The splitting problem for complex homogeneous supermanifolds

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Abstract

It is a classical result that any complex analytic Lie supergroup \( \mathcal{G} \) is split [5], that is its structure sheaf is isomorphic to the structure sheaf of a certain vector bundle. However, there do exist non-split complex analytic homogeneous supermanifolds.

We study the question how to find out whether a complex analytic homogeneous supermanifold is split or non-split. Our main result is a description of left invariant gradings on a complex analytic homogeneous supermanifold \( \mathcal{G}/\mathcal{H} \) in the terms of \( \mathcal{H} \)-invariants. As a corollary to our investigations we get some simple sufficient conditions for a complex analytic homogeneous supermanifold to be split in terms of Lie algebras.

1 Introduction

A supermanifold is called split if its structure sheaf is isomorphic to the exterior power of a certain vector bundle. By Batchelor’s Theorem any real supermanifold is non-canonically split. However, this is false in the complex analytic case. The property of a supermanifold to be split is very important for several reasons. For instance, in [2] it was shown that the moduli space of super Riemann surfaces is not projected (and in particular is not split) for genus \( g \geq 5 \). The physical meaning of this result is that [2]: ”certain approaches to superstring perturbation theory that are very powerful in low orders have no close analog in higher orders”. Another problem, when the property of a supermanifold to be split is very important, is the calculation of the cohomology group with values in a vector bundle over a supermanifold. In the split case we may use the well understood tools of complex analytic geometry. In the general case, several methods were suggested by Onishchik’s school: spectral sequences, see e.g. [12]. All these methods connect the cohomology group with values in a vector bundle with the cohomology group with values in the corresponding split vector bundle.

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How do we determine whether a complex analytic supermanifold is split or non-split? Let me describe here some results in this direction that were obtained by Green, Koszul, Onishchik and Serov. In [3] Green described a moduli space with a marked point such that any non-marked point corresponds to a non-split supermanifold while the marked point corresponds to a split one. His idea was used for instance in [2]. The calculation of the Green moduli space is a difficult problem itself, and in many cases the method is difficult to apply. Furthermore, Onishchik and Serov [9, 10, 11] considered grading derivations, which correspond to \( \mathbb{Z} \)-gradings of the structure sheaf of a supermanifold. For example, it was shown that almost all supergrassmannians do not possess such derivations, i.e. their structure sheaves do not possess any \( \mathbb{Z} \)-gradings. Hence, in particular, they are non-split. The idea of grading derivations was independently used by Koszul. In [4] the following statement was proved: if the tangent bundle of a supermanifold \( \mathcal{M} \) possesses a (holomorphic) connection then \( \mathcal{M} \) is split. (Koszul’s proof works in real and complex analytic cases.) In fact, it was shown that we can assign a grading derivation to any supermanifold with a connection and that this grading derivation is induced by a \( \mathbb{Z} \)-grading of a vector bundle.

Assume that a complex analytic supermanifold \( \mathcal{M} = (\mathcal{M}_0, \mathcal{O}_\mathcal{M}) \) is split. By definition this means that its structure sheaf \( \mathcal{O}_\mathcal{M} \) is isomorphic to \( \wedge E \), where \( E \) is a locally free sheaf on the complex analytic manifold \( \mathcal{M}_0 \). The sheaf \( \wedge E \) is naturally \( \mathbb{Z} \)-graded and the isomorphism \( \mathcal{O}_\mathcal{M} \cong \wedge E \) induces the \( \mathbb{Z} \)-grading in \( \mathcal{O}_\mathcal{M} \). We call such gradings split. The main result of our paper is a description of those left invariant split gradings on a homogeneous superspace \( \mathcal{G}/\mathcal{H} \) which are compatible with split gradings on \( \mathcal{G} \). We also give sufficient conditions for pairs \( (\mathfrak{g}, \mathfrak{h}) \), where \( \mathfrak{g} = \text{Lie} \mathcal{G} \) and \( \mathfrak{h} = \text{Lie} \mathcal{H} \), such that \( \mathcal{G}/\mathcal{H} \) is split.

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2 Complex analytic supermanifolds. Main definitions.

