Error estimation in homogenisation

Strobl, 27th of January, 2015

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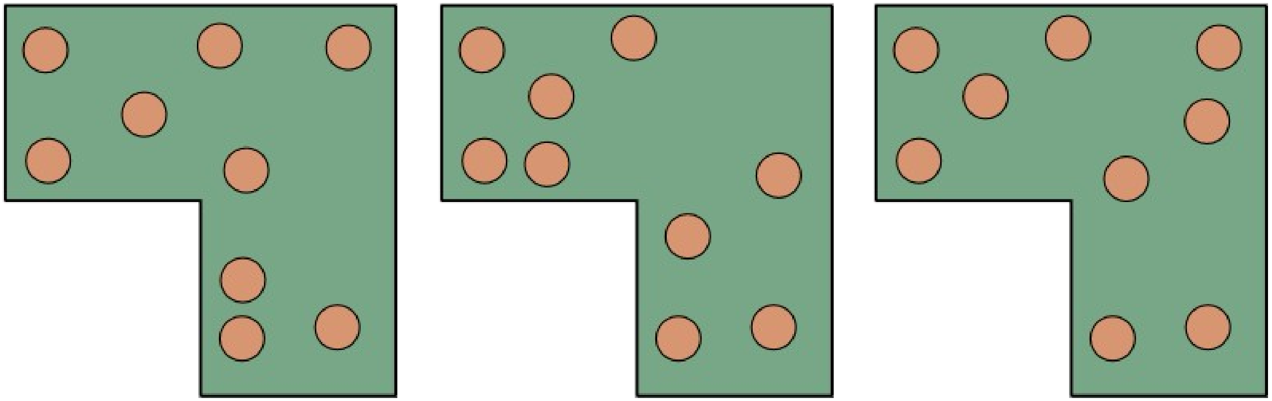
Motivation

Problem: Analysis of an heterogeneous materials. Vague information available. The position of the particles is not available.



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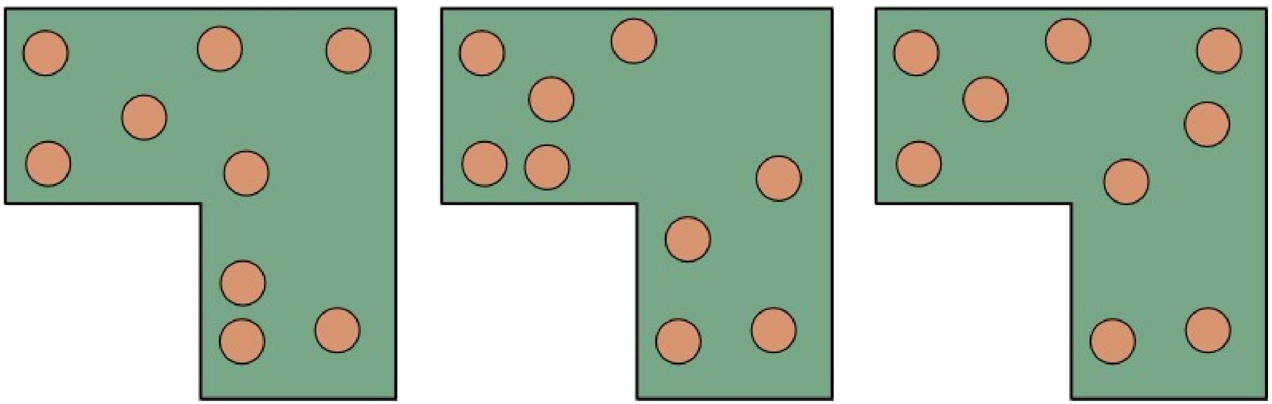

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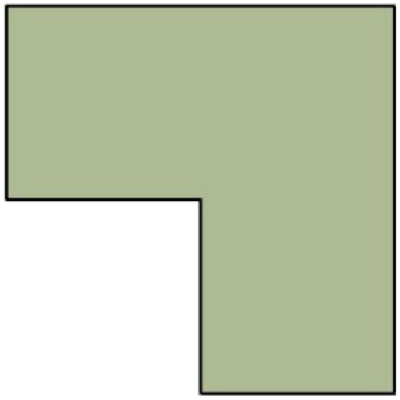

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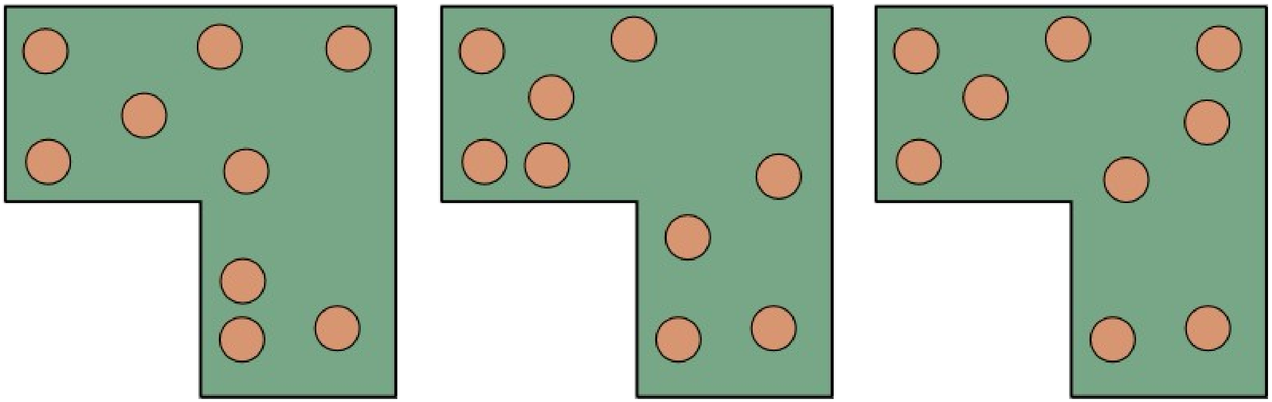

