Generalizations and variants of associativity for variadic functions: a survey

53rd ISFE

Jean-Luc Marichal

in collaboration with Bruno Teheux

University of Luxembourg
Let $X$ be a nonempty set

$F : X^2 \rightarrow X$ is \textit{associative} if

\[
F(F(x, y), z) = F(x, F(y, z))
\]

\textbf{Example:} $F(x, y) = x + y$ on $X = \mathbb{R}$
Associative binary operations

Extension to a function with an indefinite arity

\[ F: \bigcup_{n \geq 2} X^n \to X \]

\[ F(x_1, \ldots, x_n) = F(F(x_1, \ldots, x_{n-1}), x_n) \quad n \geq 3 \]

Example

\[ F(x_1, x_2) = x_1 + x_2 \]
\[ F(x_1, x_2, x_3) = F(F(x_1, x_2), x_3) = x_1 + x_2 + x_3 \]

etc.
Notation

- $X = \textit{alphabet}$

- Elements of $X$: \textit{letters} \hspace{1cm} (x, y, z, \ldots \in X)

- The set

$$X^* = \bigcup_{n \geq 0} X^n$$

is the set of all tuples on $X$, called \textit{strings over} $X$

(x, y, z, \ldots \in X^*)

\textbf{Convention:} $X^0 = \{\varepsilon\}$, where $\varepsilon = \textit{the empty string}$
Notation

- $X^*$ is endowed with concatenation ($\varepsilon = \text{neutral element}$)
  
  $x \in X^n$ and $y \in X \Rightarrow xy\varepsilon = xy \in X^{n+1}$

- Repeated strings

  \[ x^n = \underbrace{x \cdots x}_{n}, \quad x^0 = \varepsilon \]

- Length of a string

  \[ |x| = n \iff x \in X^n \]
  \[ |\varepsilon| = 0, \quad |x| = 1 \]
Notation

Let $Y$ be a nonempty set

- **$n$-ary function**
  \[ F : X^n \to Y \]

- **$*$-ary function or variadic function**
  \[ F : X^* \to Y \]

**$n$-ary part of $F$**

\[ F_n = F|_{X^n} \]

**Default value of $F$**

\[ F(\varepsilon) = F_0(\varepsilon) \]
Associativity

For any map \( F: \bigcup_{n \geq 2} X^n \to X \)

- Binary associativity: \( F(F(xy)z) = F(xF(yz)) \)
- Induction formula: \( F(xz) = F(F(x)z) \) \( |xz| \geq 3 \)

**Proposition**

Binary associativity + induction formula

\[ F(xyz) = F(xF(y)z) \quad |y| \geq 2, \quad |xz| \geq 1 \]

\[ \implies \]

Extension to functions \( F \) defined on \( X^* = \bigcup_{n \geq 0} X^n \) ?
Associativity

Definitions

- A variadic operation on $X$ is a map $F : X^* \rightarrow X \cup \{\varepsilon\}$
- A variadic operation $F : X^* \rightarrow X \cup \{\varepsilon\}$ is said to be associative if

$$F(xyz) = F(xF(y)z), \quad x, y, z \in X^*$$

We say that $F$ is $\varepsilon$-standard if

$$F(x) = \varepsilon \iff x = \varepsilon$$
Associativity

\[ F(xyz) = F(xF(y)z), \quad x, y, z \in X^* \]

**Theorem**

An \( \varepsilon \)-standard operation \( F : X^* \to X \cup \{\varepsilon\} \) is associative iff

(i) Binary associativity + induction formula

(ii) \( F_1 \circ F_1 = F_1 \)

(iii) \( F_1 \circ F_2 = F_2 \)

(iv) \( F_2(xy) = F_2(F_1(x)y) = F_2(xF_1(y)) \)

**Observation:** \( F_1 = \text{id}_X \) can always be considered
Associativity

Example

\[ F : \mathbb{R}^* \rightarrow \mathbb{R} \cup \{\varepsilon\} \]

\[ F_n(x_1 \cdots x_n) = \|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2} \quad n \geq 2 \]

\[ F_0(\varepsilon) = \varepsilon \]

\[ F_1(x) = x \text{ or } F_1(x) = \sqrt{x^2} \text{ or } \cdots \]
**Definition.** A *string function* over $X$ is a function $F : X^* \rightarrow X^*$

**Examples** (data processing tasks)

- $F(x) =$ sorting the letters of $x$ in alphabetic order
- $F(x) =$ transforming a string $x$ into upper case
- $F(x) =$ removing from $x$ all occurrences of a given letter
- $F(x) =$ removing from $x$ all repeated occurrences of letters
  
