Weighted lattice polynomials of independent random variables

Jean-Luc Marichal

Applied Mathematics Unit, University of Luxembourg
162A, avenue de la Faïencerie, L-1511 Luxembourg, Luxembourg

Abstract

We give the cumulative distribution functions, the expected values, and the moments of weighted lattice polynomials when regarded as real functions of independent random variables. Since weighted lattice polynomial functions include ordinary lattice polynomial functions and, particularly, order statistics, our results encompass the corresponding formulas for these particular functions. We also provide an application to the reliability analysis of coherent systems.

Key words: weighted lattice polynomial, lattice polynomial, order statistic, cumulative distribution function, moment, reliability.

1 Introduction

The cumulative distribution functions (c.d.f.'s) and the moments of order statistics have been discovered and studied for many years (see for instance David and Nagaraja [4]). Their generalizations to lattice polynomial functions, which are nonsymmetric extensions of order statistics, were investigated very recently by Marichal [10] for independent variables and then by Dukhovny [5] for dependent variables.

Roughly speaking, an n-ary *lattice polynomial* is a well-formed expression involving n real variables $x_1, \ldots, x_n$ linked by the lattice operations $\wedge = \min$ and $\vee = \max$ in an arbitrary combination of parentheses. In turn, such an expression naturally defines an n-ary *lattice polynomial function*. For instance,

$$p(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee x_3$$

Email address: jean-luc.marichal[at]uni.lu (Jean-Luc Marichal).

is a 3-ary (ternary) lattice polynomial function.

Lattice polynomial functions can be straightforwardly generalized by fixing certain variables as parameters, like in the 2-ary (binary) polynomial function

$$p(x_1, x_2) = (c \wedge x_1) \vee x_2,$$

where c is a real constant. Such "parameterized" lattice polynomial functions, called *weighted lattice polynomial* functions [7], play an important role in the areas of nonlinear aggregation and integration. Indeed, they include the whole class of discrete Sugeno integrals [12,13], which are very useful aggregation functions in many areas. More details about the Sugeno integrals and their applications can be found in the remarkable edited book [6].

In this paper we give closed-form formulas for the c.d.f. of any weighted lattice polynomial function in terms of the c.d.f.'s of its input variables. More precisely, considering a weighted lattice polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ and n independent random variables $X_1, \ldots, X_n$, X_i $(i = 1, \ldots, n)$ having c.d.f. $F_i(x)$, we give formulas for the c.d.f. of $Y_p := p(X_1, \ldots, X_n)$. We also yield formulas for the expected value $\mathbf{E}[g(Y_p)]$, where g is any measurable function. The special cases $g(x) = x$, x^r, $[x - \mathbf{E}(Y_p)]^r$, and e^{tx} give, respectively, the expected value, the raw moments, the central moments, and the moment-generating function of Y_p.

As the lattice polynomial functions are particular weighted lattice polynomial functions, we retrieve, as a corollary, the formulas of the c.d.f.'s and the moments of the lattice polynomial functions.

This paper is organized as follows. In Section 2 we recall the basic material related to lattice polynomial functions and their weighted versions. In Section 3 we provide the announced results. In Section 4 we investigate the particular case where the input random variables are uniformly distributed over the unit interval. Finally, in Section 5 we provide an application of our results to the reliability analysis of coherent systems.

2 Weighted lattice polynomials

In this section we recall some basic definitions and properties related to weighted lattice polynomial functions. More details and proofs can be found in [7].

As we are concerned with weighted lattice polynomial functions of random variables, we do not consider weighted lattice polynomial functions on a general lattice, but simply on a closed interval $L := [a, b]$ of the extended real number system $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Clearly, such an interval is a bounded distributive lattice, with a and b as bottom and top elements. The lattice operations $\wedge$ and $\vee$ then represent the minimum and maximum operations, respectively. To simplify the notation, we also set $[n] := \{1, \ldots, n\}$ for any integer $n \geqslant 1$.

Let us first recall the definition of lattice polynomial functions (with real variables); see for instance Birkhoff [2, §II.5].

Definition 1 *The class of* lattice polynomial functions *from L^n to L is defined as follows:*

(1) For any $k \in [n]$, the projection $(x_1, \ldots, x_n) \mapsto x_k$ is a lattice polynomial function from L^n to L.
(2) If p and q are lattice polynomial functions from L^n to L, then $p \wedge q$ and $p \vee q$ are lattice polynomial functions from L^n to L.
(3) Every lattice polynomial function from L^n to L is constructed by finitely many applications of the rules (1) and (2).