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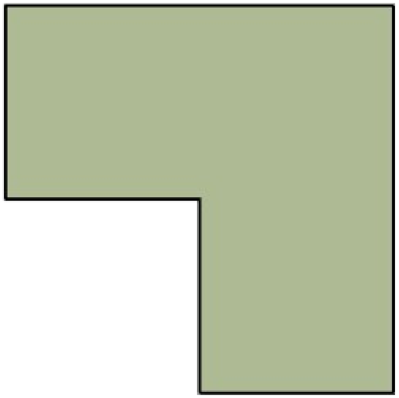
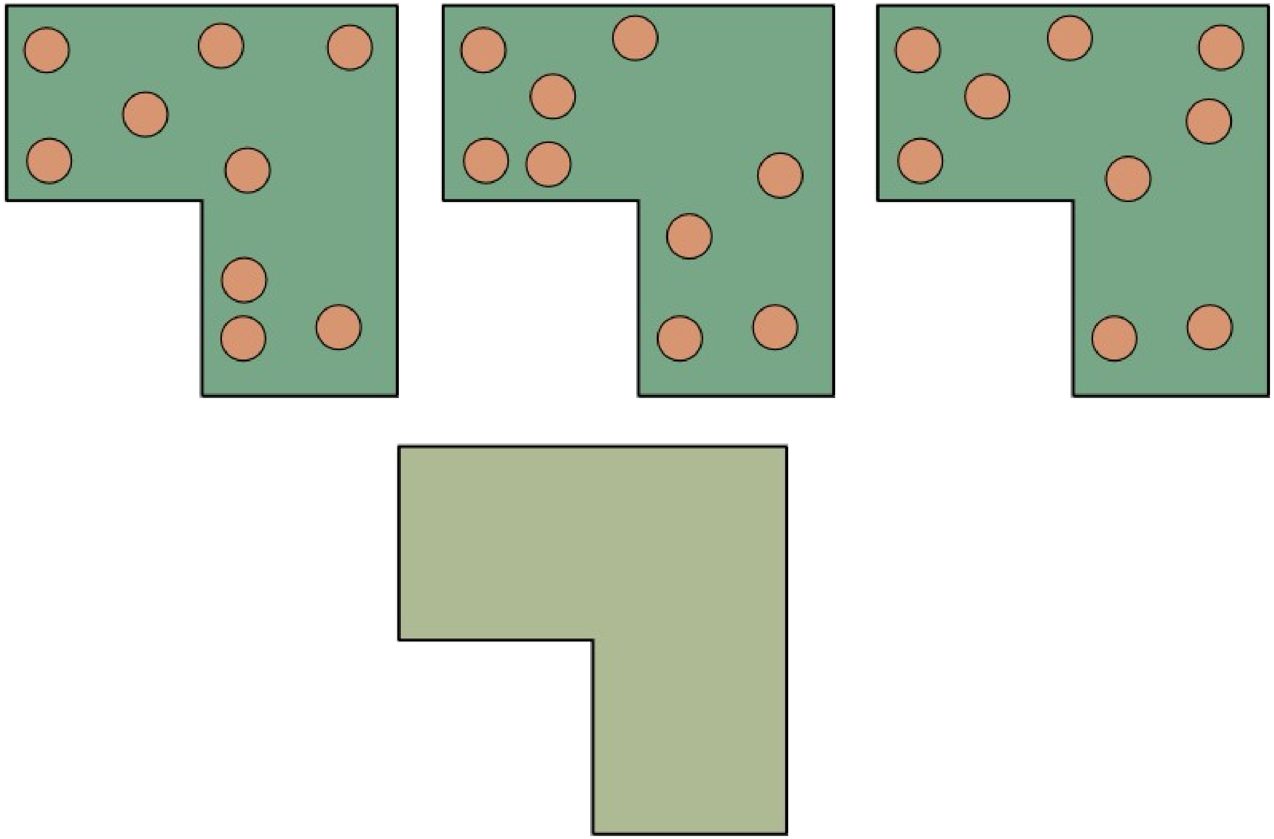


New problem:
Assess the validity of the homogenisation.

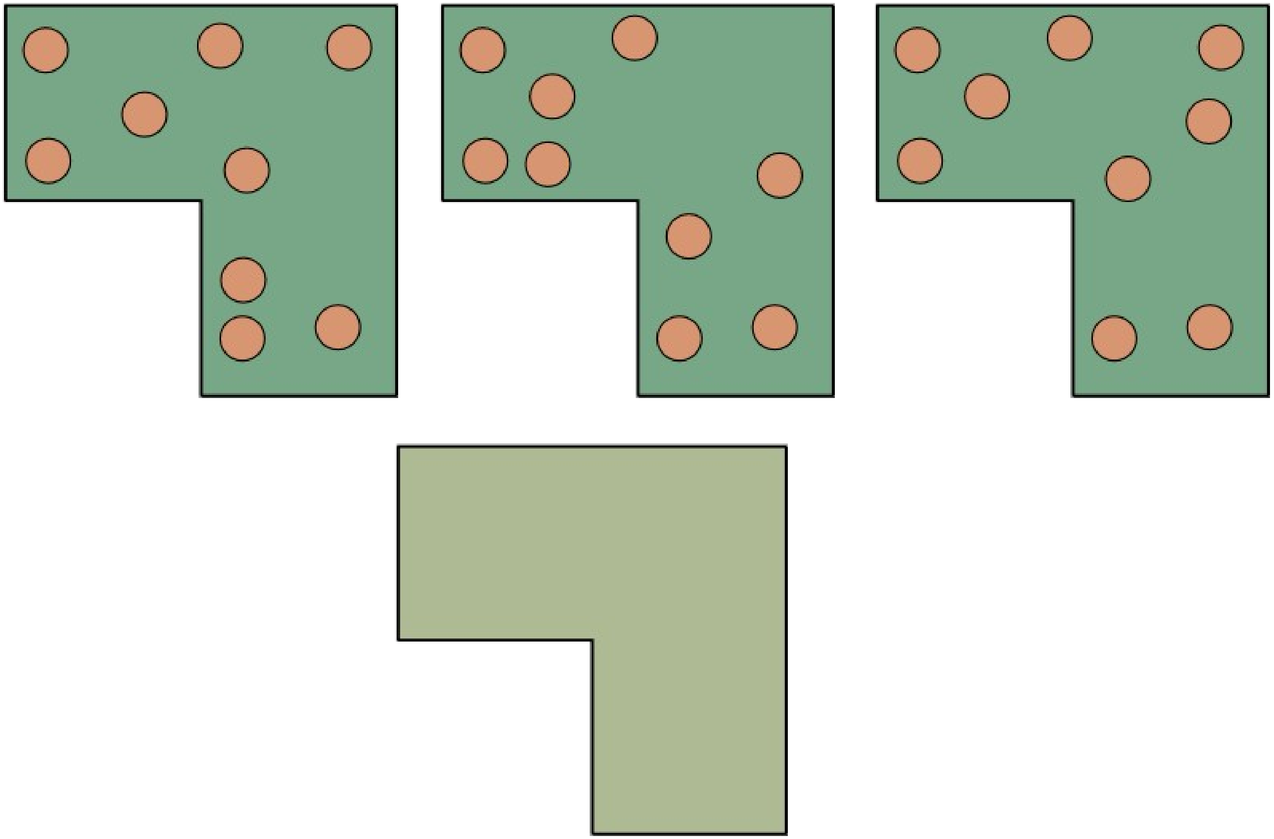
Proposed solution

Idea: Understand the original problem as an SPDE (the center of particles is a random variable) and bound the distance between both models



