W(x)} &\leq -(1-\delta)(\delta|\nabla V|^2 - \Delta V)(x)\mathbb{1}_{\{\rho_o(x)\geq R_0\}} + b\mathbb{1}_{\{\rho_o(x)<R_0\}} \\ &= -\phi(x)\mathbb{1}_{\{\rho_o(x)\geq R_0\}} + b\mathbb{1}_{\{\rho_o(x)<R_0\}}.\end{aligned}$$

We then complete the proof of (b) by using Lemma 2.4.

□

Proof of Corollary 2.3. We still consider $\rho_o \in C^2(M)$ for the sake of brevity. In case (a), for $\sigma \in (0,1)$, define the Lyapunov function by

$$W_\sigma(r) = \int_{R_0}^r h_k(s)^{1-d}\,\mathrm{e}^{\sigma V(s)}\,\mathrm{d}s + 1, \quad \text{for all} \quad r \geq R_0.$$

Then using a similar calculation as in the proof of Theorem 3.1(a), we have

$$\frac{LW_\sigma(\rho_o(x))}{W_\sigma(\rho_o(x))} \leq -\frac{(1-\sigma)h_k(\rho_o(x))^{1-d}\,\mathrm{e}^{\sigma V(\rho_o(x))}\,V'(\rho_o(x))}{W_\sigma(\rho_o(x))}$$

for all $\rho_o(x) \geq R_0$. Therefore, there exists a positive constant b such that

$$\frac{LW_\sigma(\rho_o(x))}{W_\sigma(\rho_o(x))} \leq -\,\theta(\rho_o(x))\mathbb{1}_{\{\rho_o(x)\geq R_0\}} + b\mathbb{1}_{\{\rho_o(x)<R_0\}},$$

which, together with Lemma 2.4, implies (a).

In case (b), by assumption (2.2), we have that for all $\rho_o(x) \geq R_0$,

$$\delta|\nabla V(\rho_o(x))|^2 - \Delta V(\rho_o(x)) = \delta|V'(\rho_o(x))|^2 - V'(\rho_o(x))\frac{(d-1)h_k'(\rho_o(x))}{h_k(\rho_o(x))} - V''(\rho_o(x)) > 0.$$

Combining this with Theorem 2.1(b), we complete the proof of (b). □

3 Application to convolution probability measures on $\mathbb{R}^d$

In this section, we first apply the results in Section 2 to the convolution probability measures on $\mathbb{R}^d$ and then give the proofs.

3.1 Main results

For each $x \in \mathbb{R}^d$, let

$$\nu_x(\mathrm{d}z) = \frac{1}{p_\nu(x)}\,\mathrm{e}^{-V(x-z)}\,\nu(\mathrm{d}z).$$

For any non-increasing function $\theta : [0,\infty) \to (0,\infty)$, let

$$H_\theta(r) = (\mu+\nu)\left(|x| \geq \frac{1}{2}\theta^{-1}(1/r)\right), \quad r > 0. \tag{3.1}$$

By Theorem 2.1, we have the following first main result.

Theorem 3.1. *Let $V \in C^1(\mathbb{R}^d)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ is a probability measure on $\mathbb{R}^d$, and let ν be another probability measure on $\mathbb{R}^d$ such that $p_\nu \in C^1(\mathbb{R}^d)$.*

(a) *Assume that for some positive constant R_0, one has*

$$\psi(s) := \frac{1}{s}\inf_{|x|=s}\int_{\mathbb{R}^d}\langle \nabla V(x-z), x\rangle\,\nu_x(\mathrm{d}z) > 0, \quad s \geq R_0. \tag{3.2}$$

Then for any $\sigma \in (0,1)$, $\mu * \nu$ satisfies the weak Poincaré inequality with $\alpha(r) = cH_\theta^{-1}(r)$ for some positive constant c, where

$$\theta(s) = \inf\left\{\frac{(1-\sigma)\psi(r)r^{1-d}\exp[\sigma\int_{R_0}^r \psi(u)\,\mathrm{d}u]}{\int_{R_0}^r t^{1-d}\exp[\sigma\int_{R_0}^t \psi(u)\,\mathrm{d}u]\,\mathrm{d}t + 1} : r \in [R_0, s\vee R_0]\right\}. \tag{3.3}$$

(b) *Let $V \in C^2(\mathbb{R}^d)$ such that for some constant R_0 and $\delta \in (0,1)$, one has*

$$\theta(s) = (1-\delta)\inf_{|x|\in[R_0, s\vee R_0]}\int_{\mathbb{R}^d}\big(\delta|\nabla V(x-z)|^2 - \Delta V(x-z)\big)\,\nu_x(\mathrm{d}z) > 0. \tag{3.4}$$

Then $\mu * \nu$ satisfies the weak Poincaré inequality with $\alpha(r) = cH_\theta^{-1}(r)$ for some positive constant c.