As mentioned in the introduction, weighted lattice polynomial functions generalize lattice polynomial functions by considering both variables and constants. We thus have the following definition.

Definition 2 *The class of* weighted lattice polynomial functions *from L^n to L is defined as follows:*

(1) For any $k \in [n]$ and any $c \in L$, the projection $(x_1, \ldots, x_n) \mapsto x_k$ and the constant function $(x_1, \ldots, x_n) \mapsto c$ are weighted lattice polynomial functions from L^n to L.
(2) If p and q are weighted lattice polynomial functions from L^n to L, then $p \wedge q$ and $p \vee q$ are weighted lattice polynomial functions from L^n to L.
(3) Every weighted lattice polynomial function from L^n to L is constructed by finitely many applications of the rules (1) and (2).

Because L is a distributive lattice, any weighted lattice polynomial function can be written in *disjunctive* and *conjunctive* forms as follows.

Proposition 3 *Let $p : L^n \to L$ be any weighted lattice polynomial function. Then there are set functions $\alpha : 2^{[n]} \to L$ and $\beta : 2^{[n]} \to L$ such that*

$$p(\mathbf{x}) = \bigvee_{S \subseteq [n]} \Big[\alpha(S) \wedge \bigwedge_{i \in S} x_i\Big] = \bigwedge_{S \subseteq [n]} \Big[\beta(S) \vee \bigvee_{i \in S} x_i\Big].$$

Proposition 3 naturally includes the classical lattice polynomial functions. To see this, it suffices to consider nonconstant set functions $\alpha : 2^{[n]} \to \{a, b\}$ and $\beta : 2^{[n]} \to \{a, b\}$, with $\alpha(\varnothing) = a$ and $\beta(\varnothing) = b$.

The set functions α and β that disjunctively and conjunctively define the polynomial function p in Proposition 3 are not unique. For example, we have

$$x_1 \vee (x_1 \wedge x_2) = x_1 = x_1 \wedge (x_1 \vee x_2).$$

However, it can be shown that, from among all the possible set functions that disjunctively define a given weighted lattice polynomial function, only one is nondecreasing. Similarly, from among all the possible set functions that conjunctively define a given weighted lattice polynomial function, only one is nonincreasing. These particular set functions are given by

$$\alpha(S) = p(\mathbf{e}_S) \quad \text{and} \quad \beta(S) = p(\mathbf{e}_{[n]\setminus S}),$$

where, for any $S \subseteq [n]$, $\mathbf{e}_S$ denotes the characteristic vector of S in $\{a, b\}^n$, i.e., the n-dimensional vector whose ith component is b, if $i \in S$, and a, otherwise. Thus, any n-ary weighted lattice polynomial function can always be written as

$$p(\mathbf{x}) = \bigvee_{S\subseteq[n]} \Big[p(\mathbf{e}_S) \wedge \bigwedge_{i\in S} x_i\Big] = \bigwedge_{S\subseteq[n]} \Big[p(\mathbf{e}_{[n]\setminus S}) \vee \bigvee_{i\in S} x_i\Big].$$

The best known instances of weighted lattice polynomial functions are given by the discrete *Sugeno integrals*, which consist of a nonlinear discrete integration with respect to a *fuzzy measure*.

Definition 4 *An L-valued* fuzzy measure *on $[n]$ is a nondecreasing set function $\mu : 2^{[n]} \to L$ such that $\mu(\varnothing) = a$ and $\mu([n]) = b$.*

The Sugeno integrals can be presented in various equivalent forms. The next definition introduce them in one of their simplest forms (see Sugeno [12]).

Definition 5 *Let μ be an L-valued fuzzy measure on $[n]$. The* Sugeno integral *of a function $\mathbf{x} : [n] \to L$ with respect to μ is defined by*

$$\mathcal{S}_\mu(\mathbf{x}) := \bigvee_{S\subseteq[n]} \Big[\mu(S) \wedge \bigwedge_{i\in S} x_i\Big].$$

Thus, any function $f : L^n \to L$ is an n-ary Sugeno integral if and only if it is a weighted lattice polynomial function fulfilling $f(\mathbf{e}_\varnothing) = a$ and $f(\mathbf{e}_{[n]}) = b$.