  $F($associativity$) = $asociativity$ = asocitvy$

Each of these tasks satisfies

$$F(xyz) = F(xF(y)z)$$
Definition. A string function $F : X^* \rightarrow X^*$ is said to be *associative* if

$$F(xyz) = F(xF(y)z), \quad x, y, z \in X^*$$

→ generalizes associativity for operations $F : X^* \rightarrow X \cup \{\varepsilon\}$
Note. Setting $x = z = \varepsilon$ in the identity

$$F(xyz) = F(xF(y)z)$$

we obtain $F(y) = F(F(y))$

$$F = F \circ F$$
Preassociativity

Let $Y$ be a nonempty set

**Definition.** We say that $F : X^* \to Y$ is *preassociative* if

\[
F(y) = F(y') \implies F(xyz) = F(xy'z)
\]

**Examples.** $F : \mathbb{R}^* \to \mathbb{R} \cup \{\varepsilon\}$

- $F_0 = \varepsilon, \ F_n(x) = x_1 + \cdots + x_n$
- $F_0 = \varepsilon, \ F_n(x) = x_1^2 + \cdots + x_n^2 = \|x\|_2^2$
- $F_0 = \varepsilon, \ F_n(x) = g(x_1 + \cdots + x_n), \ g$ one-to-one
Preassociativity

\[
F(y) = F(y') \implies F(xyz) = F(xy'z)
\]

**Proposition**

Let \( F : X^* \rightarrow X^* \) be a string function

\[
F \text{ associative } \iff \begin{cases} 
F \circ F = F \\
F \text{ preassociative}
\end{cases}
\]
Preassociativity

\[ F(y) = F(y') \implies F(xyz) = F(xy'z) \]

Various codomains can be considered

**Examples** \( F : X^* \to \mathbb{Z} \)
- \( F(x) = |x| \) (number of letters in \( x \))
- \( F(x) = \) number of occurrences in \( x \) of a given letter, say ‘\( z \)’
- \( F(x) = \) number of letters distinct from \( z \) minus the number of occurrences of \( z \)

**Note.** The function that outputs the number of distinct letters in \( x \) is not preassociative:

If \( a, b \in X \) are distinct, then \( F(a) = F(b) = 1 \) but \( 1 = F(aa) \neq F(ab) = 2 \)
### Theorem

Let $F: X^* \to Y$. The following assertions are equivalent:

(i) $F$ is preassociative

(ii) $F$ can be factorized into

\[ F = f \circ H \]

where $H: X^* \to X^*$ is associative

$f: \text{ran}(H) \to Y$ is one-to-one
Preassociativity

Preassociative functions

Associative string functions
Barycentric associativity

**Definition.** A variadic operation $F : X^* \rightarrow X \cup \{\varepsilon\}$ is said to be **barycentrically associative** (or **B-associative**) if

$$F(xyz) = F(xF(y)|y|z)$$

$$F(abcd) = F(F(ab)^2cd) = F(F(ab)F(ab)cd)$$

**Notes.**
- ...first considered for symmetric functions on $\bigcup_{n \geq 1} \mathbb{R}^n$ (Schimmack 1909, Kolmogoroff 1930, Nagumo 1930)
- ...can be considered also for string functions $F : X^* \rightarrow X^*$
  $F(x) =$ removing from $x$ all repeated occurrences of letters
Barycentric associativity

\[ F(xyz) = F(xF(y)\mid y\mid z) \]

Suppose \( F : X^* \rightarrow X \cup \{\varepsilon\} \) is B-associative and \( \varepsilon \)-standard. Then \( F \) remains B-associative if we modify \( F(\varepsilon) \).

\[ \Rightarrow \quad \text{The value } F(\varepsilon) \text{ is unimportant and we can assume } \text{ran}(F) \subseteq X \]
Barycentric associativity

\[ F(xyz) = F(xF(y)|y|z) \]

**Example.** Arithmetic mean \( F : \mathbb{R}^* \to \mathbb{R} \)

\[
F(x_1 \cdots x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i
\]

\[
F(x_1 \, F(x_2x_3)^2) = F\left( x_1 \, \frac{x_2+x_3}{2} \, \frac{x_2+x_3}{2} \right)
\]
\[
= \frac{1}{3} \left( x_1 + \frac{x_2+x_3}{2} + \frac{x_2+x_3}{2} \right)
\]
\[
= \frac{1}{3} \left( x_1 + x_2 + x_3 \right)
\]
\[
= F(x_1 \, x_2 \, x_3)
\]
Definition. *Quasi-arithmetic means*

$I = \text{non-trivial real interval, possibly unbounded}$

$f: I \to \mathbb{R}$ continuous and strictly monotonic

$F: I^* \to I$

\[
F(x_1 \cdots x_n) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(x_i)\right)
\]