Proposed solution

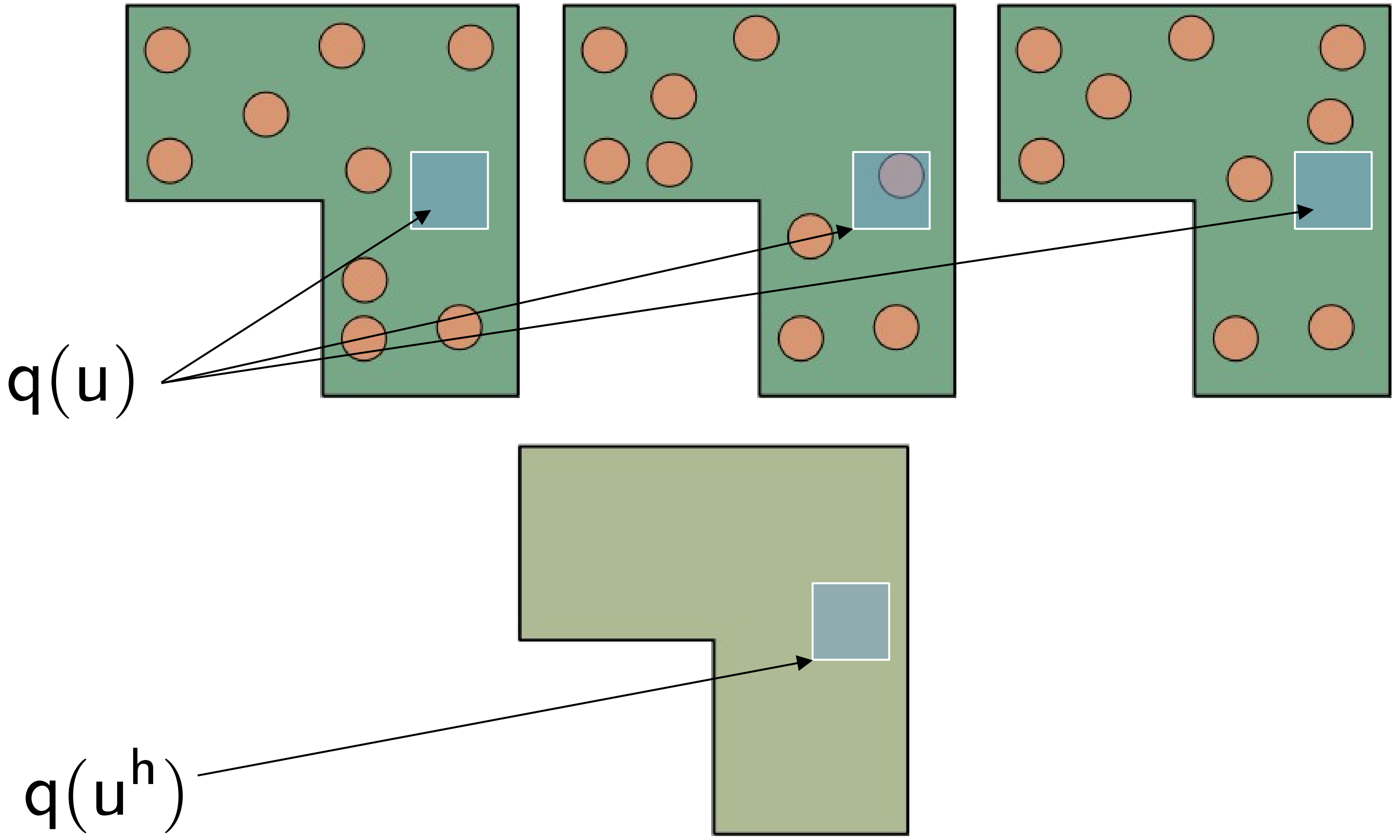
SPDE: Stochastic partial differential equation.
Collection of parametric problems + probability density function



Proposed solution

QoI: Quantity of interest. The output. Scalar that depends of the solution.

$$q(u) = \int_\Omega \int_\Theta \gamma(x) \cdot u(x, \theta) \quad \text{(linear)}$$



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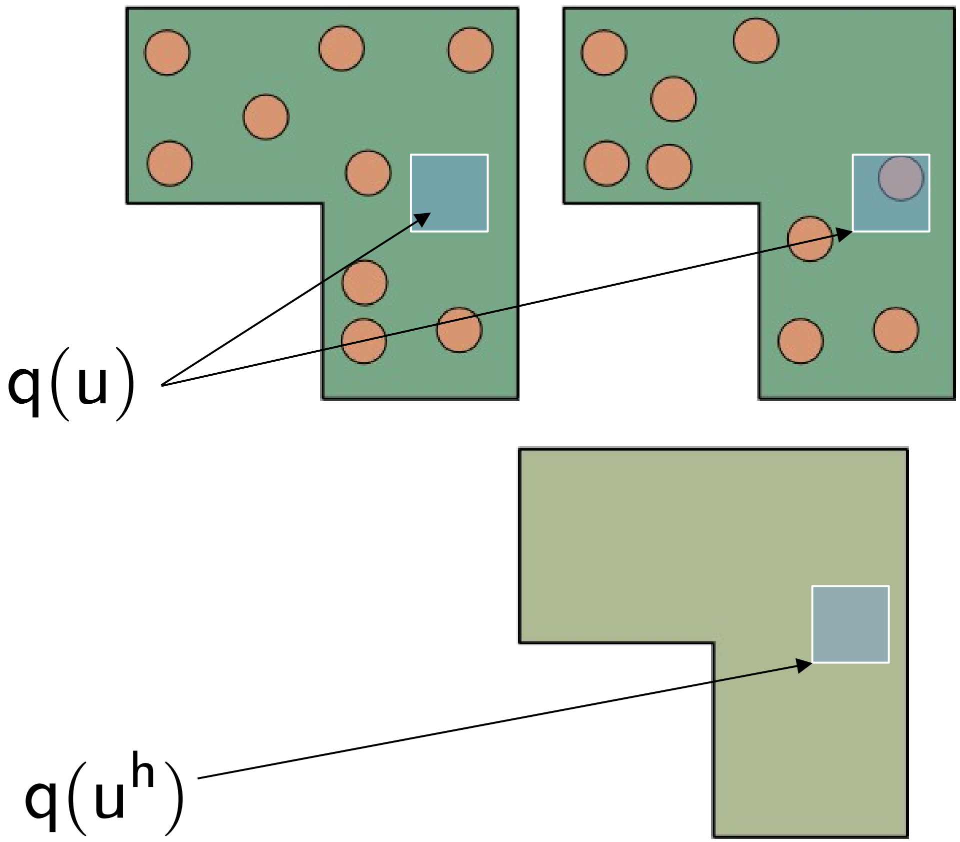