3 Cumulative distribution functions and moments

Consider n independent random variables $X_1, \ldots, X_n$, X_i $(i \in [n])$ having c.d.f. $F_i(x)$, and set $Y_p := p(X_1, \ldots, X_n)$, where $p : L^n \to L$ is any weighted lattice polynomial function. Let $H : \overline{\mathbb{R}} \to \{0,1\}$ be the Heaviside step function, defined by $H(x) = 1$, if $x \geqslant 0$, and 0, otherwise. For any $c \in \overline{\mathbb{R}}$, we also define the function $H_c(x) := H(x - c)$.

The c.d.f. of Y_p is given in the next theorem.

Theorem 6 *Let $p : L^n \to L$ be a weighted lattice polynomial function. Then, the c.d.f. of Y_p is given by each of the following formulas:*

$$F_p(y) = 1 - \sum_{S \subseteq [n]} \Big[1 - H_{p(\mathbf{e}_S)}(y)\Big] \prod_{i \in [n]\setminus S} F_i(y) \prod_{i \in S} [1 - F_i(y)], \tag{1}$$

$$F_p(y) = \sum_{S \subseteq [n]} H_{p(\mathbf{e}_{[n]\setminus S})}(y) \prod_{i \in S} F_i(y) \prod_{i \in [n]\setminus S} [1 - F_i(y)]. \tag{2}$$

Proof. Starting from the disjunctive form of p, we have

$$\begin{aligned} F_p(y) &= 1 - \Pr\Big[\bigvee_{S \subseteq [n]} \Big[p(\mathbf{e}_S) \wedge \bigwedge_{i \in S} X_i\Big] > y\Big] \\ &= 1 - \Pr\Big[\exists S \subseteq [n] \text{ such that } y < p(\mathbf{e}_S) \text{ and } y < \bigwedge_{i \in S} X_i\Big]. \end{aligned}$$

Consider the following events:

$$\begin{aligned} A &:= \Big[\exists S \subseteq [n] \text{ such that } y < p(\mathbf{e}_S) \text{ and } y < \bigwedge_{i \in S} X_i\Big], \\ B &:= \Big[\exists S \subseteq [n] \text{ such that } y < p(\mathbf{e}_S) \text{ and } \bigvee_{i \in [n]\setminus S} X_i \leqslant y < \bigwedge_{i \in S} X_i\Big]. \end{aligned}$$

Event B implies event A trivially. Conversely, noting that p is nondecreasing in each variable and replacing S with a superset $S' \supseteq S$, if necessary, we readily see that event A implies event B.

Since the events $\Big[\bigvee_{i \in [n]\setminus S} X_i \leqslant y < \bigwedge_{i \in S} X_i\Big]$ ($S \subseteq [n]$) are mutually exclusive, we have

$$F_p(y) = 1 - \sum_{S \subseteq [n]} \Pr\Big[y < p(\mathbf{e}_S)\Big] \Pr\Big[\bigvee_{i \in [n]\setminus S} X_i \leqslant y < \bigwedge_{i \in S} X_i\Big],$$

which, using independence, proves the first formula. The second one can be proved similarly by starting from the conjunctive form of p. $\square$

The expressions of $F_p(y)$, given in Theorem 6, are closely related to the following concept of *multilinear extension* of a set function, which was introduced by Owen [11] in game theory.

Definition 7 *The* multilinear extension *of a set function* $v : 2^{[n]} \to \mathbb{R}$ *is the function* $\Phi_v : [0,1]^n \to \mathbb{R}$ *defined by*

$$\Phi_v(\mathbf{x}) := \sum_{S\subseteq[n]} v(S) \prod_{i\in S} x_i \prod_{i\in[n]\setminus S} (1 - x_i).$$

Using this concept, we can immediately rewrite (1) and (2) as

$$F_p(y) = 1 - \Phi_{v_{p,y}}[1 - F_1(y), \ldots, 1 - F_n(y)],$$
$$F_p(y) = \Phi_{v^*_{p,y}}[F_1(y), \ldots, F_n(y)],$$

where, for any fixed $y \in \mathbb{R}$, the (nondecreasing) set functions $v_{p,y} : 2^{[n]} \to \{0,1\}$ and $v^*_{p,y} : 2^{[n]} \to \{0,1\}$ are defined by

$$v_{p,y}(S) := 1 - H_{p(\mathbf{e}_S)}(y) \quad \text{and} \quad v^*_{p,y}(S) := H_{p(\mathbf{e}_{[n]\setminus S})}(y).$$