Remark 3.2. (i) In [13], the authors prove that if the function ϕ in the Lyapunov condition satisfies

$$\liminf_{|x|\to\infty}\phi(x) = \infty, \tag{3.5}$$

then there exists a super Poincaré inequality with respect to $\mu*\nu$. Let $\tilde\theta(r) = \inf_{|x|\geq r\vee R_0}\phi(x)$. Then (3.5) holds if and only if $\lim_{r\to\infty}\tilde\theta(r) = \infty$. Note that in this case, to keep as much information about ϕ as possible, it is better for us to choose $\tilde\theta(|x|)$ instead of $\theta(|x|) := \inf_{|y|\in[R_0, R_0\vee|x|]}\phi(y)$ used in Theorem 3.1 to control $\phi(x)$. However, in this article, we take more consideration of the following case for weak Poincaré inequalities:

$$\liminf_{|x|\to\infty}\phi(x) = 0.$$

So in the case (3.5), we should refer the reader to [13] for super Poincaré inequalities.

(ii) In the proof of Theorem 3.1 (a), it provides a new and reasonable Lyapunov function such that [13, Theorem 4.1 (a)] can be improved as follows. Recall that μ satisfies the super Poincaré inequality with $\beta : (0,\infty) \to (0,\infty)$ if

$$\mu(f^2) \leq r\mu(|\nabla f|^2) + \beta(r)\mu(|f|)^2, \quad r > 0,\ f \in C_b^1(\mathbb{R}^d).$$

Theorem A. *Let $V \in C^1(\mathbb{R}^d)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ is a probability measure on $\mathbb{R}^d$, and let ν be another probability measure on $\mathbb{R}^d$ such that $p_\nu \in C^1(\mathbb{R}^d)$. Let*

$$\xi(r,s) = \Big(1 + s^{-\frac{d}{2}}\Big)\frac{\sup_{|x|\leq r}\mathrm{e}^{-(\frac{d}{2}+1)V(x)}}{\inf_{|x|\leq r}\mathrm{e}^{-(\frac{d}{2}+2)V(x)}}.$$

If $\liminf_{r\to\infty}\psi(r) = +\infty$*, where* ψ *is defined as in* (3.2)*, then* $\mu * \nu$ *satisfies the super Poincaré inequality with*

$$\beta(r) = c(1 + \xi(\gamma(2/r), r/2)),$$

where c *is some positive constant and*

$$\gamma(s) := \inf\left\{t > 0 : \inf_{r\ge t\vee R_0} \frac{(1-\sigma)\psi(r)r^{1-d}\exp[\sigma\int_{R_0}^r \psi(u)\,\mathrm{d}u]}{\int_{R_0}^r s^{1-d}\exp[\sigma\int_{R_0}^s \psi(u)\,\mathrm{d}u]\,\mathrm{d}s + 1} \ge s\right\}$$

for any $\sigma \in (0,1)$ *and some positive constant* R_0 *such that* $\psi(r) > 0$ *for all* $r \ge R_0$*.*

In the following subsection, we will give a brief explanation for the proof of this theorem and use the example in [13, Theorem 4.4] to show the benefit of this result.

From (3.2) and (3.4), it is easy to see that if the function p_ν has previous estimates, then Theorem 3.1 can be simplified as follows.

Theorem 3.3. *Let* $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ *be a probability measure on* $\mathbb{R}^d$ *and* ν *be another probability measure on* $\mathbb{R}^d$ *such that* $p_\nu \in C^1(\mathbb{R}^d)$*. Set*

$$\mathrm{e}^{-\tilde{V}_\nu(s)} = \inf_{|x|=s}\int_{\mathbb{R}^d} \mathrm{e}^{-V(x-z)}\,\nu(\mathrm{d}z) \quad \textit{and} \quad \mathrm{e}^{-\hat{V}_\nu(s)} = \sup_{|x|=s}\int_{\mathbb{R}^d} \mathrm{e}^{-V(x-z)}\,\nu(\mathrm{d}z).$$

(*a*) *If* $\tilde{V}_\nu \in C^1([0,\infty))$ *such that for some positive constant* R_0 *and any* $\sigma \in (0,1)$*, one has*

$$\theta(s) := \inf_{r\in[R_0, s\vee R_0]} \frac{(1-\sigma)\tilde{V}_\nu'(r)\mathrm{e}^{\sigma\tilde{V}_\nu(r)}r^{1-d}}{\int_{R_0}^r s^{1-d}\mathrm{e}^{\sigma\tilde{V}_\nu(s)}\,\mathrm{d}s + 1} > 0, \tag{3.6}$$

then $\mu * \nu$ *satisfies the weak Poincaré inequality with*

$$\alpha(r) = c\inf\left\{\frac{\sup_{0\le t<2s}\mathrm{e}^{\tilde{V}_\nu(t)-\hat{V}_\nu(t)}}{\theta(2s)} : (\mu+\nu)(|x|\ge s) \le r,\ s > 0\right\} \tag{3.7}$$

for some positive constant c*.*

(*b*) *If* $\tilde{V}_\nu \in C^2([0,\infty))$ *such that for some positive constant* R_0 *and* $\delta \in (0,1)$*,*

$$\theta(s) := (1-\delta)\inf_{r\in[R_0,s\vee R_0]}\left[\delta|\tilde{V}_\nu'(r)|^2 - \tilde{V}_\nu'(r)\frac{d-1}{r} - \tilde{V}_\nu''(r)\right] > 0, \tag{3.8}$$

then $\mu * \nu$ *satisfies the weak Poincaré inequality with* $\alpha(r)$ *defined as in* (3.7) *for some positive constant* c*.*

Next, we shall apply above results to the convolution with compactly supported probability measures. Note that, if ν is a probability measure with compact support, then the function p_ν is obviously differentiable on $\mathbb{R}^d$. Thus, by Theorem 3.1, we obtain the following corollary directly.

