Generalizations and variants of associativity for aggregation functions

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Tutorial presentation

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Part 0: Introduction and notation
Associative binary operations

Let $X$ be a nonempty set

Consider an operation $F: X^2 \to X$
Denote $F(a, b)$ by $ab$

$F$ is associative if

$$(ab)c = a(bc)$$

$\longrightarrow$ enables us to define the expression $abc$ by setting

$$abc = (ab)c$$

**Question:** How can we define $abcd$?
Associative binary operations

By associativity, we have

\[(abc)d = (a(bc))d = a(bc)d = a((bc)d) = a(bcd) = a(b(cd)) = ab(cd) = \cdots = \cdots\]

\[\rightarrow \text{we can define } abcd \text{ by setting}\]

\[abcd = (abc)d\]

Associativity shows that the expression \(ab\)\(cd\) can be computed regardless of how the parentheses are inserted.
Associative binary operations

For any $a_1, \ldots, a_n \in X$, we can set

$$a_1 \cdots a_{n-1} a_n = (a_1 \cdots a_{n-1}) a_n$$

... can be computed regardless of how parentheses are inserted.

In other words, associativity shows that

$$a_1 \cdots a_j \cdots a_k \cdots a_n = a_1 \cdots (a_j \cdots a_k) \cdots a_n$$

holds for any $1 \leq j < k \leq n$ with $1 < j$ or $k < n$

(associativity for operations with an indefinite arity)
Associative operations with an indefinite arity

We started with a binary operation $F : X^2 \rightarrow X$ and we now extend its domain to the set $X^2 \cup X^3 \cup X^4 \cup \ldots$

$$F : \bigcup_{n \geq 2} X^n \rightarrow X$$

Assume that $F$ satisfies

- $F(F(a_1, a_2), a_3) = F(a_1, F(a_2, a_3))$
- $F(a_1, \ldots, a_{n-1}, a_n) = F(F(a_1, \ldots, a_{n-1}), a_n)$, $n \geq 3$

Then (and only then)

$$F(a_1, \ldots, a_j, \ldots, a_k, \ldots, a_n) = F(a_1, \ldots, F(a_j, \ldots, a_k), \ldots, a_n)$$
Notation

- $X = \textit{alphabet}$

- Elements of $X$: letters $(x, y, z, \ldots \in X)$

- The set

$$X^* = \bigcup_{n \geq 0} X^n$$

is the set of all tuples on $X$, called \textit{strings over $X$} $(x, y, z, \ldots \in X^*)$

\textbf{Convention:} $X^0 = \{\varepsilon\}$, where $\varepsilon = \textit{the empty string}$
Notation

- $X^*$ is endowed with concatenation ($\varepsilon$ = neutral element)

\[ x \in X^n \text{ and } y \in X \implies xy\varepsilon = xy \in X^{n+1} \]

- Repeated strings

\[ x^n = x \cdots x, \quad x^0 = \varepsilon \]

- Length of a string

\[ |x| = n \iff x \in X^n \]

\[ |\varepsilon| = 0, \quad |x| = 1 \]
Let $Y$ be a nonempty set

- **$n$-ary function**
  
  $$F : X^n \rightarrow Y$$

- **$*$-ary function or variadic function**
  
  $$F : X^* \rightarrow Y$$

**$n$-ary part of $F$**

$$F_n = F|_{X^n}$$

**Default value of $F$**

$$F(\varepsilon) = F_0(\varepsilon)$$
Part 1: Associativity, generalizations, and variants
The condition

\[ F(x_1 \cdots x_n z) = F(F(x_1 \cdots x_n) z), \quad n \geq 2 \]

can be rewritten as

\[ F(xz) = F(F(x)z), \quad |xz| \geq 3 \]

( induction condition )
The condition

\[ F(x_1 \cdots x_j \cdots x_k \cdots x_n) = F(x_1 \cdots F(x_j \cdots x_k) \cdots x_n) \]

for any \( 1 \leq j < k \leq n \) with \( 1 < j \) or \( k < n \)

can be rewritten as

\[ F(xyz) = F(xF(y)z), \quad |y| \geq 2, \ |xz| \geq 1 \]

(associativity on \( \bigcup_{n \geq 2} X^n \))
Associativitiy

**Proposition**

For any map $F: \bigcup_{n \geq 2} X^n \rightarrow X$

\[
\begin{align*}
F_2 &= F|_{X^2} \text{ is associative} \\
F(xz) &= F(F(x)z), \quad |xz| \geq 3
\end{align*}
\]

\[\uparrow\]

\[
F(xyz) = F(xF(y)z), \quad |y| \geq 2, \quad |xz| \geq 1
\]

\[\rightarrow\] Extension to functions $F$ defined on $X^* = \bigcup_{n \geq 0} X^n$?
Associativity