We will use the word ”supermanifold” in the sense of Berezin and Leites, see [1], [7] and [8] for details. Throughout, we will be interested in the complex analytic version of the theory. Recall that a complex analytic superdomain
of dimension \(n|m\) is a \(\mathbb{Z}_2\)-graded ringed space

\[
U = \left(U, F_U \otimes \bigwedge (m)\right),
\]

where \(F_U\) is the sheaf of holomorphic functions on an open set \(U \subset \mathbb{C}^n\) and \(\bigwedge (m)\) is the exterior (or Grassmann) algebra with \(m\) generators. A complex analytic supermanifold of dimension \(n|m\) is a \(\mathbb{Z}_2\)-graded ringed space that is locally isomorphic to a complex superdomain of dimension \(n|m\).

Let \(\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_M)\) be a complex analytic supermanifold and

\[
\mathcal{J}_\mathcal{M} = (\mathcal{O}_M)_{\bar{1}} + (\mathcal{O}_M)_{\bar{2}}^2
\]

be the subsheaf of ideals generated by odd elements in \(\mathcal{O}_M\). We put \(\mathcal{F}_\mathcal{M} := \mathcal{O}_M/\mathcal{J}_\mathcal{M}\). Then \((\mathcal{M}_0, \mathcal{F}_\mathcal{M})\) is a usual complex analytic manifold. It is called the reduction or underlying space of \(\mathcal{M}\). We will write \(\mathcal{M}_0\) instead of \((\mathcal{M}_0, \mathcal{F}_\mathcal{M})\) for simplicity of notation. Morphisms of supermanifolds are just morphisms of the corresponding \(\mathbb{Z}_2\)-graded ringed spaces. If \(f : \mathcal{M} \to \mathcal{N}\) is a morphism of supermanifolds, then we denote by \(f_0\) the morphism of the underlying spaces \(\mathcal{M}_0 \to \mathcal{N}_0\) and by \(f^*\) the morphism of the structure sheaves \(\mathcal{O}_N \to (f_0)_*(\mathcal{O}_M)\). If \(x \in \mathcal{M}_0\) and \(m_x\) is the maximal ideal of the local superalgebra \((\mathcal{O}_M)_x\), then the vector superspace \(T_x(\mathcal{M}) := (m_x/m_x^2)^*\) is the tangent space of \(\mathcal{M}\) at \(x \in \mathcal{M}_0\).

Denote by \(\mathcal{T}_\mathcal{M}\) the tangent sheaf or the sheaf of vector fields of \(\mathcal{M}\). In other words, \(\mathcal{T}_\mathcal{M}\) is the sheaf of derivations of the structure sheaf \(\mathcal{O}_M\). Since the sheaf \(\mathcal{O}_M\) is \(\mathbb{Z}_2\)-graded, the tangent sheaf \(\mathcal{T}_\mathcal{M}\) is also \(\mathbb{Z}_2\)-graded, i.e. there is the natural decomposition \(\mathcal{T}_\mathcal{M} = (\mathcal{T}_\mathcal{M})_{\bar{0}} \oplus (\mathcal{T}_\mathcal{M})_{\bar{1}}\), where

\[
(\mathcal{T}_\mathcal{M})_{\bar{i}} := \left\{ v \in \mathcal{T}_\mathcal{M} \mid v((\mathcal{O}_M)_j) \subset (\mathcal{O}_M)_{j+i} \right\}.
\]

Let \(\mathcal{M}_0\) be a complex analytic manifold and let \(\mathcal{E}\) be the sheaf of holomorphic sections of a vector bundle over \(\mathcal{M}_0\). Then the ringed space \((\mathcal{M}_0, \bigwedge \mathcal{E})\) is a supermanifold. In this case \(\dim \mathcal{M} = n|m\), where \(n = \dim \mathcal{M}_0\) and \(m\) is the rank of the locally free sheaf \(\mathcal{E}\).

**Definition 1.** A supermanifold \((\mathcal{M}_0, \mathcal{O}_M)\) is called split if \(\mathcal{O}_M \simeq \bigwedge \mathcal{E}\) for a locally free sheaf \(\mathcal{E}\) on \(\mathcal{M}_0\). The grading of \(\mathcal{O}_M\) induces by an isomorphism \(\mathcal{O}_M \simeq \bigwedge \mathcal{E}\) and the natural \(\mathbb{Z}\)-grading of \(\bigwedge \mathcal{E} = \bigoplus_p \bigwedge^p \mathcal{E}\) is called split grading.