Corollary 3.4. *Let* $V \in C^1(\mathbb{R}^d)$ *such that* $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ *is a probability measure on* $\mathbb{R}^d$ *and let* ν *be another probability measure on* $\mathbb{R}^d$ *with* $R := \sup\{|z| : z \in \mathrm{supp}\nu\} < \infty$*.*

(a) *Assume that for some positive constant $R_0 > R$, one has*

$$\psi(s) := \frac{1}{s} \inf_{s-R\le|u|\le R+s} (\langle u, \nabla V(u)\rangle - R|\nabla V(u)|) > 0, \quad s \ge R_0. \tag{3.9}$$

For any $\sigma \in (0,1)$, let

$$\theta(s) = \inf\left\{ \frac{(1-\sigma)\psi(r) r^{1-d} \exp[\sigma \int_{R_0}^r \psi(u)\,\mathrm{d}u]}{\int_{R_0}^r (s^{1-d} \exp[\sigma \int_{R_0}^s \psi(u)\,\mathrm{d}u])\,\mathrm{d}s + 1} : r \in [R_0, s\vee R_0] \right\}. \tag{3.10}$$

Then $\mu * \nu$ satisfies the weak Poincaré inequality with $\alpha(r) = cH_\theta^{-1}(r)$ for some positive constant c.

(b) *Let $V \in C^2(\mathbb{R}^d)$ such that for some positive constants $R_0 > R$ and $\delta \in (0,1)$, one has*

$$\theta(s) := (1-\delta) \inf_{R_0-R\le|u|\le R+s\vee R_0} (\delta|\nabla V(u)|^2 - \Delta V(u)) > 0. \tag{3.11}$$

Then $\mu * \nu$ satisfies the weak Poincaré inequality with $\alpha(r) = cH_\theta^{-1}(r)$ for some positive constant c.

Remark 3.5. We remark that in Corollary 3.4, due to the compactness of ν and the monotonicity of θ, there exists a positive constant r_0 such that

$$H_\theta(r) = \mu\left(|x| \ge \frac{1}{2}\theta^{-1}(1/r)\right), \quad r \in (r_0, \infty).$$

By Theorem 3.3, we obtain the following corollary directly.

Corollary 3.6. *Let $V \in C^1(\mathbb{R}^d)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ is a probability measure on $\mathbb{R}^d$ and let ν be another probability measure on $\mathbb{R}^d$ with $R := \sup\{|z| : z \in \mathrm{supp}\nu\} < \infty$. For some constant $R_0 > R$ and any $s \ge R_0$, let*

$$\tilde V(s) = \sup_{s-R\le|x|\le R+s} V(x) \quad \text{and} \quad \hat V(s) = \inf_{s-R\le|x|\le R+s} V(x).$$

Then the assertions in Theorem 3.3(a)(b) still hold by replacing $\tilde V_\nu$ and $\hat V_\nu$ with $\tilde V$ and $\hat V$, respectively.

3.2 Proofs

Using Lemma 2.4, we complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $L_\nu = \Delta - \nabla V_\nu$. First, if the Lyapunov condition **(L)** holds for L_ν with some function $\phi = \theta(|\cdot|)$, where $\theta : (0,\infty) \to [0,\infty)$ is a non-increasing function, then we have

$$\begin{aligned}
&\mu * \nu\left(\theta(|x|) \le \frac{1}{r}\right) \\
&\le \mu * \nu(|x| \ge \theta^{-1}(1/r)) = \iint_{\{|x|\ge\theta^{-1}(1/r)\}} \mathrm{e}^{-V(x-z)}\,\nu(\mathrm{d}z)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq \iint_{\{|x-z|\geq \frac{1}{2}\theta^{-1}(1/r)\}} \mathrm{e}^{-V(x-z)}\,\nu(\mathrm{d}z)\,\mathrm{d}x + \iint_{\{|z|\geq \frac{1}{2}\theta^{-1}(1/r)\}} \mathrm{e}^{-V(x-z)}\,\nu(\mathrm{d}z)\,\mathrm{d}x \\
&= \iint_{\{|x-z|\geq \frac{1}{2}\theta^{-1}(1/r)\}} \mathrm{e}^{-V(x-z)}\,\mathrm{d}x\,\nu(\mathrm{d}z) + \int_{\{|z|\geq \frac{1}{2}\theta^{-1}(1/r)\}} \int \mathrm{e}^{-V(x-z)}\,\mathrm{d}x\,\nu(\mathrm{d}z) \\
&= \mu\left(2|x| \geq \theta^{-1}(1/r)\right) + \nu\left(2|z| \geq \theta^{-1}(1/r)\right) \\
&= H_\theta(r).
\end{aligned}
$$

Hence, by Lemma 2.4, $\mu * \nu$ satisfies a weak Poincaré inequality with $\alpha(r) = cH_\theta^{-1}(r)$ for some positive constant c. We now turn to construct some suitable Lyapunov functions.