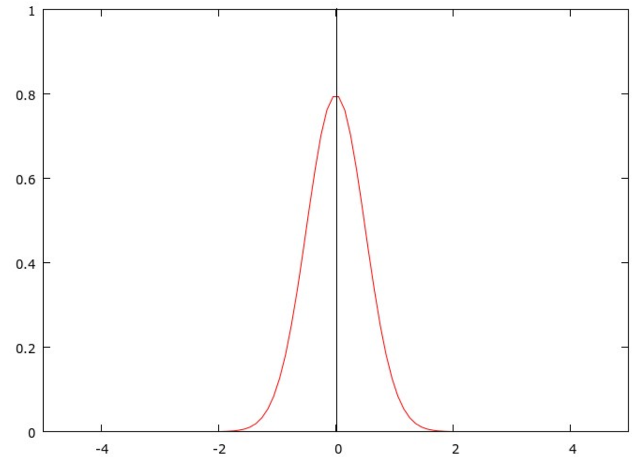

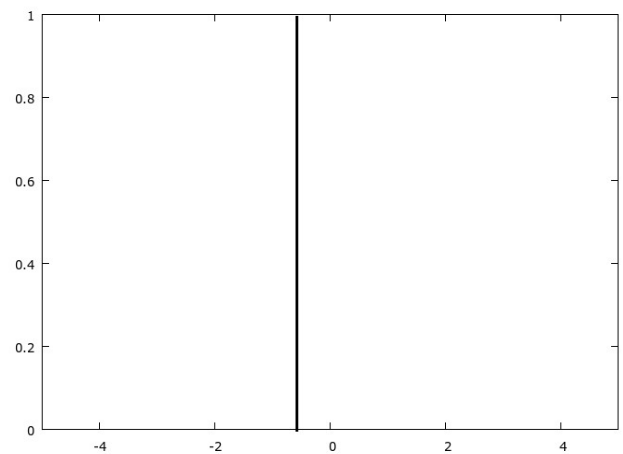

Heat equation

Heterogeneous problem

$$a(u,v) = \int_\Omega \int_\Theta k \nabla u \cdot \nabla v$$

$$l(v) = \int_\Omega \int_\Theta fv - \int_{\partial\Omega} gv$$

$$a(u,v) = l(v) \quad \forall v \in V$$

Problem statement

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Homogeneous problem

$$a_0(u_0, v) = \int_\Omega k_0 \nabla u \cdot \nabla v$$

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$$a_0(u^h, v) = l(v) \quad \forall v \in V_0^h \subseteq V_0$$

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Aim: Bound

$$q(u) - q(u^h)$$

The computation of the bound must be deterministic.

Derivation

Hypothesis

Deterministic boundary conditions

Derivation

Hypothesis

Deterministic boundary conditions

Constant volume fraction

$$\int_\Omega \mathsf{k}(\mathsf{x}, \theta) = \mathsf{v_f} \mathsf{k_I} + (1 - \mathsf{v_f}) \mathsf{k_M} \quad \forall \theta \in \Theta$$

Derivation

Hypothesis

Deterministic boundary conditions

Constant volume fraction

$$\int_{\Omega} k(x,\theta) = v_f k_I + (1 - v_f) k_M \quad \forall \theta \in \Theta$$

Constant PDF over the domain

$$\underbrace{\int_{\Theta} k(x,\theta)}_{E[k]} = v_f k_I + (1 - v_f) k_M \quad \forall x \in \Omega$$

“Flux” FE

The unknown is the flux field and

$$\nabla \cdot \hat{Q} = f \quad x \in \Omega$$

$$\hat{Q} \cdot n = g \quad x \in \partial\Omega_N$$

are fulfilled strongly.

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In order to derive bounds, we will also need to use an homogenised “flux” FE solution $\hat{Q}$

Error in the energy norm

Rewriting the problem in terms of the flux and the temperature

$$\begin{aligned}
\nabla \cdot Q &= f \quad \forall x \in \Omega \times \Theta \\
Q \cdot n &= g \quad \forall x \in \partial\Omega_N \times \Theta \\
u &= h \quad \forall x \in \partial\Omega_D \times \Theta \\
Q + k\nabla u &= 0 \quad \forall x \in \Omega \times \Theta
\end{aligned}$$

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$\hat{Q}$ will fulfill exactly the first 2 equations.

u^h will fulfill exactly the 3^{rd} equation.

In general, $\hat{Q} + k\nabla u^h \neq 0$ Discrepancy = measure of the error

Error in the energy norm

Formalizing this idea, it can be shown that

$$\|u - u^h\| = \|e\| \leq \|\hat{Q} + k\nabla u^h\|_{k^{-1}} =: \eta$$

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Expanding η^2

$$\eta^2 = \int_\Omega \int_\Theta k^{-1}\hat{Q} \cdot \hat{Q} + \int_\Omega \int_\Theta k\nabla u^h \cdot \nabla u^h + 2\int_\Omega \int_\Theta \hat{Q} \cdot \nabla u^h$$

$$= E[k^{-1}] \int_\Omega \hat{Q} \cdot \hat{Q} + E[k] \int_\Omega \nabla u^h \cdot \nabla u^h + 2\int_\Omega \hat{Q} \cdot \nabla u^h$$

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Deterministic quantity

Error in the quantity of interest

The error in energy norm is not always relevant.

Solution: Bound for the quantity of interest $q(u)$

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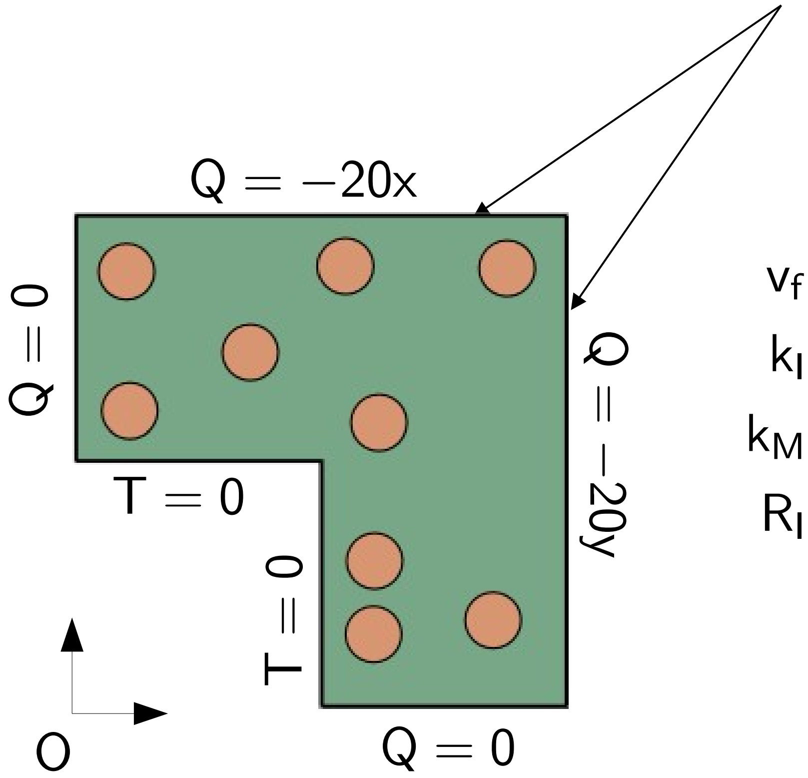
Use the bound in the energy norm,

$$R(\phi^h) - \eta\eta_\phi \leq q(u) - q(u^h) \leq R(\phi^h) + \eta\eta_\phi$$

Validation

The quantity of the interest is the average temperature in the exterior faces.