For example, all smooth supermanifolds are split by Batchelor’s Theorem. In [4] it was shown that all complex analytic Lie supergroups are split too. In this paper we study the splitting problem for complex analytic homogeneous supermanifolds.
3 Lie supergroups and their homogeneous spaces

3.1 Lie supergroups and super Harish-Chandra pairs.

A Lie supergroup is a group object in the category of supermanifolds, i.e., it is a supermanifold \( \mathcal{G} \) with three morphisms: the multiplication morphism, the inversion morphism and the identity morphism, which satisfy the usual conditions, modeling the group axioms. In this case the underlying space \( \mathcal{G}_0 \) is a Lie group. The structure sheaf of a (complex analytic) Lie supergroup can be explicitly described in terms of the corresponding Lie superalgebra and underlying Lie group using super Harish-Chandra pairs (see [5] and [14] for more details). Let us describe this construction briefly.

Definition 2. A super Harish-Chandra pair is a pair \((\mathcal{G}_0, \mathfrak{g})\) that consists of a Lie group \(\mathcal{G}_0\) and a Lie superalgebra \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) such that \(\mathfrak{g}_0 = \text{Lie} \mathcal{G}_0\) provided with a representation \(\text{Ad} : \mathcal{G}_0 \rightarrow \text{Aut} \mathfrak{g}\) of \(\mathcal{G}_0\) in \(\mathfrak{g}\) such that:

- \(\text{Ad}\) preserves the parity and induces the adjoint representation of \(\mathcal{G}_0\) on \(\mathfrak{g}_0\);
- the differential \((d \text{Ad})_e\) at the identity \(e \in \mathcal{G}_0\) coincides with the adjoint representation \(\text{ad}\) of \(\mathfrak{g}_0\) on \(\mathfrak{g}_0\).

If a super Harish-Chandra pair \((\mathcal{G}_0, \mathfrak{g})\) is given, it determines the Lie supergroup \(\mathcal{G}\) in the following way, see [5]. Let \(\mathfrak{U}(\mathfrak{g})\) be the universal enveloping superalgebra of \(\mathfrak{g}\). It is clear that \(\mathfrak{U}(\mathfrak{g})\) is a \(\mathfrak{U}(\mathfrak{g}_0)\)-module, where \(\mathfrak{U}(\mathfrak{g}_0)\) is the universal enveloping algebra of \(\mathfrak{g}_0\). Recall that we denote by \(\mathcal{F}_{\mathcal{G}_0}\) the structure sheaf of the manifold \(\mathcal{G}_0\). The natural action of \(\mathfrak{g}_0\) on the sheaf \(\mathcal{F}_{\mathcal{G}_0}\) gives rise to a structure of \(\mathfrak{U}(\mathfrak{g}_0)\)-module on \(\mathcal{F}_{\mathcal{G}_0}(U)\) for any open set \(U \subset \mathcal{G}_0\).

Putting\[ O_\mathcal{G}(U) = \text{Hom}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), \mathcal{F}_{\mathcal{G}_0}(U)) \]
for every open \(U \subset \mathcal{G}_0\), we get a sheaf \(O_\mathcal{G}\) of \(\mathbb{Z}_2\)-graded vector spaces. (Here we assume that the functions in \(\mathcal{F}_{\mathcal{G}_0}(U)\) are even.) The enveloping superalgebra \(\mathfrak{U}(\mathfrak{g})\) has a Hopf superalgebra structure. Using this structure we can define the product of elements from \(O_\mathcal{G}\) such that \(O_\mathcal{G}\) becomes a sheaf of superalgebras, see [5] and [14] for details. A supermanifold structure on \(O_\mathcal{G}\) is determined by the isomorphism \(\Phi_\mathcal{G} : O_\mathcal{G} \rightarrow \text{Hom} \left( \bigwedge (\mathfrak{g}_1), \mathcal{F}_{\mathcal{G}_0} \right), f \mapsto f \circ \gamma_\mathfrak{g} \), where

\[ \gamma_\mathfrak{g} : \bigwedge (\mathfrak{g}_1) \rightarrow \mathfrak{U}(\mathfrak{g}), \quad X_1 \wedge \cdots \wedge X_r \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^{\left| \sigma \right|} X_{\sigma(1)} \cdots X_{\sigma(r)} \quad (1) \]
The following formulas define the multiplication morphism, the inversion
morphism and the identity morphism respectively:

\[ \mu^*(f)(X \otimes Y)(g, h) = f \left( \text{Ad}(h^{-1})(X) \cdot Y \right)(gh); \]
\[ \iota^*(f)(X)(g) = f \left( \text{Ad}(g)(S(X)) \right)(g^{-1}); \]
\[ \varepsilon^*(f) = f(1(e)). \]  

(2)

Here \( X, Y \in \mathfrak{U}(g) \), \( f \in \mathcal{O}_G \), \( g, h \in G_0 \) and \( S \) is the antipode map of the Hopf
superalgebra \( \mathfrak{U}(g) \). Here we identify the enveloping superalgebra
\( \mathfrak{U}(g \oplus g) \) with the tensor product \( \mathfrak{U}(g) \otimes \mathfrak{U}(g) \).