In case (a), define

$$
W_\sigma(r) = \int_{R_0}^{r} \left(s^{1-d} \exp\left[\sigma \int_{R_0}^{s} \psi(u)\,\mathrm{d}u\right]\right)\mathrm{d}s + 1, \quad \text{for all} \quad r \geq R_0,
$$

where $0 < \sigma < 1$. Then by a similar discussion as in the proof of Corollary 2.3 for $k = 0$ and L_ν, we have that there exists a constant $b > 0$ such that

$$
\begin{aligned}
\frac{L_\nu W_\sigma(|x|)}{W_\sigma(|x|)} &\leq -\frac{(1-\sigma)\psi(|x|)|x|^{1-d}\exp\left[\sigma\int_{R_0}^{|x|}\psi(u)\,\mathrm{d}u\right]}{\int_{R_0}^{|x|} s^{1-d}\exp\left[\sigma\int_{R_0}^{s}\psi(u)\,\mathrm{d}u\right]\mathrm{d}s + 1}\mathbb{1}_{\{|x|\geq R_0\}} + b\mathbb{1}_{\{|x|<R_0\}} \\
&\leq -\theta(|x|)\mathbb{1}_{\{|x|\geq R_0\}} + b\mathbb{1}_{\{|x|<R_0\}}.
\end{aligned} \tag{3.12}
$$

In case (b), we consider a smooth function such that $W(x) = \mathrm{e}^{(1-\delta)V_\nu(x)}$ for $|x| \geq R_0$ and $W(x) \geq 1$ for all $x \in \mathbb{R}^d$. Using the same argument as in the proof of Theorem 2.1, we have

$$
\frac{L_\nu W(x)}{W(x)} \leq -(1-\delta)(\delta|\nabla V_\nu|^2 - \Delta V_\nu)\mathbb{1}_{\{|x|\geq R_0\}} + b\mathbb{1}_{\{|x|<R_0\}}. \tag{3.13}
$$

Moreover, for any $|x| \geq R_0$,

$$
\begin{aligned}
\delta|\nabla V_\nu(x)|^2 - \Delta V_\nu(x) &= \int_{\mathbb{R}^d} (|\nabla V(x-z)|^2 - \Delta V(x-z))\nu_x(\mathrm{d}z) - (1-\delta)|\nabla V_\nu(x)|^2 \\
&\geq \int_{\mathbb{R}^d} (\delta|\nabla V(x-z)|^2 - \Delta V(x-z))\nu_x(\mathrm{d}z) \\
&\geq \frac{1}{1-\delta}\theta(|x|).
\end{aligned} \tag{3.14}
$$

Combining this with (3.13), we complete the proof of (b). □

Proof of Theorem A. Let

$$
\theta(r) = \frac{(1-\sigma)\psi(r)r^{1-d}\exp[\sigma\int_{R_0}^{r}\psi(u)\,\mathrm{d}u]}{\int_{R_0}^{r} s^{1-d}\exp[\sigma\int_{R_0}^{s}\psi(u)\,\mathrm{d}u]\,\mathrm{d}s + 1}.
$$

By (3.12), we know that L_ν satisfies

$$
\frac{L_\nu W_\sigma(|x|)}{W_\sigma(|x|)} \leq -\theta(|x|)\mathbb{1}_{\{|x|\geq R_0\}} + b\mathbb{1}_{\{|x|<R_0\}}.
$$

It is easy to see that if $\liminf_{r\to\infty}\psi(r) = +\infty$, then $\liminf_{r\to\infty}\theta(r) = +\infty$. Thus by [13, Lemma 4.2], we complete the proof directly. □

Now we use Theorem A to prove the following result.

Example 3.7. *Let $V(x) = c + |x|^p$ for some $p > 1$ and $c \in \mathbb{R}$ such that $\mu(\mathrm{d}x) := \mathrm{e}^{-V(x)}\,\mathrm{d}x$ is a probability measure on $\mathbb{R}^d$. Let ν be any compactly supported probability measure. Then there exists a constant $c > 0$ such that $\mu * \nu$ satisfies the super Poincaré inequality with*

$$\beta(r) = \exp(cr^{-\frac{p}{2(p-1)}}), \quad r > 0.$$

Proof. Suppose that ν is supported on $\{x : |x| \le R\}$ for some positive constant R. Then

$$\psi(s) \ge \frac{1}{s} \inf_{|x|=s, |z| \le R} \langle \nabla V(x-z), x \rangle \ge \frac{p|s-R|^{p-1}s}{s+R}. \tag{3.15}$$