Definitions

- A variadic operation on \( X \) is a map \( F : X^* \rightarrow X \cup \{ \varepsilon \} \)
- A variadic operation \( F : X^* \rightarrow X \cup \{ \varepsilon \} \) is said to be associative if

\[
F(xyz) = F(xF(y)z), \quad x, y, z \in X^*
\]

We often consider the condition

\[
F(x) = \varepsilon \iff x = \varepsilon
\]

(\( F \) is \( \varepsilon \)-standard)
**Associativity**

\[ F \text{ is associative on } X^* \]

\[ + \]

\[ \varepsilon\text{-standard} \]

\[ F \text{ is associative on } \bigcup_{n \geq 2} X^n \]

\[ F_2 \text{ associative} \]

\[ + \]

\[ \text{Induction condition} \]

\[ F_1(F_1(x)) = F_1(x) \]

\[ F_1(F_2(xy)) = F_2(xy) \]

\[ F_2(x F_1(y)) = F_2(xy) \]

\[ F_2(F_1(x)y) = F_2(xy) \]
Suppose $F_2 : X^2 \rightarrow X$ associative is given

How can we extend $F_2$ to an associative and $\varepsilon$-standard operation $F : X^* \rightarrow X \cup \{\varepsilon\}$?

Induction condition $\rightarrow$ determine $F_3, F_4, \ldots, F_n, \ldots$ uniquely

What about $F_1$?

$F_1(F_1(x)) = F_1(x)$

$F_1(F_2(xy)) = F_2(xy) = F_2(x F_1(y)) = F_2(F_1(x)y)$

Note that $F_1 = \text{id}_X$ is a possible solution
Associativity

Example (sum). Take $X = \mathbb{R}$ and

\[
F_2(x_1x_2) = x_1 + x_2
\]

Induction $\rightarrow$ $F_n(x) = F(x_1 \cdots x_n) = x_1 + \cdots + x_n$, \hspace{1cm} n \geq 2

$F_1 = ?$

We have $F_1(x) = F_1(x + 0) = F_1(F_2(x0)) = F_2(x0) = x$

$\Rightarrow$ we get $F_1 = \text{id}_\mathbb{R}$. 
**Associativity**

**Example (Euclidean norm).** Take $X = \mathbb{R}$ and

\[
F_2(x_1 x_2) = \sqrt{x_1^2 + x_2^2}
\]

Induction $\Rightarrow F_n(x) = \sqrt{x_1^2 + \cdots + x_n^2} = \|x\|_2 \quad n \geq 2$

$F_1$ must satisfy

\[
\begin{align*}
F_1(F_1(x)) &= F_1(x) \\
F_1(F_2(xy)) &= F_2(xy) = F_2(xF_1(y)) = F_2(F_1(x)y)
\end{align*}
\]

$\Rightarrow F_1(x) = x$ and $F_1(x) = \sqrt{x^2}$ are possible solutions
Example (t-norm). A t-norm is an associative binary operation \( T : [0, 1]^2 \rightarrow [0, 1] \) that is symmetric, nondecreasing in each argument, and satisfies \( T(x_1) = T(1x) = x \).

→ Extension to a variadic t-norm: \( T : [0, 1]^* \rightarrow [0, 1] \cup \{\varepsilon\} \)

We have \( T_1(x) = T_1(T_2(x_1)) = T_2(x_1) = x \)

Example. If \( T(x_1x_2) = \max(0, x_1 + x_2 - 1) \), then

\[
T_n(x_1 \cdots x_n) = \max(0, \sum_{i=1}^n x_i - n + 1)
\]

Note. Same approach for t-conorms, uninorms,...
There are also associative operations $F : X^* \to X \cup \{\varepsilon\}$ that are not $\varepsilon$-standard.

**Example.** Let $a \in X$ and define

$$F(x) = \begin{cases} a, & \text{if } x = uav \text{ for some } uv \in X^* \\ \varepsilon, & \text{otherwise} \end{cases}$$

Associativity is satisfied

$$F(xyz) = F(xF(y)z)$$
Definition. A *string function* over $X$ is a function $F : X^* \rightarrow X^*$ → not an aggregation procedure

**Examples** (data processing tasks)

- $F(x) =$ sorting the letters of $x$ in alphabetic order
- $F(x) =$ transforming a string $x$ into upper case
- $F(x) =$ removing from $x$ all occurrences of a given letter
- $F(x) =$ removing from $x$ all repeated occurrences of letters