Sometimes we will identify the Lie superalgebra \( g \) of a Lie supergroup \( G \)
with the tangent space \( T_e(G) \) at \( e \in G_0 \). The corresponding to \( T \in T_e(G) \) left
invariant vector field on \( G \) is given by

\[ (\text{id} \otimes T) \circ \mu^*, \]  

(3)

where \( \mu \) is the multiplication morphism of \( G \). (Recall that a vector field \( Y \)
on \( G \) is called left invariant if \( (\text{id} \otimes Y) \circ \mu^* = \mu^* \circ Y \).) Denote by \( l_g \) and by
\( r_g \) the left and right translations with respect to \( g \in G_0 \), respectively. The
morphisms \( l_g \) and \( r_g \) are given by the following formulas:

\[ l_g^*(f)(X)(h) = f(X)(gh); \quad r_g^*(f)(X)(h) = f \left( \text{Ad}(g^{-1})X \right)(hg), \]

(4)

where \( f \in \mathcal{O}_G \), \( X \in \mathfrak{U}(g) \) and \( g, h \in G_0 \).

3.2 Homogeneous supermanifolds.

An action of a Lie supergroup \( G \) on a supermanifold \( M \) is a morphism \( \nu : \)
\( G \times M \rightarrow M \) such that the usual conditions modeling group action axioms
hold. Any vector \( X \in T_e(G) \) defines the vector field on \( M \) by the following
formula:

\[ X \mapsto (X \otimes \text{id}) \circ \nu^*. \]

(5)

Definition 3. An action \( \nu \) is called transitive if \( \nu_0 \) is a transitive action of
the Lie group \( G_0 \) on \( M_0 \) and the vector fields (5) generates the tangent space
\( T_x(M) \) at any point \( x \in M_0 \). In this case the supermanifold \( M \) is called
\( G \)-homogeneous. A supermanifold \( M \) is called homogeneous, if it possesses a
transitive action of a Lie supergroup.

If a supermanifold \( M \) is \( G \)-homogeneous and \( \nu : G \times M \rightarrow M \) is the
corresponding transitive action, then \( M \) is isomorphic to the supermanifold
\( G/H \), where \( H \) is the isotropy subsupergroup of a certain point (see [16] for
details). Recall that the underlying space of $\mathcal{G}/\mathcal{H}$ is the complex analytic manifold $\mathcal{G}_0/\mathcal{H}_0$ and the structure sheaf $\mathcal{O}_{\mathcal{G}/\mathcal{H}}$ of $\mathcal{G}/\mathcal{H}$ is given by

$$
\mathcal{O}_{\mathcal{G}/\mathcal{H}} = \left\{ f \in (\pi_0)_*(\mathcal{O}_\mathcal{G}) \mid \mu_{\mathcal{G} \times \mathcal{H}}^*(f) = \text{pr}^*(f) \right\},
$$

where $\pi_0 : \mathcal{G}_0 \to \mathcal{G}_0/\mathcal{H}_0$ is the natural map, $\mu_{\mathcal{G} \times \mathcal{H}}$ is the restriction of the multiplication map on $\mathcal{G} \times \mathcal{H}$ and $\text{pr} : \mathcal{G} \times \mathcal{H} \to \mathcal{G}$ is the natural projection. Using (2) we can rewrite the condition $\mu_{\mathcal{G} \times \mathcal{H}}^*(f) = \text{pr}^*(f)$ in the following way:

$$
f\left(\text{Ad} (h^{-1}) (X)Y\right)(gh) = \begin{cases} 
 f(X)(g), & Y \in \mathbb{C}; \\
 0, & Y \notin \mathbb{C}; 
\end{cases}
$$

where $X \in \mathfrak{U}(\mathfrak{g})$, $Y \in \mathfrak{U}(\mathfrak{h})$, $\mathfrak{h} = \text{Lie} \mathcal{H}$, $g \in \mathcal{G}_0$ and $h \in \mathcal{H}_0$.

