FROM HYPERCOMPLEX TO HOLOMORPHIC SYMPLECTIC STRUCTURES

WEI HONG AND MATHIEU STIÉNON

Abstract. The notions of holomorphic symplectic structures and hypercomplex structures on Courant algebroids are introduced and then proved to be equivalent. These generalize hypercomplex structures and holomorphic symplectic 2-forms on manifolds respectively. Basic properties of such structures are established.

1. Introduction

This paper is an extension of [19]. Here, we make the case that, when seen in the framework of Courant algebroids, hypercomplex structures and holomorphic symplectic structures are one and the same concept.

A *hypercomplex manifold* is a smooth manifold M endowed with three complex structures $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ (regarded as endomorphisms of the tangent bundle of M) that satisfy the quaternionic relations $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. A characteristic feature of hypercomplex manifolds discovered by Obata in 1956 is the existence of a unique torsion-free connection ∇ that satisfies $\nabla\boldsymbol{i} = \nabla\boldsymbol{j} = \nabla\boldsymbol{k} = 0$ [16]. Hypercomplex manifolds have been the subject of much attention in the past. Noteworthy are the constructions of left-invariant hypercomplex structures on compact Lie groups and homogeneous spaces due to Spindel, Sevrin, Troos & Van Proeyen (in 1988) and also to Joyce (in 1992). Moreover, important examples of hypercomplex manifolds arose in mathematical physics in the form of hyper-Kähler manifolds. Hyper-Kähler manifolds are hypercomplex manifolds $(M; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ endowed with a Riemannian metric g with respect to which $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are covariantly constant and mutually orthogonal.

A *holomorphic symplectic manifold* is a complex manifold $(M; \boldsymbol{j})$ endowed with a closed nondegenerate holomorphic 2-form ω. Hyper-Kähler manifolds, which carry three symplectic 2-forms each of which is holomorphic with respect to one of the three complex structures, constitute again a special subclass.

The generalized complex geometry introduced in the last decade by Hitchin [11] and Gualtieri [8] provides the motivation for attempting to unify hypercomplex and holomorphic symplectic structures. A generalized complex structure on a manifold M is an endomorphism J of the vector bundle $TM \oplus T^*M$, skew-symmetric with respect to a natural symmetric pairing, and satisfying $J^2 = -1$ and $\mathcal{N}(J, J) = 0$, where $\mathcal{N}(J, K)$ is the Frölicher-Nijenhuis bracket of a pair (J, K) of endomorphisms of the Courant algebroid $TM \oplus T^*M$. A generalized complex structure on a manifold M can thus be seen as a complex structure on the corresponding (standard) Courant algebroid $TM \oplus T^*M$. Complex structures have been defined on arbitrary Courant algebroids in a similar fashion [14, 17].

Three new concepts are introduced in the present paper. They generalize hypercomplex manifolds, the Obata connection, and holomorphic symplectic 2-forms to the realm of Courant algebroids:

(1) A *hypercomplex structure on a Courant algebroid E* is defined as a triple of complex structures I, J, K on E satisfying the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$. Hypercomplex manifolds, holomorphic symplectic 2-forms, and hyper-Poisson manifolds

provide particular examples. The notion of hyper-Poisson structure, also introduced in this paper, can be seen as a degenerate analogue of hyper-Kähler structures.

(2) The analogue of the Obata connection for a Courant algebroid E endowed with a hypercomplex structure (I, J, K) is called a *hypercomplex connection*. Though a hypercomplex connection is not itself a connection in the usual sense, its restrictions to all Dirac subbundles of E stable under I, J, K are torsion-free (Lie algebroid) connections.

(3) A *holomorphic symplectic structure on a Courant algebroid E relative to a complex structure J on E* is a section Ω of $\wedge^2 L_J$ such that $\Omega^\sharp\overline{\Omega}^\sharp = -\operatorname{id}_{L_J}$ (Ω is nondegenerate) and $d_{L_J^*}\Omega = 0$ (Ω is closed). Here L_J and L_J^* denote the eigenbundles of J. Given a complex manifold $(M; \boldsymbol{j})$, let $J = \begin{pmatrix} \boldsymbol{j} & 0 \\ 0 & -\boldsymbol{j}^* \end{pmatrix}$ be the corresponding complex structure on the standard Courant algebroid $TM \oplus T^*M$. The holomorphic symplectic structures on $(TM \oplus T^*M; J)$ are instances of extended Poisson structures in the sense of [6].

We prove the following three theorems:

(1) A Courant algebroid endowed with a hypercomplex structure admits a unique hypercomplex connection (see Theorems 3.13 and 3.14).
(2) There exists a one-to-one correspondence between the hypercomplex structures and the holomorphic symplectic structures on a Courant algebroid (see Theorem 4.6).
(3) Given a holomorphic symplectic structure Ω on a Courant algebroid E relative to a complex structure J on E with eigenbundles L_J and L_J^*, the restriction of the hypercomplex connection on E to any Lie subalgebroid of L_J^* maximal isotropic with respect to Ω is a flat torsion-free (Lie algebroid) connection (see Theorem 5.5).

Finally, given a complex Lagrangian foliation of a complex manifold $(M; \boldsymbol{j})$ endowed with a holomorphic symplectic 2-form ω, we apply the third result above to the special case in which $E = TM \oplus T^*M$, $J = \begin{pmatrix} \boldsymbol{j} & 0 \\ 0 & -\boldsymbol{j}^* \end{pmatrix}$, and $\Omega = \omega + \overline{\omega}^{-1}$, and thereby recover a connection on the Lagrangian foliation, as discovered by Behrend & Fantechi [3].

2. Complex structures on Courant algebroids

A *Courant algebroid* (see [14, 17]) consists of a vector bundle $\pi : E \to M$, a nondegenerate symmetric pairing $\langle , \rangle$ on the fibers of π, a bundle map $\rho : E \to TM$ called the anchor, and an $\mathbb{R}$-bilinear operation $\circ$ on $\Gamma(E)$ called the Dorfman bracket, which, for all $f \in C^\infty(M)$ and $x, y \in \Gamma(E)$, satisfy the relations

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z),$$
$$x \circ fy = (\rho(x)f)y + f(x \circ y),$$
$$\rho(x)\langle y, y\rangle = 2\langle x, y \circ y\rangle = 2\langle x \circ y, y\rangle.$$

Consider the $\mathbb{R}$-linear map $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ defined by $\langle \mathcal{D}f, x\rangle = \rho(x)f$. It follows from the relation above that, for all $f \in C^\infty(M)$ and $x, y, z \in \Gamma(E)$,

$$x \circ y + y \circ x = \mathcal{D}\langle x, y\rangle,$$
$$\mathcal{D}f \circ x = 0,$$
$$\rho \circ \mathcal{D} = 0,$$
$$\rho(x \circ y) = [\rho(x), \rho(y)],$$
$$\rho(x)\langle y, z\rangle = \langle x \circ y, z\rangle + \langle y, x \circ z\rangle$$

(see [20]).