$F($associativity$) = $asociativity$ = asocitvy$

Each of these tasks satisfies

$$F(xyz) = F(xF(y)z)$$
**Definition.** A string function $F : X^* \to X^*$ is said to be **associative** if

$$F(xyz) = F(xF(y)z), \quad xyz \in X^*$$

→ generalizes associativity for operations $F : X^* \to X \cup \{\varepsilon\}$

i.e., the properties of associativity for string functions also hold for variadic operations

**Proposition**

The definition above remains equivalent if we assume $|xz| \leq 1$
Note. Setting $x = z = \varepsilon$ in the identity

$$F(xyz) = F(xF(y)z)$$

we obtain $F(y) = F(F(y))$

$$F = F \circ F$$
Associativity for string functions

Further equivalent forms

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<td>The following conditions are equivalent:</td>
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<td>(i) $F$ is associative</td>
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<td>(iv) $xyz = uvw \Rightarrow F(xF(y)z) = F(uF(v)w)$</td>
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Associativity for string functions

Definition. Let $m \geq 0$ be an integer. A function $F : X^* \rightarrow X^*$ is said to be $m$-bounded if $|F(x)| \leq m$ for every $x \in X^*$

Example. The variadic operations $F : X^* \rightarrow X \cup \{\varepsilon\}$ are exactly the 1-bounded string functions over $X$

Proposition

Assume that $F : X^* \rightarrow X^*$ is associative

(a) $F$ is $m$-bounded if and only if $F_0, \ldots, F_{m+1}$ are $m$-bounded
(b) If $m$-bounded, $F$ is uniquely determined by $F_0, \ldots, F_{m+1}$

Example (cont.) An associative function $F : X^* \rightarrow X^*$ is 1-bounded iff $F_0, F_1$, and $F_2$ range in $X \cup \{\varepsilon\}$

In this case, $F$ is completely determined by $F_0, F_1$, and $F_2$
Preassociativity

Let \( Y \) be a nonempty set

**Definition.** We say that \( F : X^* \rightarrow Y \) is **preassociative** if

\[
F(y) = F(y') \implies F(xyz) = F(xy'z)
\]

(we can assume \(|xz| \leq 1\))

**Examples.** \( F : \mathbb{R}^* \rightarrow \mathbb{R} \cup \{\varepsilon\} \)

- \( F_0 = \varepsilon, \ F_n(x) = x_1 + \cdots + x_n \)
- \( F_0 = \varepsilon, \ F_n(x) = x_1^2 + \cdots + x_n^2 = \|x\|_2^2 \)
- \( F_0 = \varepsilon, \ F_n(x) = g(x_1 + \cdots + x_n), \ g \) one-to-one

\( F \) preassociative \( g, h \) one-to-one \( \implies \ g \circ F \) and \( F \circ (h, \ldots, h) \) preassociative
Preassociativity

\[ F(y) = F(y') \implies F(xyz) = F(xy'z) \]

Equivalent definition

\[
\begin{align*}
F(x) &= F(x') \\
F(y) &= F(y')
\end{align*}
\implies F(xy) = F(x'y').
\]
Preassociativity

\[ F(y) = F(y') \implies F(xyz) = F(xy'z) \]

Proposition

Let \( F : X^* \rightarrow X^* \) be a string function

\[ F \text{ associative} \iff \begin{cases} F \circ F = F \\ F \text{ preassociative} \end{cases} \]

Proof. \((\Rightarrow)\) We have \( F \circ F = F \) by associativity.
Assume that \( F(y) = F(y') \). Then

\[ F(xyz) = F(xF(y)z) = F(xF(y')z) = F(xy'z) \]

\((\Leftarrow)\) We have \( F(F(y)) = F(y) \), and hence \( F(xF(y)z) = F(xyz) \) by preassociativity.
Preassociativity

\[ F(y) = F(y') \Rightarrow F(xyz) = F(xy'z) \]

Aggregation version:

**Proposition**

Let \( F : X^* \rightarrow X \cup \{\varepsilon\} \) be \( \varepsilon \)-standard

\[ F \text{ associative} \iff \begin{cases} F_1 \circ F^+ = F^+ \\ F \text{ preassociative} \end{cases} \]

where \( F^+ = F|_{\bigcup_{n \geq 1} X^n} \)

**Note.** Contrary to associativity, preassociativity does not involve any composition of functions.
Preassociativity

\[ F(y) = F(y') \Rightarrow F(xyz) = F(xy'z) \]

Various codomains can be considered

**Examples** \( F : X^* \to \mathbb{Z} \)

- \( F(x) = |x| \) (number of letters in \( x \))
- \( F(x) = \) number of occurrences in \( x \) of a given letter, say ‘z’
- \( F(x) = \) number of letters distinct from \( z \) minus the number of occurrences of \( z \)