Thus there exists a positive constant R_0, for $r > R_0$,

$$\theta(r) = \frac{(1-\sigma)\psi(r) r^{1-d} \exp\left[\sigma \int_{R_0}^r \psi(u)\,\mathrm{d}u\right]}{\int_{R_0}^r s^{1-d} \exp\left[\sigma \int_{R_0}^s \psi(u)\,\mathrm{d}u\right] \mathrm{d}s + 1} \ge c_1 r^{2(p-1)}$$

for some positive constant c_1. Thus $\gamma(u) \le c_2(1 + u^{\frac{1}{2(p-1)}}), u > 0$ holds for some positive constant c_2. Moreover, as explained in the proof of [13, Example 4.4], one has

$$\xi(r, s) \le c_3(1 + s^{-d/2})\,\mathrm{e}^{c_4 r^p}, \quad s, r > 0$$

for some positive constants c_3, c_4. So the desired assertion follows by using Theorem A. □

However, by [13, Theorem 4.1 (a)], it is easy to calculate that $\mu * \nu$ satisfies the super Poincaré inequality with

$$\beta(r) = \exp(cr^{-\frac{p}{p-1}}), \quad r > 0,$$

which is less sharp than that presented in this example.

Let us continue with the proofs of main results in Subsection 3.1.

Proof of Theorem 3.3. Let $\tilde{L} = \Delta - \nabla \tilde{V}_\nu$ and $\tilde{\mu}(\mathrm{d}x) = \mathrm{e}^{-\tilde{V}_\nu(|x|)}\,\mathrm{d}x$. First, if the Lyapunov condition **(L)** holds for $\tilde{L}$ with some function $\phi(\cdot) = \theta(|\cdot|)$, where θ is a positive and non-decreasing function on $\mathbb{R}^+$, then for any $f \in C_b^1(\mathbb{R}^d)$ with $\mu(f) = 0$ and $x_0 \in \mathbb{R}^d$ such that $f(x_0) = \frac{1}{\tilde{\mu}(B_{r_0})}(\int_{B_{r_0}} f\,\mathrm{d}\tilde{\mu})$,

$$\begin{aligned}
\mu * \nu(f^2) &\le \inf_{c \in \mathbb{R}} \mu * \nu(f - c)^2 \\
&\le \int_{\phi > 1/s} (f - f(x_0))^2\,\mathrm{d}\mu * \nu + \int_{\phi \le 1/s} (f - f(x_0))^2\,\mathrm{d}\mu * \nu \\
&\le s \int_{\phi > 1/s} \phi (f - f(x_0))^2\,\mathrm{d}\mu * \nu + \mu * \nu\,(\phi \le 1/s)\,\mathrm{Osc}(f)^2 \\
&\le s \sup_{\phi > 1/s} \mathrm{e}^{\tilde{V}_\nu(|x|) - \hat{V}_\nu(|x|)} \int \phi (f - f(x_0))^2\,\mathrm{d}\tilde{\mu} + \mu * \nu\,(\phi \le 1/s)\,\mathrm{Osc}(f)^2 \\
&\le s \sup_{0 \le t < \theta^{-1}(1/s)} \mathrm{e}^{\tilde{V}_\nu(t) - \hat{V}_\nu(t)} \int \phi (f - f(x_0))^2\,\mathrm{d}\mu * \nu
\end{aligned}$$

$$+ (\mu+\nu)\left(|x| \geq \frac{1}{2}\theta^{-1}(1/s)\right)\mathrm{Osc}(f)^2, \qquad 1/s > \inf\theta.$$

Let $r = \frac{1}{2}\theta^{-1}(1/s)$. Then, using a similar argument as in the inequality (2.5) we obtain

$$\mu*\nu(f^2) \leq \frac{\sup_{0\leq t<2r} \mathrm{e}^{\tilde{V}_\nu(t)-\hat{V}_\nu(t)}}{\theta(2r)} \int |\nabla f|^2 \,\mathrm{d}\mu*\nu + (\mu+\nu)(|x|\geq r)\,\mathrm{Osc}(f)^2, \qquad r>0.$$

It follows that $\mu*\nu$ satisfies a weak Poincaré inequality with

$$\alpha(s) = c\inf\left\{\frac{\sup_{0\leq t<2r}\mathrm{e}^{\tilde{V}_\nu(t)-\hat{V}_\nu(t)}}{\theta(2r)} : (\mu+\nu)(|x|\geq r)\leq s,\ r>0\right\}$$

for some positive constant c. Now it suffices for us to construct some suitable Lyapunov functions.

In case (a), define the Lyapunov function by

$$W_\sigma(|x|) = \int_{R_0}^{|x|} s^{1-d}\,\mathrm{e}^{\sigma\tilde{V}_\nu(s)}\,\mathrm{d}s + 1, \quad \text{for all } \ |x|\geq R_0.$$

Then by a similar calculation as in the proof of Corollary 2.3 (a), we have that for all $|x|\geq R_0$, there exists a positive constant b such that

$$\frac{\tilde{L}W_\sigma(|x|)}{W_\sigma(|x|)} \leq -\,\theta(|x|)\mathbb{1}_{\{|x|\geq R_0\}} + b\mathbb{1}_{\{|x|<R_0\}}.$$