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$k_I = 0.5$

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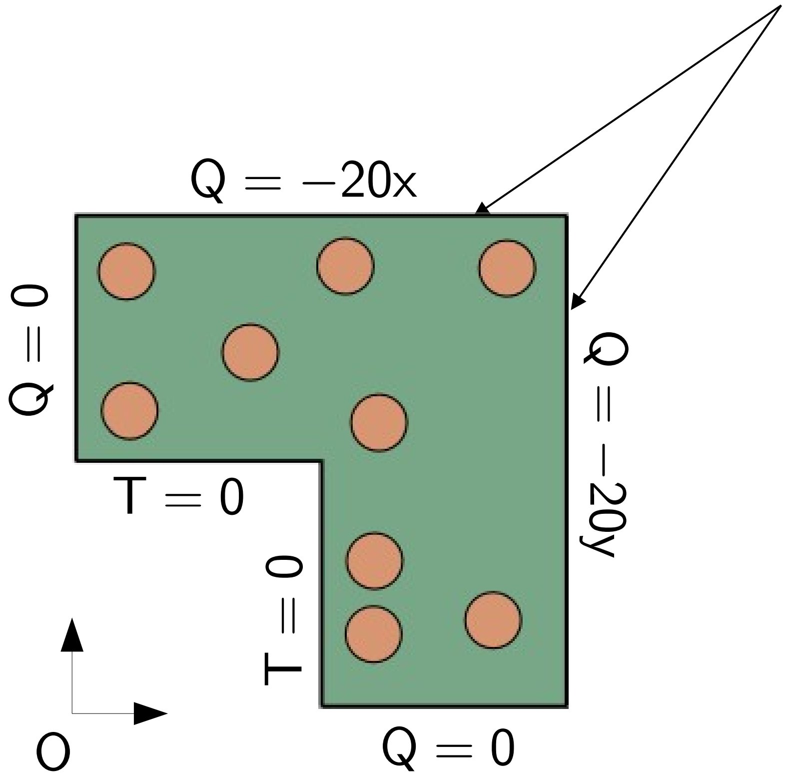

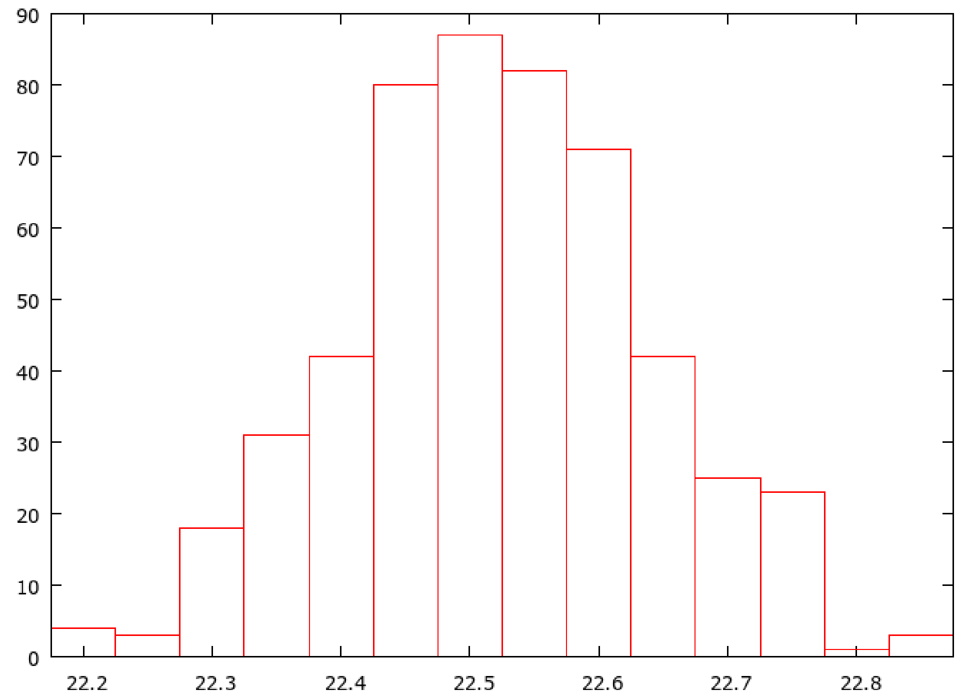

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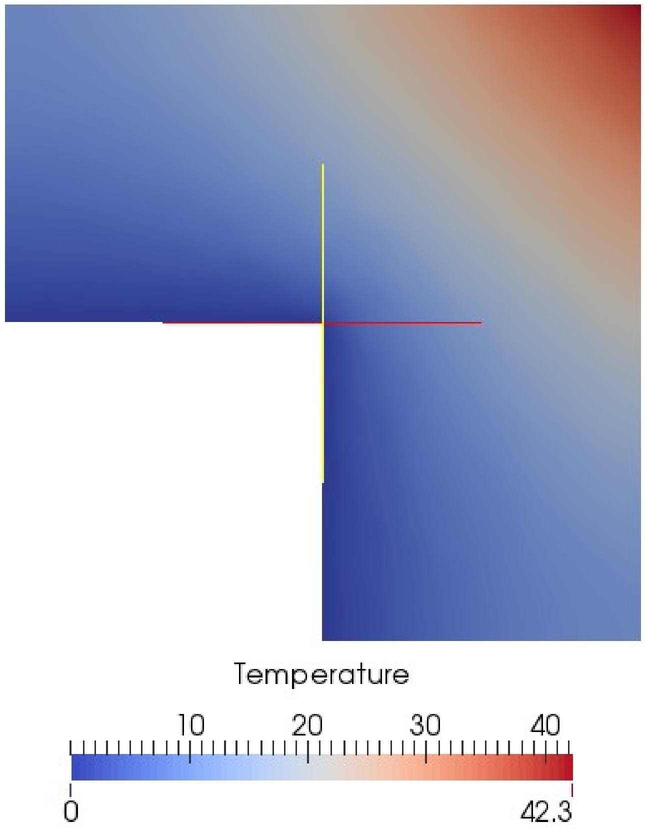