**Note.** The function that outputs the number of distinct letters in \( x \) is not preassociative:

If \( a, b \in X \) are distinct, then \( F(a) = F(b) = 1 \) but

\[ 1 = F(aa) \neq F(ab) = 2 \]
Preassociativity

**Theorem**

Let \( F: X^* \rightarrow Y \). The following assertions are equivalent:

(i) \( F \) is preassociative

(ii) \( F \) can be factorized into

\[
F = f \circ H
\]

where \( H: X^* \rightarrow X^* \) is associative

\( f: \text{ran}(H) \rightarrow Y \) is one-to-one

**Example.** \( F(x) = |x| \)

\[
H(x) = x, \quad f(x) = |x| ? \quad \text{No}!
\]
Preassociativity

Example. $F(x) = |x|$

Fix $a \in X$ and define $H(x) = a^{|x|}$

$$\text{ran}(H) = \{a^n | n \geq 0\}$$

$H$ is associative

$$H(xH(y)z) = H(xa^{|y|}z) = a^{|xyz|} = H(xyz)$$

Define $f : \text{ran}(H) \rightarrow \mathbb{N}$ by $f(a^n) = n$

$\rightarrow f$ is one-to-one and

$$F = f \circ H$$
Preassociativity

Preassociative functions

Associative string functions
Preassociativity

Recall that a function $F : X^* \rightarrow X^*$ is said to be $\varepsilon$-standard if

$$F(x) = \varepsilon \iff x = \varepsilon$$

**Definition**

A variadic function $F : X^* \rightarrow Y$ is said to be **standard** if

$$F(x) = F(\varepsilon) \iff x = \varepsilon$$

This means that $\text{ran}(F_0) \cap \text{ran}(F^+) = \emptyset$
Preassociativity

Aggregation version:

**Theorem**

Assume that $F : X^* \rightarrow Y$ is standard.

The following assertions are equivalent:

(i) $F$ is preassociative and satisfies $\text{ran}(F_1) = \text{ran}(F^+)$

(ii) $F$ can be factorized into

$$F^+ = f \circ H^+$$

where $H : X^* \rightarrow X \cup \{\varepsilon\}$ is associative and $\varepsilon$-standard.

$f : \text{ran}(H^+) \rightarrow Y$ is one-to-one.

This result enables us to generate preassociative functions from known associative variadic operations.
Preassociativity

Example. Let $V = d$-dim. vector space on $\mathbb{R}$ and $B$ = basis for $V$. $(v)_B$ = coordinate vector for $v \in V$ relative to $B$.

Note. $(\cdot)_B : V \to \mathbb{R}^d$ is one-to-one.

Consider the operation $H : V^* \to V \cup \{e\}$ defined by

$$H(e) = e \quad \text{and} \quad H(v_1 \cdots v_n) = \sum_{i=1}^{n} v_i$$

Fix $e \notin \mathbb{R}^d$. The function $F : V^* \to \mathbb{R}^d \cup \{e\}$ defined by

$$F(e) = e \quad \text{and} \quad F(v_1 \cdots v_n) = (\sum_{i=1}^{n} v_i)_B$$

is preassociative, standard, and $\text{ran}(F_1) = \mathbb{R}^d = \text{ran}(F^+)$.
Preassociativity

This factorization result also enables us to produce axiomatizations of classes of preassociative functions from known axiomatizations of classes of associative functions.

A class of associative binary operations \( H : X^2 \rightarrow X \)

\[\Downarrow \quad \text{(extension)}\]

A class of associative variadic operations \( H : X^* \rightarrow X \cup \{\varepsilon\} \)

\[\Downarrow \quad \text{(factorization)}\]

A class of preassociative variadic functions \( F : X^* \rightarrow Y \)
Preassociatitivity

**Theorem** (Aczél 1949)

\[ H: \mathbb{R}^2 \rightarrow \mathbb{R} \text{ is} \]

- continuous
- one-to-one in each argument
- associative

if and only if

\[ H(xy) = \varphi^{-1}(\varphi(x) + \varphi(y)) \]

where \( \varphi: \mathbb{R} \rightarrow \mathbb{R} \) is continuous and strictly monotone

\[ H(x_1 \cdots x_n) = \varphi^{-1}(\sum_{i=1}^{n} \varphi(x_i)) \]
Preassociativity

Theorem

Assume that \( F : \mathbb{R}^* \rightarrow \mathbb{R} \cup \{ F(\varepsilon) \} \) is standard

The following assertions are equivalent:

(i) \( F \) is preassociative and satisfies \( \text{ran}(F_1) = \text{ran}(F^+) \),
\( F_1 \) and \( F_2 \) are continuous and one-to-one in each argument

(ii) we have
\[
F(x_1 \cdots x_n) = \psi \left( \sum_{i=1}^{n} \varphi(x_i) \right)
\]
where \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) and \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) are continuous and strictly monotone
Preassociativity