In case (b), we consider a smooth function such that $W(x) = c\,\mathrm{e}^{(1-\delta)\tilde{V}_\nu(|x|)}$ for $|x|\geq R_0$. Then,

$$\begin{aligned}\frac{\tilde{L}W(x)}{W(x)} &\leq -(1-\delta)\left[\delta|\tilde{V}_\nu'(|x|)|^2 - \tilde{V}_\nu'(|x|)\frac{d-1}{|x|} - \tilde{V}_\nu''(|x|)\right]\mathbb{1}_{\{|x|\geq R_0\}} + b\mathbb{1}_{\{|x|<R_0\}}\\ &= -\theta(|x|)\mathbb{1}_{\{|x|\geq R_0\}} + b\mathbb{1}_{\{|x|<R_0\}}.\end{aligned}$$

□

Proof of Corollary 3.4. In case (a). It is easy to see that

$$\begin{aligned}&\int_{\mathbb{R}^d}\langle x,\nabla V(x-z)\rangle\,\nu_x(\mathrm{d}z)\\ &\quad= \int_{\mathbb{R}^d}\big(\langle x-z,\nabla V(x-z)\rangle + \langle z,\nabla V(x-z)\rangle\big)\nu_x(\mathrm{d}z)\\ &\quad\geq \int_{\mathbb{R}^d}\big(\langle x-z,\nabla V(x-z)\rangle - R|\nabla V(x-z)|\big)\nu_x(\mathrm{d}z)\\ &\quad= \int_{\{|z|\leq R\}}\big(\langle x-z,\nabla V(x-z)\rangle - R|\nabla V(x-z)|\big)\nu_x(\mathrm{d}z).\end{aligned}$$

Then according to the definitions of ψ, we have that for any $s\geq R_0(>R)$,

$$\begin{aligned}&\inf_{|x|=s}\int_{\mathbb{R}^d}\langle x,\nabla V(x-z)\rangle\,\nu_x(\mathrm{d}z)\\ &\geq \inf_{|x|=s}\int_{\{|z|\leq R\}}(\langle x-z,\nabla V(x-z)\rangle - R|\nabla V(x-z)|)\,\nu_x(\mathrm{d}z)\\ &\geq \inf_{s-R\leq|u|\leq s+R}(\langle u,\nabla V(u)\rangle - R|\nabla V(u)|)\\ &= s\psi(s).\end{aligned}$$

Then, we complete the proof of (a) due to Theorem 3.1(a).

In case (b). For any $s \geq R_0\,(> R)$, we have that for $s \geq R_0$,

$$\begin{aligned}
&\inf_{R_0\leq |x|\leq s}\int_{\mathbb{R}^d}\Big(\delta|\nabla V|^2(x-z)-\Delta V(x-z)\Big)\nu_x(\mathrm{d}z)\\
&=\inf_{R_0\leq |x|\leq s}\int_{\{|z|\leq R\}}\Big(\delta|\nabla V|^2(x-z)-\Delta V(x-z)\Big)\nu_x(\mathrm{d}z)\\
&\geq \inf_{(R_0-R)\leq |u|\leq R+s}\big(\delta|\nabla V|^2(u)-\Delta V(u)\big)>0,
\end{aligned}$$

which leads to complete the proof by Theorem 3.1(b). $\square$

Proof of Corollary 3.6. The results follow from Theorem 3.3 and the following fact: there exists constant $R_0 > R$ such that for $s \geq R_0$,

$$\begin{aligned}
&\sup_{|x|=s}\int_{\mathbb{R}^d}\mathrm{e}^{-V(x-z)}\ \nu(\mathrm{d}z)\leq \sup_{|x|=s}\sup_{|z|\leq R}\mathrm{e}^{-V(x-z)}\leq \sup_{s-R\leq |u|\leq s+R}\mathrm{e}^{-V(u)}=\mathrm{e}^{-\tilde V(s)};\\
&\inf_{|x|=s}\int_{\mathbb{R}^d}\mathrm{e}^{-V(x-z)}\ \nu(\mathrm{d}z)\geq \inf_{|x|=s}\inf_{|z|\leq R}\mathrm{e}^{-V(x-z)}\geq \inf_{s-R\leq |u|\leq s+R}\mathrm{e}^{-V(u)}=\mathrm{e}^{-\hat V(s)}.\ \square
\end{aligned}$$

4 Examples

In this section, we present the following examples to illustrate the results obtained in Section 3. As an application of Theorem 3.3, we present below an example where the support of ν is unbounded and disconnected.

Example 4.1. *Let $d = 1$. For $0 < \delta < 1$ and $p > 0$, let $V(x) = c + (1 + x^2)^{\frac{\delta}{2}}$ and*

$$\nu(\mathrm{d}z)=\frac{1}{\gamma}\sum_{i\in\mathbb{Z}}\frac{\delta_i(\mathrm{d}z)}{1+|z|^{1+p}},$$

where

$$c=\log\int_{\mathbb{R}}\mathrm{e}^{-(1+x^2)^{\frac{\delta}{2}}}\ \mathrm{d}x\quad\text{and}\quad \gamma=\sum_{i\in\mathbb{Z}}\frac{1}{1+|i|^{1+p}}.$$

Then there exists a positive constant C such that the weak Poincaré inequality for $\mu * \nu$ holds with $\alpha(s) = Cs^{-2/p}$ for all $s > 0$.