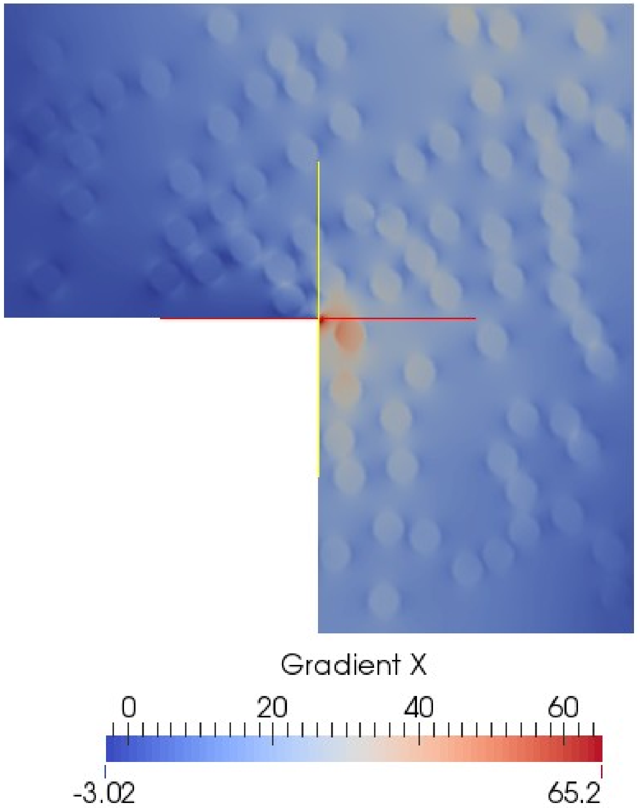

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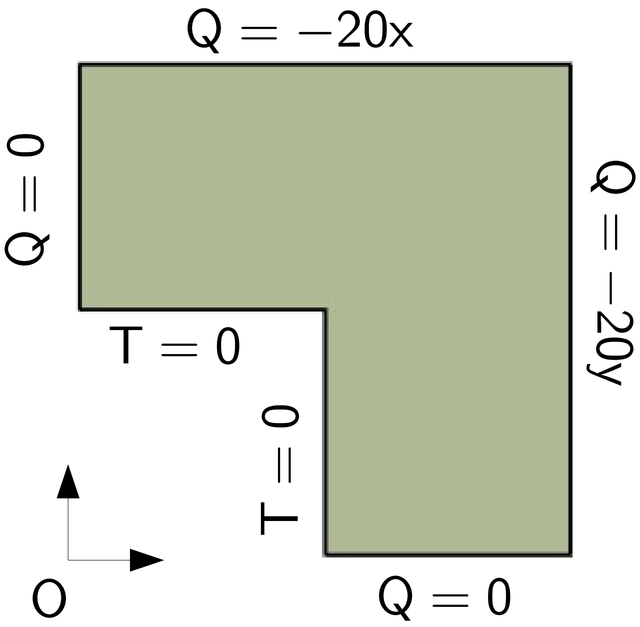

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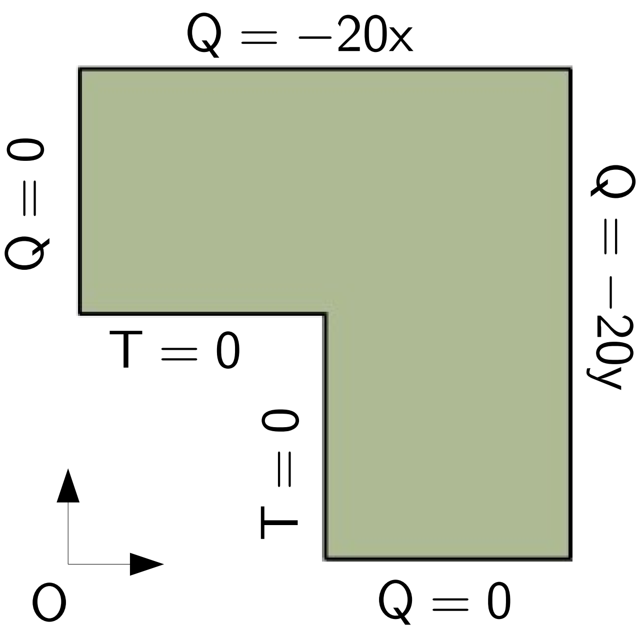

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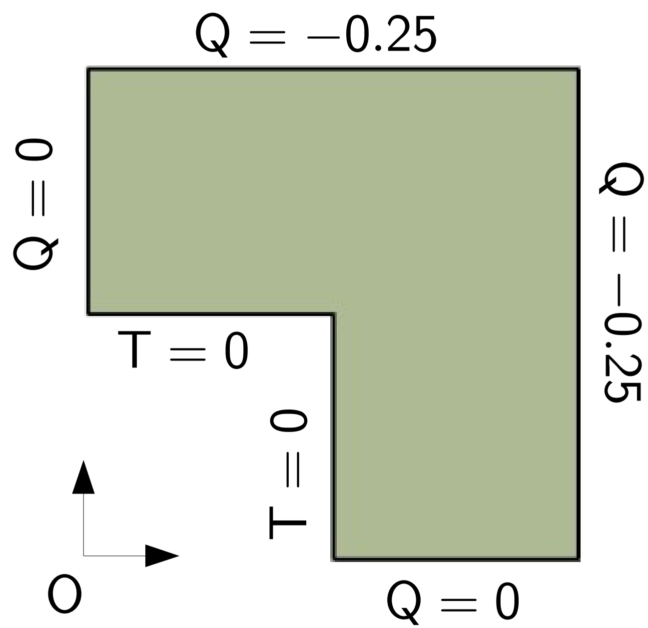

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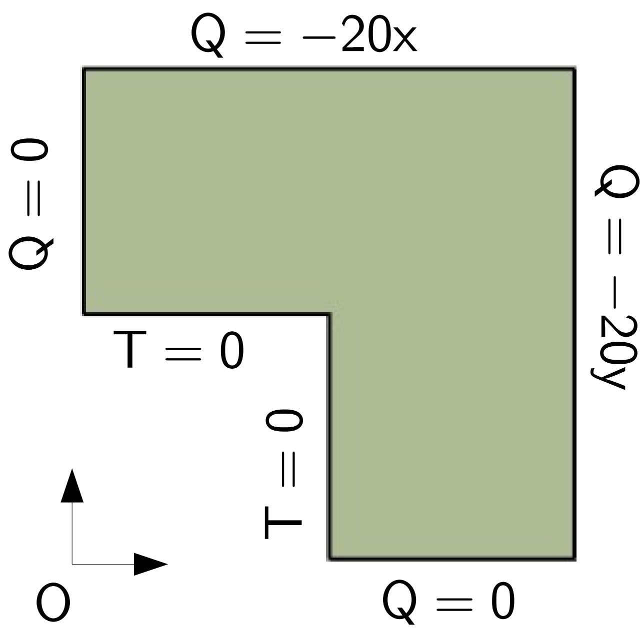