Owen [11] showed that the function Φ_v, being a multilinear polynomial, has the form

$$\Phi_v(\mathbf{x}) = \sum_{S\subseteq[n]} m_v(S) \prod_{i\in S} x_i, \tag{3}$$

where the set function $m_v : 2^{[n]} \to \mathbb{R}$, called the *Möbius transform* of v, is defined as

$$m_v(S) = \sum_{T\subseteq S} (-1)^{|S|-|T|} v(T). \tag{4}$$

Using this polynomial form of Φ_v, we can immediately derive two further formulas for $F_p(y)$, namely

$$F_p(y) = 1 - \sum_{S\subseteq[n]} m_{v_{p,y}}(S) \prod_{i\in S} [1 - F_i(y)], \tag{5}$$

$$F_p(y) = \sum_{S\subseteq[n]} m_{v^*_{p,y}}(S) \prod_{i\in S} F_i(y). \tag{6}$$

Formulas (1)–(2) and (5)–(6) thus provide four equivalent expressions for $F_p(y)$. As particular cases, we retrieve the c.d.f. of any lattice polynomial function. For example, using formula (1) leads to the following corollary (see [10]).

Corollary 8 *Let $p : L^n \to L$ be a lattice polynomial function. Then, the c.d.f. of Y_p is given by*

$$F_p(y) = 1 - \sum_{\substack{S\subseteq[n] \\ p(\mathbf{e}_S)=b}} \prod_{i\in[n]\setminus S} F_i(y) \prod_{i\in S}[1 - F_i(y)].$$

Let us now consider the expected value $\mathbf{E}[g(Y_p)]$, where $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is any measurable function. For instance, when $g(x) = x^r$, we obtain the raw moments of Y_p.

By definition, we simply have

$$\mathbf{E}[g(Y_p)] = \int_{-\infty}^{\infty} g(y)\,\mathrm{d}F_p(y) = -\int_{-\infty}^{\infty} g(y)\,\mathrm{d}[1 - F_p(y)].$$

Using integration by parts, we can derive alternative expressions of $\mathbf{E}[g(Y_p)]$. We then have the following immediate result.

Proposition 9 *Let $p : L^n \to L$ be any weighted lattice polynomial function and let $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be any measurable function of bounded variation.*

(1) If $\lim_{y\to\infty} g(y)[1 - F_i(y)] = 0$ for all $i \in [n]$, then

$$\mathbf{E}[g(Y_p)] = g(-\infty) + \int_{-\infty}^{\infty} [1 - F_p(y)]\,\mathrm{d}g(y).$$

(2) If $\lim_{y\to-\infty} g(y)F_i(y) = 0$ for all $i \in [n]$, then

$$\mathbf{E}[g(Y_p)] = g(\infty) - \int_{-\infty}^{\infty} F_p(y)\,\mathrm{d}g(y).$$

Clearly, combining Proposition 9 with formulas (1)–(2) and (5)–(6) immediately leads to various explicit expressions of $\mathbf{E}[g(Y_p)]$. For instance, if

$$\lim_{y\to\infty} g(y)[1 - F_i(y)] = 0, \qquad \forall\, i \in [n],$$

then

$$\mathbf{E}[g(Y_p)] = g(-\infty) + \sum_{S\subseteq[n]} \int_{-\infty}^{p(\mathbf{e}_S)} \prod_{i\in[n]\setminus S} F_i(y) \prod_{i\in S}[1 - F_i(y)]\,\mathrm{d}g(y). \tag{7}$$

It is noteworthy that Eq. (7) can also be established without the knowledge of the c.d.f. of Y_p. As the proof is very informative, we give it in the appendix.

Remark 10 *We can also retrieve the c.d.f. of Y_p directly from formula (7). Indeed, rewriting (7) with the function $g(y) = H(z-y)$, we immediately obtain*

$$\begin{aligned} F_p(z) = \Pr[z - Y_p \geqslant 0] &= \Pr[H(z - Y_p) = 1] = \mathbf{E}[H(z - Y_p)] \\ = 1 + \int_{-\infty}^{\infty} \sum_{S\subseteq[n]} \Big[1 - H_{p(\mathbf{e}_S)}(y)\Big] \prod_{i\in[n]\setminus S} F_i(y) \prod_{i\in S}[1 - F_i(y)]\, \mathrm{d}H(z - y) \end{aligned}$$

and hence we retrieve (1).