Proof. We use Theorem 3.3(a) to give the proof. First we need to estimate p_ν. It is easy to see that

$$p_\nu(x)=\frac{\mathrm{e}^{-c}}{\gamma}\sum_{i\in\mathbb{Z}}\frac{\mathrm{e}^{-[1+(x-i)^2]^{\frac{\delta}{2}}}}{1+|i|^{1+p}}=\frac{\mathrm{e}^{-c}}{\gamma}\sum_{k\in\mathbb{Z}}\frac{\mathrm{e}^{-[1+((x)-k)^2]^{\frac{\delta}{2}}}}{1+|[x]-k|^{1+p}},$$

where $x = [x] + (x)$ and $[x]$ is the integral part of x. Moreover, as

$$\frac{1}{2}k^2\leq 1+((x)-k)^2\leq 2+k^2,$$

we have

$$\sum_{k\in\mathbb{Z}}\frac{\mathrm{e}^{-(2+k^2)^{\frac{\delta}{2}}}}{1+|[x]-k|^{1+p}}\leq\sum_{k\in\mathbb{Z}}\frac{\mathrm{e}^{-[1+((x)-k)^2]^{\frac{\delta}{2}}}}{1+|[x]-k|^{1+p}}\leq\sum_{k\in\mathbb{Z}}\frac{\mathrm{e}^{-(\frac{1}{2})^{\frac{\delta}{2}}k^\delta}}{1+|[x]-k|^{1+p}}. \tag{4.1}$$

To deal with the terms on the both sides of the inequality above, we need the following estimates:

$$\frac{|[x]|^{p+1}}{1+|[x]-k|^{p+1}} \le \frac{2^p\Big(|[x]-k|^{p+1}+|k|^{p+1}\Big)}{1+|[x]-k|^{p+1}} \le 2^p\Big(1+|k|^{p+1}\Big).$$

Using these inequalities and the dominated convergence theorem, we have

$$\lim_{|x|\to+\infty}\sum_{k\in\mathbb{Z}}\frac{|[x]|^{p+1}\,\mathrm{e}^{-(\frac{1}{2})^{\frac{\delta}{2}}k^\delta}}{1+|[x]-k|^{p+1}}=\sum_{k\in\mathbb{Z}}\mathrm{e}^{-(\frac{1}{2})^{\frac{\delta}{2}}k^\delta};$$

and

$$\lim_{|x|\to+\infty}\sum_{k\in\mathbb{Z}}\frac{|[x]|^{p+1}\,\mathrm{e}^{-(2+k^2)^{\frac{\delta}{2}}}}{1+|[x]-k|^{p+1}}=\sum_{k\in\mathbb{Z}}\mathrm{e}^{-(2+k^2)^{\frac{\delta}{2}}}.$$

Combining these with (4.1) yields

$$p_\nu(x)\asymp\frac{1}{|x|^{1+p}}.$$

Here and in what follows, for any functions f and g, we write "$f\asymp g$" if there exist positive constants c_1 and c_2 such that $c_2 f\le g\le c_1 f$. It then follows that

$$\hat{V}_\nu(s)=\tilde{V}_\nu(s)=\log(1+s)^{1+p}+\mathrm{o}(\log(1+s)).$$

By this and the definition of θ in (3.6), there exists some positive constant R_0 such that

$$\theta(s):=\inf_{r\in[R_0,s\vee R_0]}\frac{(1-\sigma)\tilde{V}_\nu'(r)\mathrm{e}^{\sigma\tilde{V}_\nu(r)}r^{1-d}}{\int_{R_0}^r t^{1-d}\mathrm{e}^{\sigma\tilde{V}_\nu(t)}\,\mathrm{d}t+1}\asymp\frac{1}{s^2}. \tag{4.2}$$

Moreover, it is easy to calculate that for large r,

$$(\mu+\nu)(|x|\ge r)\asymp r^{-p}.$$

By this and (4.2), we conclude that there exists some positive constant C such that

$$H_\theta(r)\le Cr^{p/2},$$

which completes the proof by Theorem 3.3 (a). □

Example 4.2. *Let $V(x)=c+|x|^p$ for some $0<p<1$, and $\mu(\mathrm{d}x)=\mathrm{e}^{-V(x)}\,\mathrm{d}x$. Then for any probability measure ν with $R:=\sup\{|z|:z\in\operatorname{supp}\nu\}<\infty$, there exists some positive constant C such that the weak poincaré inequality for $\mu*\nu$ holds with*

$$\alpha(s)=C\Big[1+\log\Big(1+\frac{1}{s}\Big)\Big]^{\frac{2(1-p)}{p}},\quad s>0.$$

Proof. a) *Method 1.* It is easy to see that

$$\inf_{|x|=s}\left(\langle\nabla V(x),x\rangle-R|\nabla V(x)|\right)=ps^{p-1}(s-R).$$