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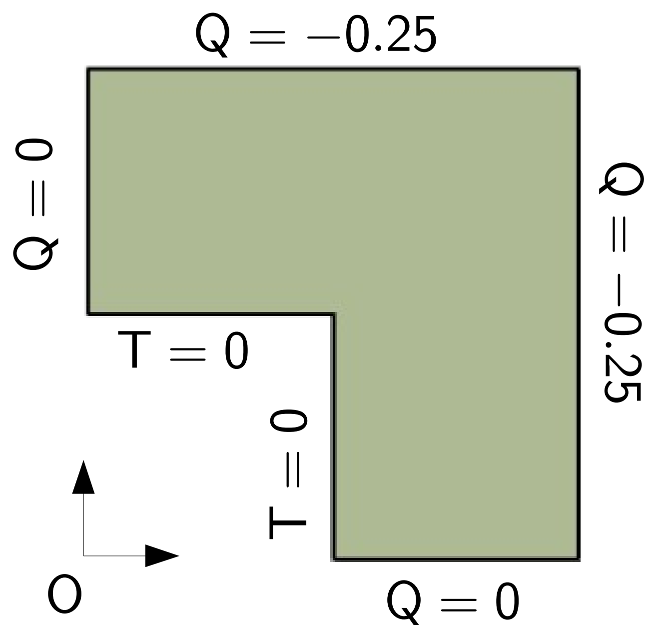

Two problems solved twice:

- Using "temperature" FE u^h, ϕ^h
- Using "flux" FE $\hat{Q}, \hat{Q}_\phi$

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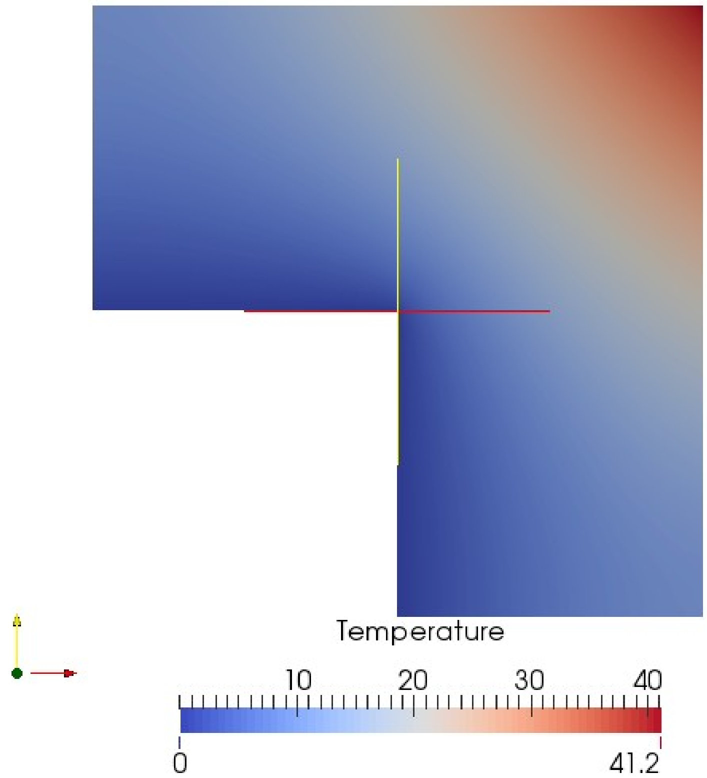

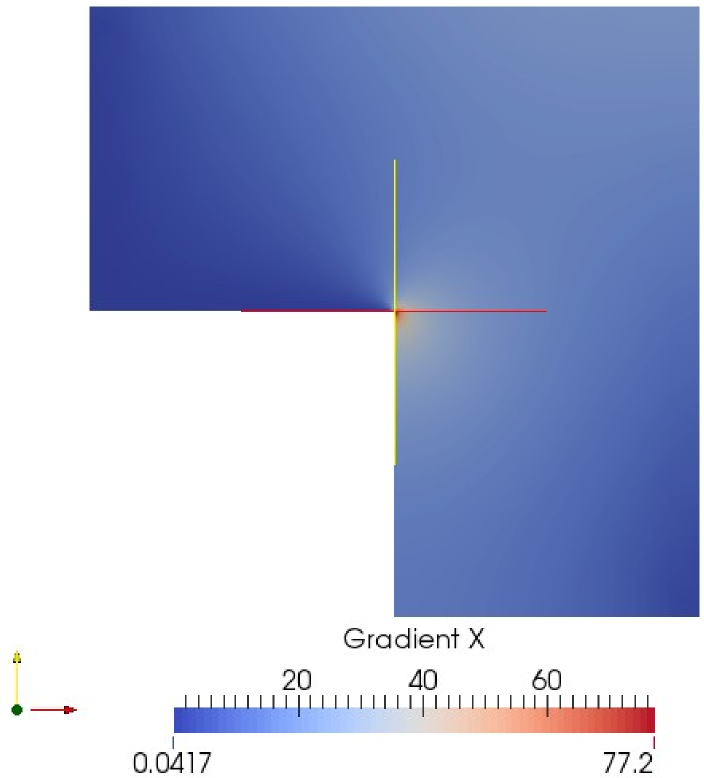

$q(u^h)$	$\lvert q(u) - q(u^h) \rvert$	$\leq \zeta_u$	$\zeta_l + q(u^h) \leq$	$q(u)$	$\leq \zeta_u + q(u^h)$
21.92	0.63	1.84	20.08	22.55	23.76