Remark 11 *Suppose that some variables X_i are constants, say, $X_k = c_k$ for all $k \in K$ for a given $K \subseteq [n]$. Then, the weighted lattice polynomial function p reduces to an $(n - |K|)$-ary weighted lattice polynomial function and it is easy to see that*

$$\bigvee_{S\subseteq[n]} \Big[p(\mathbf{e}_S) \wedge \bigwedge_{i\in S} X_i\Big] = \bigvee_{S\subseteq[n]\setminus K} \Big[\alpha_p^K(S) \wedge \bigwedge_{i\in S} X_i\Big]$$

where

$$\alpha_p^K(S) := \bigvee_{T\subseteq K} \Big[p(\mathbf{e}_{S\cup T}) \wedge \bigwedge_{j\in T} c_j\Big] \qquad (S \subseteq [n] \setminus K).$$

Thus, the conditional expectation $\mathbf{E}[g(Y_p) \mid X_k = c_k \, \forall k \in K]$ can be immediately calculated by Proposition 9.

4 The case of uniformly distributed variables on the unit interval

We now examine the case where the random variables $X_1, \ldots, X_n$ are uniformly distributed on $[0, 1]$. We also assume $L = [0, 1]$.

Recall that the *incomplete beta function* is defined, for any $u, v > 0$, by

$$B_z(u, v) := \int_0^z t^{u-1}(1 - t)^{v-1}\, \mathrm{d}t \qquad (z \in \mathbb{R}),$$

and the *beta function* is defined, for any $u, v > 0$, by $B(u, v) := B_1(u, v)$.

According to Eq. (7), for any weighted lattice polynomial function $p : [0, 1]^n \to [0, 1]$ and any measurable function $g : [0, 1] \to \overline{\mathbb{R}}$ of bounded variation, we have

$$\mathbf{E}[g(Y_p)] = g(0) + \sum_{S\subseteq[n]} \int_0^{p(\mathbf{e}_S)} y^{n-|S|}(1 - y)^{|S|}\, \mathrm{d}g(y).$$

If, furthermore, $g(y) = \frac{1}{r} y^r$, with $r \in \mathbb{N} \setminus \{0\}$, then

$$\frac{1}{r}\, \mathbf{E}[Y_p^r] = \sum_{S\subseteq[n]} B_{p(\mathbf{e}_S)}(n - |S| + r, |S| + 1). \tag{8}$$

Remark 12 *Considering Eq. (8), with $p(\mathbf{e}_S) = z$, if $S = [n]$, and 0, otherwise, we obtain*

$$B_z(r, n+1) = \frac{1}{r}\int_{[0,1]^n} \Big[z \wedge \bigwedge_{i\in[n]} x_i\Big]^r \,\mathrm{d}\mathbf{x}$$

and hence the following identity

$$\frac{1}{r}\int_{[0,1]^n} \Big(\bigvee_{S\subseteq[n]} \Big[p(\mathbf{e}_S)\wedge\bigwedge_{i\in S} x_i\Big]\Big)^r \mathrm{d}\mathbf{x} = \sum_{S\subseteq[n]} \frac{1}{n-|S|+r}\int_{[0,1]^n} \Big[p(\mathbf{e}_S)\wedge\bigwedge_{i\in S} x_i\Big]^{n-|S|+r} \mathrm{d}\mathbf{x},$$

which shows that computing the raw moments of any weighted lattice polynomial reduces to computing the raw moments of bounded minima.

Let us now examine the case of the Sugeno integrals. As these integrals are usually considered over the domain $[0,1]^n$, we naturally calculate their expected values when their input variables are uniformly distributed over $[0,1]^n$. Since any Sugeno integral is a particular weighted lattice polynomial, by Eq. (8), its expected value then writes

$$\begin{aligned}\int_{[0,1]^n} \mathcal{S}_\mu(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \sum_{S\subseteq[n]} B_{\mu(S)}(n-|S|+1, |S|+1)\\ &= \sum_{S\subseteq[n]} \int_0^{\mu(S)} x^{n-|S|}(1-x)^{|S|}\,\mathrm{d}\mathbf{x}\\ &= \sum_{S\subseteq[n]} \sum_{i=0}^{|S|} \binom{|S|}{i}(-1)^i\, \frac{\mu(S)^{n-|S|+i+1}}{n-|S|+i+1}.\end{aligned}$$

Surprisingly, this expression is very close to that of the expected value of the Choquet integral with respect to the same fuzzy measure.