Extension of a binary t-norm to an ($\varepsilon$-standard) variadic t-norm

$$T: [0, 1]^2 \to [0, 1] \quad \longrightarrow \quad T: [0, 1]^* \to [0, 1] \cup \{\varepsilon\}$$

**Theorem**

Suppose $F: [0, 1]^* \to \mathbb{R} \cup \{F(\varepsilon)\}$ is standard + $F_1$ str. $\nearrow$

The following assertions are equivalent:

(i) $F$ is preassociative and satisfies $\text{ran}(F_1) = \text{ran}(F^+)$, $F_2$ is symmetric, nondecreasing in each argument

$F(x_1) = F(1x) = F(x)$

(ii) we have

$$F(x_1 \cdots x_n) = (f \circ T)(x_1 \cdots x_n)$$

where $f: [0, 1] \to \mathbb{R}$ is strictly increasing

$T: [0, 1]^* \to [0, 1] \cup \{\varepsilon\}$ is a variadic t-norm
Preassociativity

Open questions

1. Find new axiomatizations of classes of preassociative functions from existing axiomatizations of classes of associative operations
2. Find interpretations of the preassociativity property in aggregation function theory and/or fuzzy logic
**Strong preassociativity**

**Definition.** We say that $F : X^* \rightarrow Y$ is *strongly preassociative* if

\[
F(xz) = F(x'z') \implies F(xyz) = F(x'yz')
\]

We can insert letters anywhere

\[
F(abc) = F(abc) \implies F(axbc) = F(abcx)
\]

**Proposition**

Let $F : X^* \rightarrow Y$

$F$ strongly preassociative $\iff \begin{cases} 
F \text{ preassociative} \\
F_n \text{ is symmetric } \forall \ n \geq 1
\end{cases}$
Part 2: Barycentric associativity, generalizations, and variants
**Barycentric associativity**

**Definition.** A variadic operation $F : X^* \to X \cup \{\varepsilon\}$ is said to be \textit{barycentrically associative} (or \textit{B-associative}) if

$$F(xyz) = F(xF(y)|y|z)$$

$$F(abcd) = F(F(ab)^2cd) = F(F(ab)F(ab)cd)$$

**Notes.**

- ...first considered for symmetric functions on $\bigcup_{n \geq 1} \mathbb{R}^n$ (Schimmack 1909, Kolmogoroff 1930, Nagumo 1930)

- ...can be considered also for string functions $F : X^* \to X^*$
  $F(x) =$ removing from $x$ all repeated occurrences of letters
Suppose $F : X^* \rightarrow X \cup \{\varepsilon\}$ is B-associative and $\varepsilon$-standard. Then $F$ remains B-associative if we modify $F(\varepsilon)$.

Proof. Define $G : X^* \rightarrow X$ by $G(\varepsilon) = e \in X$ and $G^+ = F^+$.

$$G(xG(y)^y|z) = ?$$

- If $y = \varepsilon$, then $G(xG(y)^y|z) = G(x\varepsilon z) = G(xyz)$.
- If $y \neq \varepsilon$, then

  $$G(xG(y)^y|z) = F(xF(y)^y|z) = F(xyz) = G(xyz)$$

$\Rightarrow$ The value $F(\varepsilon)$ is unimportant and we can assume $\text{ran}(F) \subseteq X$. 

\[ . \]
**Barycentric associativity**

Consider $X = \mathbb{R}^n$ as an infinite set of identical homogeneous balls, i.e., each ball is identified by the coordinates $x \in \mathbb{R}^n$ of its center.

Define $F : X^* \to X$ as

$$F(x_1 \cdots x_n) = \text{barycenter of the balls } x_1, \ldots, x_n$$

![Diagram](image.png)

$$F(xyz) = F(xF(y)|y|z)$$
Barycentric associativity

\[ F(xyz) = F(xF(y)|y|z) \]

**Example.** Arithmetic mean \( F : \mathbb{R}^* \rightarrow \mathbb{R} \)

\[
F(x_1 \cdots x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i
\]

\[
F(x_1 F(x_2 x_3)^2) = F(x_1 \frac{x_2 + x_3}{2} \frac{x_2 + x_3}{2})
\]
\[
= \frac{1}{3} \left( x_1 + \frac{x_2 + x_3}{2} + \frac{x_2 + x_3}{2} \right)
\]
\[
= \frac{1}{3} \left( x_1 + x_2 + x_3 \right)
\]
\[
= F(x_1 x_2 x_3)
\]
Barycentric associativity