Note. $F$ is $B$-associative
**Theorem** (Kolmogoroff-Nagumo, 1930)

\[ I = \text{non-trivial real interval, possibly unbounded} \]

Let \( F: I^* \rightarrow I \)

The following assertions are equivalent:

(i)  
- \( F \) is B-associative
- \( F_n \) symmetric
- \( F_n \) continuous
- \( F_n \) strictly increasing in each argument
- \( F_n \) reflexive, i.e., \( F_n(x \cdots x) = x \)

(ii) \( F \) is a quasi-arithmetic mean

**Note.** One can show that reflexivity is redundant
Barycentric associativity

Further examples of real B-associative functions

- \( F_n(x) = \min(x_1, \ldots, x_n) \)
- \( F_n(x) = \max(x_1, \ldots, x_n) \)
- \( F_n(x) = x_1 \)
- \( F_n(x) = x_n \)
- \( F_n(x) = \sum_{i=1}^{n} \frac{2^{i-1}}{2^n-1} x_i \)

\[
F_n^\alpha(x) = \frac{\sum_{i=1}^{n} \alpha^{n-i}(1-\alpha)^{i-1} x_i}{\sum_{i=1}^{n} \alpha^{n-i}(1-\alpha)^{i-1}}, \quad \alpha \in \mathbb{R}
\]

Take \( \alpha = 1, \alpha = 0, \alpha = 1/3, \) etc.
Barycentric preassociativity

Definition. We say that $F : X^* \to Y$ is *barycentrically preassociative* (or *B-preassociative*) if

$$F(y) = F(y') \quad |y| = |y'| \quad \Rightarrow \quad F(xyz) = F(xy'z)$$

Notes

- ...inspired from the following property by de Finetti (1931)

$$F(y) = F(u|y|) \quad \Rightarrow \quad F(xyz) = F(xu|y|z) \quad (|y|, |xz| \geq 1)$$

- Preassociativity $\Rightarrow$ B-preassociativity
- The value $F(\varepsilon)$ is unimportant
Barycentric preassociativity

B-preassociative functions

Preassociative functions

Associative string functions
Barycentric preassociativity

\[
\begin{align*}
F(y) &= F(y') \\
|y| &= |y'| \\
\end{align*}
\Rightarrow 
F(xyz) &= F(xy'z)
\]

Interpretations

- **Decision making**: if we express an indifference when comparing two profiles, then this indifference is preserved when adding identical pieces of information to these profiles.

- **Aggregation function theory**: the aggregated value of a series of numerical values remains unchanged when modifying a bundle of these values without changing their partial aggregation.
Barycentric preassociativity

\[
F(y) = F(y') \quad |y| = |y'| \quad \Rightarrow \quad F(xyz) = F(xy'z)
\]

Let \( F : X^* \to X^* \)

\[
F \text{ associative} \iff \begin{cases} 
F(x) = F(F(x)) \\
F \text{ preassociative}
\end{cases}
\]

**Proposition**

Let \( F : X^* \to X \cup \{\varepsilon\} \)

\[
F \text{ B-associative} \quad \Rightarrow \quad \begin{cases} 
F(x) = F(F(x)|x|) \\
F \text{ B-preassociative}
\end{cases}
\]

The converse holds whenever \( F(x) \in X \) for all \( x \neq \varepsilon \)
Barycentric preassociativity

B-preassociative functions

B-associative variadic operations
Definition. Quasi-arithmetic pre-means

\( f : \mathbb{I} \to \mathbb{R} \) and \( f_n : \mathbb{R} \to \mathbb{R} \) continuous and strictly increasing \((n \geq 1)\)

\[
F : \mathbb{I}^* \to \mathbb{R} \\
F(x_1 \cdots x_n) = f_n\left(\frac{1}{n} \sum_{i=1}^{n} f(x_i)\right)
\]

Note. \( F \) is B-preassociative
Barycentric preassociativity

Quasi-arithmetic pre-means

\[ F(x_1 \cdots x_n) = f_n \left( \frac{1}{n} \sum_{i=1}^{n} f(x_i) \right) \]

\( F \) quasi-arithmetic pre-mean
\( F_n \) reflexive \( \forall n \)

\( \iff \)

\( F \) quasi-arithmetic mean

Non-reflexive examples

\( f_n(x) = nx \) and \( f(x) = x \) \( \Rightarrow \) \( F(x) = \sum_{i=1}^{n} x_i \)

\( f_n(x) = e^{nx} \) and \( f(x) = \ln x \) \( \Rightarrow \) \( F(x) = \prod_{i=1}^{n} x_i \)
Barycentric preassociativity

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Selected references