Let us recall the definition of the Choquet integrals [3]. Just as for the Sugeno integrals, the Choquet integrals can be expressed in various equivalent forms. We present them in one of their simplest forms (see for instance [8]).

Definition 13 *Let μ be a $[0,1]$-valued fuzzy measure on $[n]$. The* Choquet integral *of a function $\mathbf{x} : [n] \to [0,1]$ with respect to μ is defined by*

$$\mathcal{C}_\mu(\mathbf{x}) := \sum_{S\subseteq[n]} m_\mu(S) \bigwedge_{i\in S} x_i,$$

where $m_\mu : 2^{[n]} \to \mathbb{R}$ is the Möbius transform (cf. (4)) of μ.

For comparison purposes, we recall the expected value of $\mathcal{C}_\mu$ (see for instance [9]):

$$
\begin{aligned}
\int_{[0,1]^n} \mathcal{C}_\mu(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \sum_{S\subseteq[n]} \mu(S)\, B(n-|S|+1, |S|+1)\\
&= \sum_{S\subseteq[n]} \mu(S) \int_0^1 x^{n-|S|}(1-x)^{|S|}\,\mathrm{d}\mathbf{x}\\
&= \sum_{S\subseteq[n]} \mu(S)\,\frac{(n-|S|)!\,|S|!}{(n+1)!}.
\end{aligned}
$$

5 Application to reliability theory

In this final section we show how the results derived here can be applied to the reliability analysis of coherent systems. For a reference on reliability theory, see for instance Barlow and Proschan [1].

Consider a system made up of n independent components, each component C_i ($i \in [n]$) having a lifetime X_i and a reliability $r_i(t) := \Pr[X_i > t]$ at time $t > 0$. Additional components, with constant lifetimes, may also be considered.

We assume that, when components are connected in series, the lifetime of the subsystem they form is simply given by the minimum of the component lifetimes. Similarly, for a parallel connection, the subsystem lifetime is the maximum of the component lifetimes.

It follows immediately that, for a system mixing series and parallel connections, the system lifetime is given by a weighted lattice polynomial function

$$
Y_p = p(X_1, \ldots, X_n)
$$

of the component lifetimes. Our results then provide explicit formulas for the c.d.f., the expected value, and the moments of the system lifetime.

For example, the system reliability at time $t > 0$ is given by

$$
R_p(t) := \Pr[Y_p > t] = \Phi_{v_{p,t}}[r_1(t), \ldots, r_n(t)]. \tag{9}
$$

Moreover, for any measurable function $g : [0, \infty] \to \overline{\mathbb{R}}$ of bounded variation and such that $\lim_{y\to\infty} g(y) r_i(y) = 0$ for all $i \in [n]$, we have, by Proposition 9,

$$
\mathbf{E}[g(Y_p)] = g(0) + \int_0^\infty R_p(t)\,\mathrm{d}g(t).
$$

As an example, the following proposition yields the *mean time to failure* $\mathbf{E}[Y_p]$ in the special case of the exponential reliability model.

Proposition 14 *If* $r_i(t) = e^{-\lambda_i t}$ *for all* $i \in [n]$*, we have*

$$\mathbf{E}[Y_p] = p(\mathbf{e}_\varnothing) + \sum_{\substack{S\subseteq[n]\\ S\neq\varnothing}} \sum_{T\subseteq S} (-1)^{|S|-|T|} \, \frac{1 - e^{-\lambda(S)\, p(\mathbf{e}_T)}}{\lambda(S)} \, ,$$

where $\lambda(S) := \sum_{i\in S} \lambda_i$.