**Definition.** *Quasi-arithmetic means*
\[ \mathbb{I} = \text{non-trivial real interval, possibly unbounded} \]
\[ f : \mathbb{I} \to \mathbb{R} \text{ continuous and strictly monotonic} \]

\[ F : \mathbb{I}^* \to \mathbb{I} \]

\[ F(x_1 \cdots x_n) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(x_i)\right) \]

**Note.** *F is B-associative*
Theorem (Kolmogoroff-Nagumo, 1930)

\[ \mathbb{I} = \text{non-trivial real interval, possibly unbounded} \]

Let \( F : \mathbb{I}^* \rightarrow \mathbb{I} \)

The following assertions are equivalent:

(i) \( F \) is B-associative
   - \( F_n \) symmetric
   - \( F_n \) continuous
   - \( F_n \) strictly increasing in each argument
   - \( F_n \) idempotent, i.e., \( F_n(x \cdots x) = F(x^n) = x \)

(ii) \( F \) is a quasi-arithmetic mean

Let us show that idempotence is redundant
i.e., the other assumptions imply that \( \delta_{F_n} = \text{id}_{\mathbb{I}} \)

\[ \delta_{F_n}(x) = F_n(x^n) \]
Barycentric associativity

Setting $xz = \varepsilon$ in the identity

$$F(xyz) = F(xF(y)^{|y|}z)$$

we obtain

$$F(y) = F(F(y)^{|y|})$$

For $y = x^n$, we get

$$\delta_{F_n}(x) = F(\delta_{F_n}(x)^n) = \delta_{F_n} \circ \delta_{F_n}(x)$$

that is

$$\delta_{F_n} = \delta_{F_n} \circ \delta_{F_n}$$

Applying $\delta_{F_n}^{-1}$ to both sides gives ($\delta_{F_n}$ is one-to-one)

$$\text{id}_{\mathbb{I}} = \delta_{F_n}$$
Barycentric associativity

Further examples of real B-associative functions

\[ F_n(x) = \min(x_1, \ldots, x_n) \]
\[ F_n(x) = \max(x_1, \ldots, x_n) \]
\[ F_n(x) = x_1 \]
\[ F_n(x) = x_n \]
\[ F_n(x) = \sum_{i=1}^{n} \frac{2^{i-1}}{2^{n-1}} x_i \]

\[ F_n^\alpha(x) = \frac{\sum_{i=1}^{n} \alpha^{n-i}(1 - \alpha)^{i-1} x_i}{\sum_{i=1}^{n} \alpha^{n-i}(1 - \alpha)^{i-1}} , \quad \alpha \in \mathbb{R} \]

Take \( \alpha = 1, \alpha = 0, \alpha = 1/3, \) etc.
Barycentric associativity

\[ F(xyz) = F(xF(y)|y|z) \]

**Proposition**

The definition above remains equivalent if we assume \(|xz| \leq 1\)

**Proposition**

The following conditions are equivalent:

(i) \( F \) is B-associative

(ii) \( F(xy) = F(F(x)|x|F(y)|y|) \)

(iii) \( F(F(xy)|xy|z) = F(xF(yz)|yz|) \)

(iv) \( xyz = uvw \) \( \Rightarrow \) \( F(xF(y)|y|z) = F(uF(v)|v|w) \)
Open questions

1. Find new axiomatizations of classes of B-associative operations

2. Prove or disprove: If an operation $F : X^* \rightarrow X \cup \{\varepsilon\}$ is B-associative, then there exists a B-associative and idempotent operation $G : X^* \rightarrow X \cup \{\varepsilon\}$ such that $F_n = \delta F_n \circ G_n$ for every $n \geq 1$

3. Prove or disprove: Let $F : X^* \rightarrow X \cup \{\varepsilon\}$ be a B-associative operation. If $F_{n+1}$ is idempotent for some $n \geq 1$, then so is $F_n$
Strong barycentric associativity

Definition. A variadic operation $F : X^* \to X \cup \{\varepsilon\}$ is said to be strongly barycentrically associative (or strongly B-associative) if, for every $x \in X^*$, the value $F(x)$ does not change if we

1. select a number of letters in $x$
2. replace each of them by their aggregated value

$$F(abcd) = F(F(ac)b F(ac)d)$$

Notes

- Strong B-associativity $\Rightarrow$ B-associativity
- B-associativity $+$ $F_n$ symmetric $\forall n$ $\Rightarrow$ strong B-assoc.
- $F_n(x) = x_1 \forall n :$ strongly B-associative
- $F_n(x) = \sum_{i=1}^{n} \frac{2^{i-1}}{2^n-1} x_i \forall n :$ B-associative but not strongly
Proposition

Assume that $F : X^* \rightarrow X \cup \{ \varepsilon \}$ is strongly B-associative. Then, for every integer $k \geq 1$ and every $x, z \in X$, the function $y \in X^k \mapsto F_{k+2}(xyz)$ is symmetric.
Proposition