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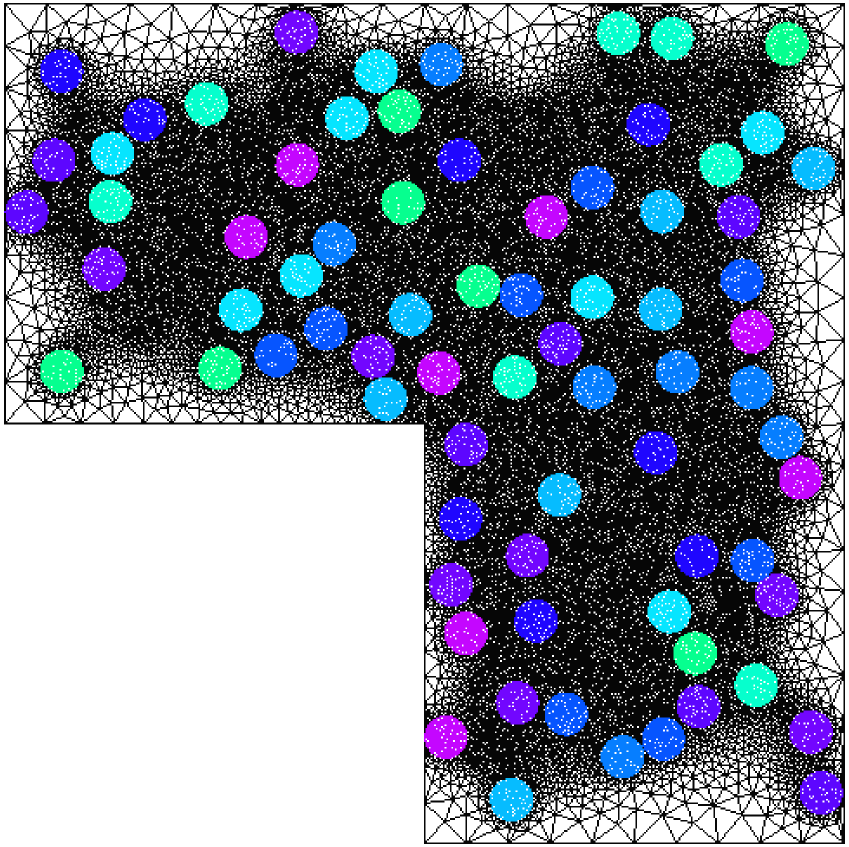
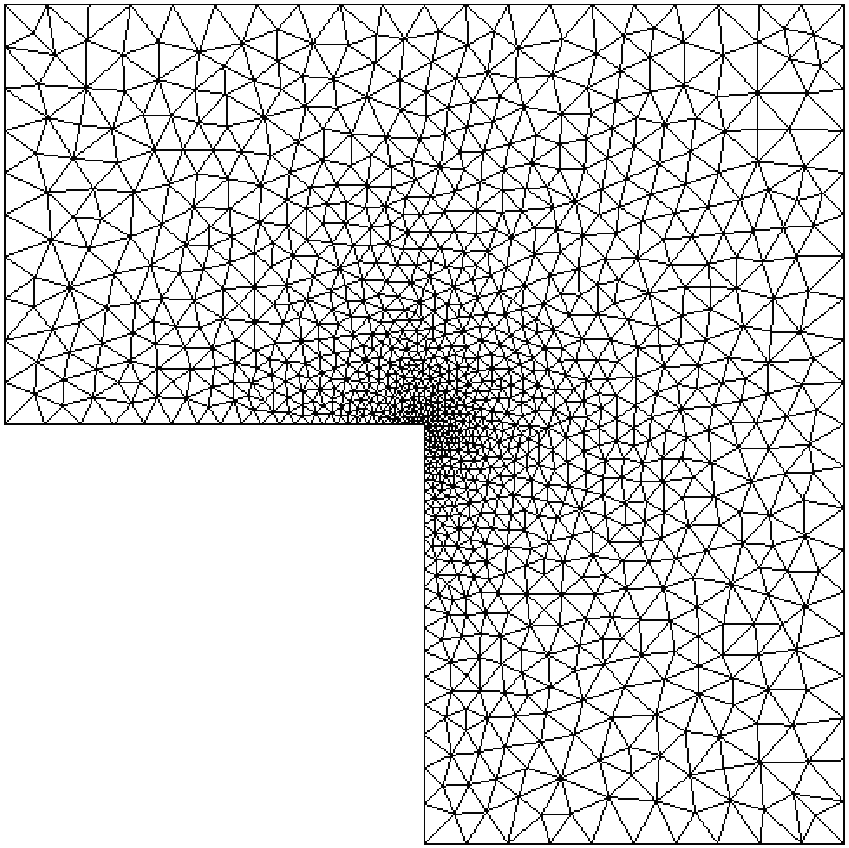


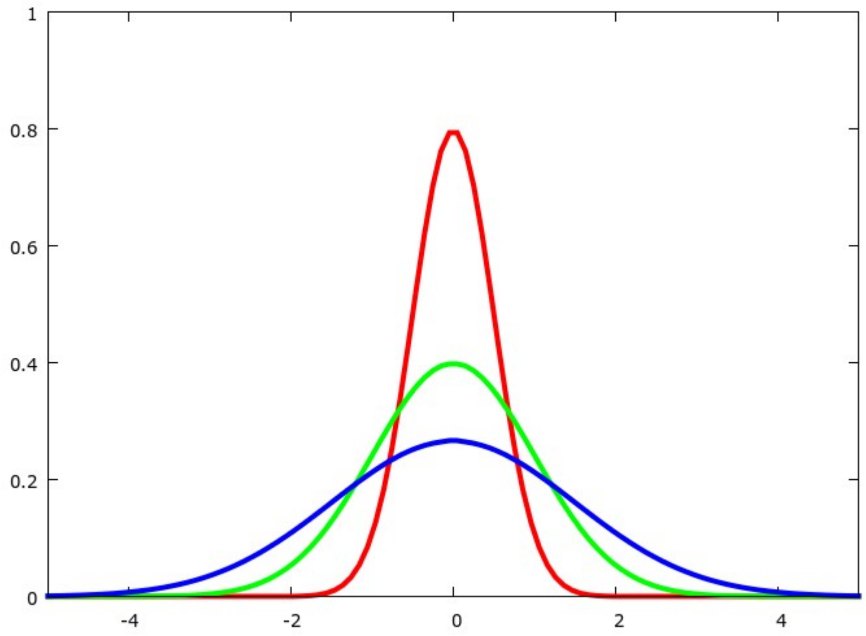


Around 60 000 elements
512 problems, 512 different meshes
Full PDF

Around 2000 elements
4 problems, 1 mesh
Bounds on the expectation

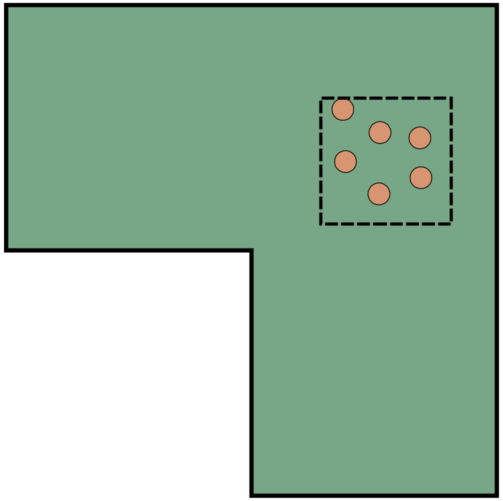
Work in progress

– A bound on the variance.



Work in progress

– Enhanced model. Insert patches with particles in parts of the domain.



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Thank you for your attention.