Proof. Using (9) and then (3)–(4), we obtain

$$R_p(t) = \sum_{S\subseteq[n]} m_{v_{p,t}}(S)\, e^{-\lambda(S)t} = \sum_{S\subseteq[n]} \sum_{T\subseteq S} (-1)^{|S|-|T|}\, v_{p,t}(T)\, e^{-\lambda(S)t}$$

and hence

$$\mathbf{E}[Y_p] = \int_0^\infty R_p(t)\, \mathrm{d}t = \sum_{S\subseteq[n]} \sum_{T\subseteq S} (-1)^{|S|-|T|} \int_0^{p(\mathbf{e}_T)} e^{-\lambda(S)t}\, \mathrm{d}t. \qquad \square$$

6 Conclusion

We have extended the c.d.f.'s and moments of lattice polynomial functions to the weighted case. At first glance, this extension may appear as a simple exercise. However, it led us to nontrivial formulas, which can be directly applied to qualitative aggregation functions such as the Sugeno integrals and their particular cases: the weighted minima, the weighted maxima, and their ordered versions.

Appendix: Alternative proof of Eq. (7)

In this appendix we present a proof of Eq. (7) without using the c.d.f. of Y_p. The main idea of this proof is based on the fact that the expected value $\mathbf{E}[g(Y_p)]$ can be expressed as an n-dimensional integral, namely

$$\mathbf{E}[g(Y_p)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g[p(\mathbf{x})]\, \mathrm{d}F_1(x_1) \cdots \mathrm{d}F_n(x_n). \tag{10}$$

At first glance, the evaluation of this expression requires a difficult or intractable integration. However, as we will now see, any weighted lattice polynomial function fulfills a remarkable decomposition formula, which will enable us to calculate $\mathbf{E}[g(Y_p)]$ in a straightforward manner.

Given a weighted lattice polynomial function $p : L^n \to L$ and an index $k \in [n]$, define the weighted lattice polynomial functions $p_k^a : L^n \to L$ and $p_k^b : L^n \to L$ as

$$\begin{aligned} p_k^a(\mathbf{x}) &:= p(x_1, \ldots, x_{k-1}, a, x_{k+1}, \ldots, x_n), \\ p_k^b(\mathbf{x}) &:= p(x_1, \ldots, x_{k-1}, b, x_{k+1}, \ldots, x_n). \end{aligned}$$

Then, it can be shown [7] that

$$p(\mathbf{x}) = \text{median}\Big[p_k^a(\mathbf{x}), x_k, p_k^b(\mathbf{x})\Big] \qquad (\mathbf{x} \in L^n). \tag{11}$$

This decomposition formula expresses that, for any index k, the variable x_k can be totally isolated in $p(\mathbf{x})$ by means of a median calculated over the variable x_k and the two functions p_k^a and p_k^b, which are independent of x_k.

This interesting property leads to the following lemma.

Lemma 15 *Let $p : L^n \to L$ be any weighted lattice polynomial function, let $k \in [n]$, and let $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be any measurable function of bounded variation and such that $g(-\infty)$ is finite and $\lim_{y\to\infty} g(y)[1 - F_k(y)] = 0$. Then*

$$\mathbf{E}[g(Y_p)] = \mathbf{E}[g_a(Y_{p_k^a})] + \mathbf{E}[g_b(Y_{p_k^b})], \tag{12}$$

where

$$\begin{aligned} g_a(x) &:= \int_{-\infty}^{x} F_k(t)\,\mathrm{d}g(t), \\ g_b(x) &:= g(x) - \int_{-\infty}^{x} F_k(t)\,\mathrm{d}g(t). \end{aligned}$$

Proof. Let $k \in [n]$ and fix x_j for all $j \neq k$. Assume $u := p_k^a(\mathbf{x})$ and $v := p_k^b(\mathbf{x})$ are finite. The other cases can be dealt with similarly. Then, we have $u \leqslant v$ and, by (11),

$$p(\mathbf{x}) = \text{median}[u, x_k, v].$$

Hence, for any measurable function $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ of bounded variation, we have

$$\begin{aligned} \int_{-\infty}^{\infty} g[p(\mathbf{x})]\,\mathrm{d}F_k(x_k) &= \int_{-\infty}^{u} g(u)\,\mathrm{d}F_k(x_k) + \int_{u}^{v} g(x_k)\,\mathrm{d}F_k(x_k) + \int_{v}^{\infty} g(v)\,\mathrm{d}F_k(x_k) \\ &= g(v) - \int_{u}^{v} F_k(t)\,\mathrm{d}g(t) \\ &= g_a(u) + g_b(v). \end{aligned}$$

Finally, integrating with respect to the other variables x_j, $j \neq k$, and then using (10), we get the result. $\square$

Observe that formula (12), when considered for every index k and every function g, completely determines the expected value of $g(Y_p)$. Indeed, repeated applications of that formula will eventually lead to integration of transformed weighted lattice polynomial functions all of whose variables are set to either a or b.