Let \( F : X^* \rightarrow X \cup \{ \varepsilon \} \)

The following assertions are equivalent:

(i) \( F \) is strongly B-associative

(ii) \( F(xyz) = F(F(xz)^{|x|} y F(xz)^{|z|}) \)

(iii) \( F(xyz) = F(F(xz)^{|x|} F(y)^{|y|} F(xz)^{|z|}) \)

Moreover, we may assume that \( |y| \leq 1 \) in assertions (ii) and (iii)
Strong barycentric associativity

In Kolmogoroff-Nagumo’s characterization, B-associativity and symmetry can be replaced with strong B-associativity.

**Theorem**

\[ \mathbb{I} = \text{non-trivial real interval, possibly unbounded} \]

Let \( F : \mathbb{I}^* \to \mathbb{I} \)

The following assertions are equivalent:

(i) \( F \) is strongly B-associative
   - \( F_n \) continuous
   - \( F_n \) strictly increasing in each argument

(ii) \( F \) is a quasi-arithmetic mean
Barycentric preassociativity

**Definition.** We say that $F : X^* \to Y$ is *barycentrically preassociative* (or *B-preassociative*) if

$$
\begin{align*}
F(y) &= F(y') \\
|y| &= |y'|
\end{align*}
\implies F(xyz) = F(xy'z)
$$

(we can assume $|xz| = 1$)

**Notes.**

- ...inspired from the following property by de Finetti (1931)

$$
F(y) = F(u|y|) \implies F(xyz) = F(xu|y|z) \quad (|y|, |xz| \geq 1)
$$

- Preassociativity $\implies$ B-preassociativity
- The value $F(\varepsilon)$ is unimportant
Barycentric preassociativity

B-preassociative functions

Preassociative functions

Associative string functions
Barycentric preassociativity

\[ F(y) = F(y') \]
\[ |y| = |y'| \]
\[ \Rightarrow F(xyz) = F(xy'z) \]

Interpretations

- **Decision making**: if we express an indifference when comparing two profiles, then this indifference is preserved when adding identical pieces of information to these profiles.

- **Aggregation function theory**: the aggregated value of a series of numerical values remains unchanged when modifying a bundle of these values without changing their partial aggregation.
Barycentric preassociativity

\[
\begin{align*}
F(y) &= F(y') \\
|y| &= |y'| \\
\end{align*}
\] \Rightarrow F(xyz) = F(xy'z)

Equivalent definition

\[
\begin{align*}
F(x) &= F(x') \quad \text{and} \quad F(y) &= F(y') \\
|x| &= |x'| \quad \text{and} \quad |y| &= |y'| \\
\end{align*}
\] \Rightarrow F(xy) = F(x'y')
Barycentric preassociativity

\[
F(y) = F(y') \\
|y| = |y'|
\]

\[\Rightarrow F(xyz) = F(xy'z)\]

Let \( F : X^* \rightarrow X^* \)

\( F \) associative \iff \{ \begin{align*}
F(x) &= F(F(x)) \\
F \text{ preassociative}
\end{align*} \}

**Proposition**

Let \( F : X^* \rightarrow X \cup \{ \varepsilon \} \)

\( F \) B-associative \iff \{ \begin{align*}
F(x) &= F(F(x)|x|) \\
F \text{ B-preassociative}
\end{align*} \}

The converse holds whenever \( \text{ran}(F^+) \subseteq X \).
Barycentric preassociativity

B-preassociative functions

B-associative variadic operations
Examples

- \( F(x) = |x| \) (preassociative)
- ...

Proposition

\[
\begin{align*}
F & \text{ B-preassociative} \\
g_n & \text{ one-to-one } \forall n \\
\end{align*}
\]

\[ \Rightarrow \quad g_n \circ F_n \quad \text{B-preassociative} \]

- \( F \) defined by \( F_n = g_n\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) \)
- ...

Barycentric preassociativity
Barycentric preassociativity

Theorem
Let $F : X^* \to Y$

The following assertions are equivalent:

(i) $F$ is B-preassociative and satisfies $\text{ran}(\delta F_n) = \text{ran}(F_n) \\forall \ n \geq 1$

(ii) $F$ can be factorized into

$$F_n = f_n \circ H_n \quad \forall \ n \geq 1$$

where $H : X^* \to X \cup \{\varepsilon\}$ is B-associative

$f_n : \text{ran}(H_n) \to Y$ is one-to-one

... enables us to generalize Kolmogoroff-Nagumo’s characterization
Barycentric preassociativity

