A tutorial on multiple crack growth and intersections with XFEM

Danas Sutula
Prof. Stéphane Bordas
Dr. Pierre Kerfriden

24/03/2015
Content

1. Problem statement
2. Crack growth
3. XFEM discretization
4. Results/verification
5. Summary
6. Appendix: Review of crack intersection management
Problem statement

• Consider a cracked linear-elastic isotropic solid subject to an external load whose quasistatic behavior can be described by the following total Lagrangian form:

\[
\mathcal{L}(u, a) = \Pi(u, a) + \sum_{i=1}^{n_{\text{tip}}} \int_{a_i} G^i_c \, da_i
\]
Problem statement

Consider a cracked linear-elastic isotropic solid subject to an external load whose quasistatic behavior can be described by the following total Lagrangian form:

\[ \mathcal{L}(u, a) = \Pi(u, a) + \sum_{i=1}^{n_{\text{tip}}} \int_{a_i} G_c^i \, da_i \]

The solution for \( u(a) \) and \( a(t) \) are obtained by satisfying the stationarity of \( L(u, a) \) during the evolution of \( t \), subject to \( \Delta a_i \geq 0 \):

\[ \delta \mathcal{L}(u, a) = \delta_u \Pi(u, a) + \sum_{i=1}^{n_{\text{tip}}} \left[ \frac{\partial \Pi(u, a)}{\partial a_i} + G_c^i \right] \delta a_i = 0 \]

Danas Sutula (MMAM)  Global energy minimization for multi-crack growth in linear elastic fracture using XFEM
Crack growth
maximum hoop stress

- Post processing of solution to evaluate SIF

\[ I^{(1+2)} = \int_{\Omega} \left( \sigma^{(1)}_{ij} \frac{\partial u^{(2)}_i}{\partial x_1} + \sigma^{(2)}_{ij} \frac{\partial u^{(1)}_i}{\partial x_1} - W^{(1+2)} \delta_{1j} \right) \frac{\partial q}{\partial x_j} d\Omega = \frac{2}{E'} \left( K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)} \right) \]
Crack growth
maximum hoop stress

- Post processing of solution to evaluate SIF

\[ I^{(1+2)} = \int_{\Omega} \left( \sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1+2)} \delta_{ij} \right) \frac{\partial q}{\partial x_j} d\Omega = \frac{2}{E'} (K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)}) \]

- Crack growth direction

\[ \theta_c(K_I, K_{II}) = 2 \tan^{-1} \left[ \frac{1}{4} \left( \frac{K_I}{K_{II}} - \text{sign}(K_{II}) \sqrt{\left( \frac{K_I}{K_{II}} \right)^2 + 8} \right) \right] \]
Crack growth
maximum hoop stress

• Post processing of solution to evaluate SIF

\[ I^{(1+2)} = \int_{\Omega} \left( \sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1+2)} \delta_{ij} \right) \frac{\partial q}{\partial x_j} \, d\Omega = \frac{2}{E'} (K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)}) \]

• Crack growth direction

\[ \theta_c(K_I, K_{II}) = 2 \tan^{-1} \left[ \frac{1}{4} \left( \frac{K_I}{K_{II}} - \text{sign}(K_{II}) \sqrt{\left( \frac{K_I}{K_{II}} \right)^2 + 8} \right) \right] \]

• Crack growth criterion

\[ \frac{k_I(K_I, K_{II}, \theta_c)^2 + k_{II}(K_I, K_{II}, \theta_c)^2}{E'} = G_c \]
• Energy release rate w.r.t. crack increment direction, $\theta_i$: 

![Diagram showing crack growth energy minimization](image-url)
• Energy release rate w.r.t. crack increment direction, $\theta_i$:

$$G_i = -\frac{\partial \Pi(u, a + \Delta a)}{\partial \theta_i}$$
• Energy release rate w.r.t. crack increment direction, $\theta_i$:

$$G_i = -\frac{\partial \Pi(u, a + \Delta a)}{\partial \theta_i}$$

• The rates of energy release rate:

$$H_{ij} = \frac{\partial G_i}{\partial \theta_j}$$

• Updated directions:

$$\theta^{i+1} = \theta^i - H^{-1}G$$
Discretization
The eXtended Finite Element Method (XFEM)