Proof of Eq. (7). For any fixed $S \subseteq [n]$, define recursively the sequence $\{g_S^k\}_{k=0}^n$ of functions $g_S^k : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ as $g_S^0 := g$ and, for $k \geqslant 1$,

$$g_S^k(x) := \begin{cases} \int_{-\infty}^{x} F_k(t)\, \mathrm{d}g_S^{k-1}(t), & \text{if } k \notin S, \\ g_S^{k-1}(x) - \int_{-\infty}^{x} F_k(t)\, \mathrm{d}g_S^{k-1}(t), & \text{if } k \in S. \end{cases}$$

Repeated applications of Lemma 15 eventually lead to

$$\mathbf{E}[g(Y_p)] = \sum_{S \subseteq [n]} g_S^n[p(\mathbf{e}_S)].$$

Let us now show that

$$g_S^n(z) = g_S^n(-\infty) + \int_{-\infty}^{z} \prod_{k \in [n] \setminus S} F_k(x) \prod_{k \in S} [1 - F_k(x)]\, \mathrm{d}g(x).$$

For any $S \subseteq [n]$ and any $k \in [n]$, we have

$$\mathrm{d}g_S^k(x) = \begin{cases} F_k(x)\, \mathrm{d}g_S^{k-1}(x), & \text{if } k \notin S, \\ [1 - F_k(x)]\, \mathrm{d}g_S^{k-1}(x), & \text{if } k \in S, \end{cases}$$

and hence

$$\mathrm{d}g_S^n(x) = \prod_{k \in [n] \setminus S} F_k(x) \prod_{k \in S} [1 - F_k(x)]\, \mathrm{d}g(x),$$

which proves the result since $g_S^n(-\infty) = g(-\infty)$, if $S = N$, and 0, otherwise. $\square$

References

[1] R. Barlow and F. Proschan. *Statistical theory of reliability and life testing.* To Begin With, Silver Spring, MD, 1981.

[2] G. Birkhoff. *Lattice theory.* Third edition. American Mathematical Society Colloquium Publications, Vol. XXV. American Mathematical Society, Providence, R.I., 1967.

[3] G. Choquet. Theory of capacities. *Ann. Inst. Fourier, Grenoble*, 5:131–295 (1955), 1953–1954.

[4] H. David and H. Nagaraja. *Order statistics. 3rd ed.* Wiley Series in Probability and Statistics. Chichester: John Wiley & Sons., 2003.

[5] A. Dukhovny. Lattice polynomials of random variables. *Statistics & Probability Letters*, 77(10):989–994, 2007.

[6] M. Grabisch, T. Murofushi, and M. Sugeno, editors. *Fuzzy measures and integrals*, volume 40 of *Studies in Fuzziness and Soft Computing.* Physica-Verlag, Heidelberg, 2000.

[7] J.-L. Marichal. Weighted lattice polynomials. *Discrete Mathematics*, submitted for revision. `http://arxiv.org/abs/0706.0570`

[8] J.-L. Marichal. Aggregation of interacting criteria by means of the discrete Choquet integral. In *Aggregation operators: new trends and applications*, pages 224–244. Physica, Heidelberg, 2002.

[9] J.-L. Marichal. Tolerant or intolerant character of interacting criteria in aggregation by the Choquet integral. *European J. Oper. Res.*, 155(3):771–791, 2004.

[10] J.-L. Marichal. Cumulative distribution functions and moments of lattice polynomials. *Statistics & Probability Letters*, 76(12):1273–1279, 2006.

[11] G. Owen. Multilinear extensions of games. *Management Sci.*, 18:P64–P79, 1972.

[12] M. Sugeno. *Theory of fuzzy integrals and its applications.* PhD thesis, Tokyo Institute of Technology, Tokyo, 1974.

[13] M. Sugeno. Fuzzy measures and fuzzy integrals—a survey. In *Fuzzy automata and decision processes*, pages 89–102. North-Holland, New York, 1977.