Quasi-arithmetic means
\(\mathbb{I} = \text{non-trivial real interval, possibly unbounded}\)
\(f : \mathbb{I} \to \mathbb{R}\) continuous and strictly monotonic

\[
F : \mathbb{I}^* \to \mathbb{I}
\]
\[
F(x_1 \cdots x_n) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(x_i)\right)
\]

Definition. **Quasi-arithmetic pre-means**
\(f : \mathbb{I} \to \mathbb{R}\) and \(f_n : \mathbb{R} \to \mathbb{R}\) continuous and strictly increasing \((n \geq 1)\)

\[
F : \mathbb{I}^* \to \mathbb{R}
\]
\[
F(x_1 \cdots x_n) = f_n\left(\frac{1}{n} \sum_{i=1}^{n} f(x_i)\right)
\]

Note. \(F\) is B-preassociative
Barycentric preassociativity

Quasi-arithmetic pre-means

\[ F(x_1 \cdots x_n) = f_n\left(\frac{1}{n} \sum_{i=1}^{n} f(x_i)\right) \]

\( F \) quasi-arithmetic pre-mean
\( F_n \) idempotent \( \forall n \) \( \iff \) \( F \) quasi-arithmetic mean

Non-idempotent examples

- \( f_n(x) = nx \) and \( f(x) = x \) \( \Rightarrow \) \( F(x) = \sum_{i=1}^{n} x_i \)
- \( f_n(x) = e^{nx} \) and \( f(x) = \ln x \) \( \Rightarrow \) \( F(x) = \prod_{i=1}^{n} x_i \)
Barycentric preassociativity

**Theorem**

\[ \mathbb{I} = \text{non-trivial real interval, possibly unbounded} \]

Let \( F : \mathbb{I}^* \to \mathbb{R} \)

The following assertions are equivalent:

(i) \( F \) is B-preassociative
   - \( F \) is symmetric
   - \( F \) is continuous
   - \( F \) strictly increasing in each argument

(ii) \( F \) is a quasi-arithmetic pre-mean function

**Open question.** Find a characterization of those quasi-arithmetic pre-mean functions which are preassociative
Barycentric preassociativity

We would like to have...

**Theorem**

Let $F : X^* \rightarrow Y$

The following assertions are equivalent:

(i) $F$ is B-preassociative

(ii) $F$ can be factorized into ...

??
Barycentric preassociativity

Definition. A string function $F : X^* \rightarrow X^*$ is said to be \textit{length-preserving} if $|F(x)| = |x|$ for every $x \in X^*$

Examples

- $F = \text{id}_{X^*}$
- $F(x) =$ sorting the letters of $x$ in alphabetic order
- $F(x) =$ transforming a string $x$ into upper case
- NOT: $F(x) =$ removing from $x$ all occurrences of ‘z’

Proposition

Let $F : X^* \rightarrow X^*$ be length-preserving

$$F \text{ associative } \iff \left\{ \begin{array}{l} F_n \circ F_n = F_n \quad \forall \ n \geq 1 \\ F \text{ B-preassociative} \end{array} \right.$$
Barycentric preassociativity

**Theorem**

Let $F : X^* \rightarrow Y$

The following assertions are equivalent:

(i) $F$ is B-preassociative

(ii) $F$ can be factorized into

$$F_n = f_n \circ H_n \quad \forall \ n \geq 1$$

where $H : X^* \rightarrow X^*$ is associative and length-preserving

$f_n : \text{ran}(H_n) \rightarrow Y$ is one-to-one
Up to one-to-one unary maps, any of these nested classes can be described in terms of the smallest one.
**Definition.** A function $F : X^* \rightarrow Y$ is *strongly B-preassociative* if

$$F(xz) = F(x'z') \quad \left\{ \begin{array}{l} |x| = |x'| \quad \text{and} \quad |z| = |z'| \end{array} \right\} \Rightarrow F(xyz) = F(x'yz')$$

Moreover, we may assume that $|y| = 1$.

**Notes**

- Strong B-preassociativity $\Rightarrow$ B-preassociativity
- B-preassoc. $\perp F_n$ symmetric $\forall n \Rightarrow$ strong B-preassoc.
- Factorization results exist ...
Strong barycentric preassociativity

B-preassociativity and symmetry can be replaced with strong B-preassociativity in the axiomatization of the class of quasi-arithmetic pre-mean functions

**Theorem**

\( \mathbb{I} = \) non-trivial real interval, possibly unbounded  
Let \( F : \mathbb{I}^* \rightarrow \mathbb{R} \)  
The following assertions are equivalent:  
(i) \( F \) is strongly B-preassociative  
\( F_n \) continuous  
\( F_n \) strictly increasing in each argument  
(ii) \( F \) is a quasi-arithmetic pre-mean function
Thank you for your attention!