• Approximation function (single crack)

\[ u^h(x) = \sum_{I \in \mathcal{N}_I} N_I(x)u^I + \sum_{J \in \mathcal{N}_J} N_J(x)H(x)a^J + \sum_{K \in \mathcal{N}_K} N_K(x) \sum_{\alpha=1}^{4} f_\alpha(x)b^K_\alpha \]

- standard part
- discontinuous enrichment
- singular tip enrichment

\[ H(x) = \begin{cases} 
+1 & \text{if } x \text{ above crack} \\
-1 & \text{if } x \text{ below crack} 
\end{cases} \]

\[ \{f_\alpha(r, \theta), \alpha = 1, 4\} = \left\{ \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2} \sin \theta, \sqrt{r} \cos \frac{\theta}{2} \sin \theta \right\} \]
Discretization
The eXtended Finite Element Method (XFEM)

Differentiation of the stiffness matrix w.r.t. crack increment direction

\[ \Pi = \frac{1}{2} u^T Ku - u^T f \]
Discretization
The eXtended Finite Element Method (XFEM)

\[
\delta K_e = \int_{\Omega_e} (\delta B^T D B + B^T D \delta B) \det(J) \, d\Omega + \int_{\Omega_e} B^T D B \delta \det(J) \, d\Omega
\]

\[
\delta^2 K_e = \int_{\Omega_e} (\delta^2 B^T D B + 2 \delta B^T D \delta B + B^T D \delta^2 B) \det(J) \, d\Omega + \int_{\Omega_e} 2 (\delta B^T D B + B^T D \delta B) \delta \det(J) \, d\Omega + \int_{\Omega_e} B^T D B \delta^2 \det(J) \, d\Omega
\]

Differentiation of the stiffness matrix
w.r.t. crack increment direction

\[
\Pi = \frac{1}{2} u^T K u - u^T f
\]

\[
\delta K_e = T^T K_e + K_e T
\]

\[
\delta^2 K_e = 2(T^T K_e T - K_e)
\]
Results
pressure driven growth of 10 random cracks

Fracture paths by different criteria
(square plate with 10 randomly distributed cracks that are subjected to internal pressure)

Max hoop stress
Global energy min.

$n_{\text{mesh}} = 300 \times 300, \Delta a \propto h_e$
Results

pressure driven growth of 10 random cracks

Fracture paths by different criteria
(square plate with 10 randomly distributed cracks that are subjected to internal pressure)

Max hoop stress
Global energy min.

$n_{\text{mesh}} = 600 \times 600$, $\Delta a \propto h_e$
Results
pressure driven growth of 10 random cracks

Fracture paths by different criteria
(square plate with 10 randomly distributed cracks that are subjected to internal pressure)

Max hoop stress
Global energy min.

\[ n_{\text{mesh}} = 1200 \times 1200, \Delta a \propto \frac{h}{e} \]
Results

pressure driven growth of 10 random cracks

Convergence to same fracture path by hoop-stress and energy-min. criteria
(square plate with 10 randomly distributed cracks that are subjected to internal pressure)
Results

tension splitting of plate containing 10 cracks

Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

Max hoop stress
Global energy min.

$n_{nod} = 80 \times 160$, $\Delta a \sim h_e$
Results
tension splitting of plate containing 10 cracks

Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

$\Delta a \propto h_e$
$n_{nod} = 160 \times 320$
Results
tension splitting of plate containing 10 cracks

Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

Max hoop stress
Global energy min. min.

\[ n_{\text{nod}} = 160 \times 320, \Delta a \propto h_e \]
Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

- Max hoop stress
- Global energy min.

$n_{\text{nod}} = 160 \times 320$, $\Delta a \propto h_e$
Results
tension splitting of plate containing 10 cracks

Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

Max hoop stress
Global energy min.

\( n_{\text{nod}} = 160 \times 320, \Delta a \propto h_e \)
Results

tension splitting of plate containing 10 cracks

Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

$n_{nod} = 160 \times 320$, $\Delta a \propto h_e$
Results
tension splitting of plate containing 10 cracks

Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

$\eta_{nod} = 160 \times 320$, $\Delta a \propto h_e$
Results
tension splitting of plate containing 10 cracks

Convergence to same fracture path by hoop-stress and energy-min. criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

\[ L_2 \text{-norm of distance between fracture surfaces} \]

\[ \text{best fit (slope = 0.93)} \]

Global energy minimization for multi-crack growth in linear elastic fracture using XFEM
Summary

1. Robust approach to determining multiple crack growth directions based on the principle of minimum global energy

2. The criteria converge to fracture paths that are, in the global sense, in close agreement. Consequence of local-symmetry.