Then there exists $R_0 > R$ such that for $|x| \geq R_0$,

$$\inf_{|x|-R\leq s\leq |x|+R} ps^{p-1}(s-R) \asymp |x|^p.$$

Thus, we can choose $\psi(|x|) := c|x|^{p-1}$ and then have that for $|x| \geq R_0$,

$$\frac{c(1-\sigma)|x|^{p-1}|x|^{1-d}\,\mathrm{e}^{c\sigma|x|^p}}{\int_{R_0}^{|x|} u^{1-d}\,\mathrm{e}^{c\sigma u^p}\,\mathrm{d}u + 1} \geq C\frac{|x|^{p-d}\,\mathrm{e}^{c\sigma|x|^p}}{|x|^{2-d-p}\,\mathrm{e}^{c\sigma|x|^p}} = C|x|^{2(p-1)}.$$

It follows from the definition of θ in (3.10) that

$$\theta(|x|) \asymp |x|^{2(p-1)}, \quad \text{for all} \quad |x| \geq R_0.$$

From this, we obtain that for any $r > 0$,

$$H_\theta(r) = \mu\Big(2|x| \geq \theta^{-1}(1/r) \vee R_0\Big) \leq C\int_{cr^{\frac{1}{2(1-p)}}}^{\infty} \mathrm{e}^{-u^p}\,u^{d-1}\,\mathrm{d}u \leq C\,\mathrm{e}^{-cr^{\frac{p}{2(1-p)}}}\,r^{\frac{d-p}{2(1-p)}}.$$

Now using Corollary 3.4 (a), we conclude that there exists some positive constant C such that

$$\alpha(s) = C\Big[1 + \log\Big(1 + \frac{1}{s}\Big)\Big]^{\frac{2(1-p)}{p}}.$$

b) *Method 2.* It is easy to calculate that for $\delta > 0$ and $|x| > 0$,

$$\delta|\nabla V(x)|^2 - \Delta V(x) = \delta p^2|x|^{2(p-1)} - p(d+p-2)|x|^{p-2}.$$

Thus, there exists some constant $R_0 > 0$ such that for all $|x| \geq R_0$,

$$\inf_{|u|\leq |x|+R} |u|^{2(p-1)} \geq (|x|+R)^{2(p-1)}.$$

So the function θ in Corollary 3.4 (b) satisfies

$$\theta(r) \asymp r^{2(p-1)}, \ r \geq R_0.$$

The rest of the proof is similar by using Corollary 3.4 (b), so we omit it. □

Next, the following examples are to illustrate Corollary 3.6.

Example 4.3. *For $p > 0$, let $V(x) = c + (d+p)\log(1+|x|)$. Then for any probability measure ν with $R := \sup\{|z| : z \in \operatorname{supp}\nu\} < \infty$, there exists some positive constant C such that $\mu * \nu$ satisfies the weak poincaré inequality with*

$$\alpha(s) = Cs^{-\frac{2}{p}}, \quad s > 0.$$

Proof. We use Corollary 3.6 to give the proof. It is easy to see that for $s > R$,

$$\tilde{V}(s) = \sup_{s-R\leq |x|\leq R+s} V(x) = c + (d+p)\log(1+R+s),$$
$$\hat{V}(s) = \inf_{s-R\leq |x|\leq R+s} V(x) = c + (d+p)\log(1+s-R).$$

Thus, there exists a positive constant C such that

$$\mathrm{e}^{\tilde{V}(s)-\hat{V}(s)} = \frac{(1+R+s)^{d+p}}{(1+s-R)^{d+p}} \leq C, \quad s > R.$$

Moreover, for $\sigma \in (\frac{d-2}{d+p} \vee 0, 1)$, let θ be in (3.10). Then there exists a positive constant $R_0 > R$ such that

$$\theta(r) = \frac{c(1-\sigma)(d+p)(1+R+r)^{\sigma(d+p)-1} r^{1-d}}{\int_{R_0}^{r}(1+R+s)^{\sigma(d+p)} s^{1-d}\,\mathrm{d}s + 1} \leq c(1+r)^{-2}, \quad r \geq R_0\,(> R).$$

Therefore, by Corollary 3.6, we obtain the results directly. This result also can be proved in a similar way by using θ constructed in (3.11) and Corollary 3.6. □

Similarly, we have

Example 4.4. *Let $p > 1$ and $V(x) = c + d\log(1+|x|) + p\log\log(\mathrm{e}+|x|)$. Then for any probability measure ν with $R := \sup\{|z| : z \in \operatorname{supp}\,\nu\} < \infty$, there exist some positive constants c_1, c_2 such that the weak poincaré inequality holds for $\mu * \nu$ with*

$$\alpha(r) = c_1 \exp[c_2 r^{-1/(p-1)}], \quad r > 0.$$

Remark 4.5. When $\nu = \delta_0$, i.e. $R = 0$, Examples 4.3–4.4 have been treated in [12]. Compared with the results in [12], the results presented above are more precise. We would like to indicate that by [12, Corollary 4.2.2 (1)], the α in Example 4.3 implies the exact main order of $\mu * \nu(|x| > N)$ as $N \to \infty$. Hence, using Lyapunov conditions seems to be able to get better convergence or decay rates for diffusion processes.

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