3. The criteria can be used to estimate the upper/lower bound of the true fracture path for smooth crack growth problems
Elements that feature crack intersections need multiple jump enrichments to capture the kinematics of crack opening correctly (Daux et al. 2000, Budyn et al. 2004)

Multiple jump enrichments are superposed on an element by extending the element’s approximation space for each crack that the element contains whilst averting linear dependence between approximations.

A dedicated book-keeping practice is required to efficiently manage multi-layer enrichments (e.g. as pertains to intersections) in a topologically consistent manner with regard to each crack occurrence.
Managing crack intersections
merging cracks in 2D

• Minimum distance criterion:
  – intersection of crack A onto crack B is forced once the distance between A’s tip and B’s surface becomes less than a prescribed tolerance (e.g. typically, the size of A’s tip enrichment radius)
  – crack A is extended normal to B’s surface and deflected along it

• As intersection happens:
  – intersection between crack A and crack B is registered once it is detected that A’s tip increment crosses B thereby forming an ‘X’-type intersection
  – crack A is pulled back until it’s tip lies on B; A is then extended along B
Managing crack intersections
blending crack intersections

• The deflected fracture extent of A is called the blending region of the fracture junction. The size of it needs to be large enough to smoothly merge A onto B.

• Crack merging is accommodated by blending elements that lie on the blending region a short distance from the junction.

• Blending elements are jump-enriched elements that are cut by the blending region and that have at least one node whose support is cut by the part of A that can be thought as the relative complement of A with respect to B.

• In a 3D context, where cracks are polygonal surfaces, the approach to crack intersections is analogous to 2D in that the surface of crack A needs to be deflected along surface B followed by an effective blending procedure.
Managing crack intersections
multi-jump enrichment

• The discontinuous part of the displacement approximation over an element cut by several cracks can be expressed (using shifted enrichment) as follows:

\[ u_{\text{disc}}^h(x) = \sum_{i=1}^{n_{\text{crk}}} \sum_{I \in \mathcal{N}^i_H} N_I(x) \left( H_i(x) - H_i(x_I) \right) a_{iI} \]

• If the element serves to blend a particular crack (e.g. A) onto another crack (e.g. B), some of the enriched DOFs pertaining to A will have to be set to zero.

• This is to prevent linear dependence between the enriched shapes relating to A with those relating to B, because they are identical by virtue of A perfectly overlying B in the blending region.

• The blending zero-degrees of freedom are determined as those DOFs whose corresponding nodal support is not cut by the main branch of crack A, yet the DOF belongs to at least one element that has at least one of its nodal supports cut by the main branch of A.
Managing crack intersections
system updating

• Evaluate the fracture growth criterion to determine which crack grow

A snap-shot during crack evolution
(fig. shows enriched elements)
Managing crack intersections
system updating

- Evaluate the fracture growth criterion to determine which crack grow
- Advance cracks

A snap-shot during crack evolution
(fig. shows enriched elements)
Managing crack intersections
system updating

- Evaluate the fracture growth criterion to determine which crack grow
- Advance cracks
- Use an intersection criterion to merge cracks (e.g. min. dist.)

A snap-shot during crack evolution (fig. shows enriched elements)
Managing crack intersections
system updating

• Evaluate the fracture growth criterion to determine which crack grow
• Advance cracks
• Use an intersection criterion to merge cracks (e.g. min. dist.)
• Update enrichment topology

A snap-shot during crack evolution
(fig. shows enriched elements)
Managing crack intersections
system updating

• Evaluate the fracture growth criterion to determine which crack grow
• Advance cracks
• Use an intersection criterion to merge cracks (e.g. min. dist.)
• Update enrichment topology

A snap-shot during crack evolution
(fig. shows enriched elements)
Managing crack intersections
system updating (close-up view)
Managing crack intersections
system updating (close-up view)

crack-A is deflected along crack-B to from a blending region

crack (A)

crack (B)
Managing crack intersections
system updating (close-up view)

Doubly jump-enriched elements for crack A,B. Full element support for both cracks

Doubly jump-enriched elements for crack A,B. Partial element support for crack A (a blending element)

crack (A)

crack (B)
Managing crack intersections
system updating (close-up view)

Enriched nodes for crack-A: element nodal support is cut by the main branch of A

Blending nodes for crack-A: nodal support is entirely within blending region (set DOFs to zero)

crack (A)

crack (B)