Let $Y \in \mathfrak{g}$ and $f \in \mathcal{O}_\mathcal{G}$. Then the operator defined by the formula

$$
Y(f)(X) = (-1)^{p(Y)} f(XY),
$$

where $p(Y)$ is the parity of $Y$, is a left invariant vector field on $\mathcal{G}$. From (4), (7) and (8) it follows that

$f \in \mathcal{O}_{\mathcal{G}/\mathcal{H}}$ if and only if $f$ is $\mathcal{H}_0$-right invariant, i.e. $r_h^*(f) = f$ for any $h \in \mathcal{H}_0$, and $Y(f) = 0$ for all $Y \in \mathfrak{h}_1$.

Sometimes we will consider also the left action $\mathcal{H} \times \mathcal{G} \to \mathcal{G}$ of a subsupergroup $\mathcal{H}$ on a Lie supergroup $\mathcal{G}$. The corresponding quotient supermanifold we will denote by $\mathcal{H}\mathcal{G}$.

### 3.3 More about split supermanifolds.

Recall that a supermanifold $\mathcal{M}$ is called split if its structure sheaf $\mathcal{O}_{\mathcal{M}}$ is isomorphic to $\bigwedge \mathcal{E}$, where $\mathcal{E}$ is a locally free sheaf on $\mathcal{M}_0$. In this case, $\mathcal{O}_{\mathcal{M}}$ possesses the $\mathbb{Z}$-grading induced by the natural $\mathbb{Z}$-grading of $\bigwedge \mathcal{E} = \bigoplus_p \bigwedge^p \mathcal{E}$ and by isomorphism $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$. Such gradings of $\mathcal{O}_{\mathcal{M}}$ we call split.

**Proposition 1.** Any Lie supergroup $\mathcal{G}$ is split.

This statement follows from the fact that any Lie supergroup is determined by its super Harish-Chandra pair. A different proof of this result (probably the first one) was given in [4]. For completeness we give here another proof.

**Proof.** The underlying space $\mathcal{G}_0$ is a closed Lie subsupergroup of $\mathcal{G}$. Hence, there exists the homogeneous space $\mathcal{G}/\mathcal{G}_0$, which is isomorphic to the supermanifold $\mathcal{N}$ such that $\mathcal{N}_0$ is a point $\mathcal{N}_0 = \mathcal{G}_0/\mathcal{G}_0$ and $\mathcal{O}_{\mathcal{N}} \simeq \bigwedge (m)$, where $m = \text{dim} \mathfrak{g}_1$. By definition, the structure sheaf $\mathcal{O}_{\mathcal{N}}$ consists of all
$r_g$-invariant functions, $g \in G_0$. We have the natural map $\varphi : G \to G/G_0$, where $\varphi_0 : G_0 \to \ast$ and $\varphi^* : \mathcal{O}_N \to (\varphi_0)_*(\mathcal{O}_G)$ is the inclusion. It is known that $\varphi : G \to G/G_0$ is a principal bundle (see [16]). Using the fact that the underlying space of $G/G_0$ is a point we get $G \simeq N \times G_0$. Note that this is an isomorphism of supermanifolds but not of Lie supergroups. □

Example 1. As an example of a homogeneous non-split supermanifold we can cite the super-grassmannian $\text{Gr}_{m|n,r|s}$ for $0 < r < m$ and $0 < s < n$. Super-grassmannians of other types are split (see Example 3).

Denote by $\text{SSM}$ the category of split supermanifolds. Objects $\text{Ob} \text{SSM}$ in this category are all split supermanifolds $M$ with fixed split gradings. Further if $X, Y \in \text{Ob} \text{SSM}$, we put

$$\text{Hom}(X, Y) = \{ \text{morphisms from } X \text{ to } Y \text{ preserving the split gradings} \}$$

As in the category of supermanifolds, we can define in $\text{SSM}$ a group object (split Lie supergroup), an action of a split Lie supergroup on a split supermanifold (split action) and a split homogeneous supermanifold.

There is a functor $\text{gr}$ from the category of supermanifolds to the category of split supermanifolds. Let us briefly describe this construction. Let $M$ be a supermanifold. Denote by $\mathcal{J}_M \subset \mathcal{O}_M$ the subsheaf of ideals generated by odd elements of $\mathcal{O}_M$. Then by definition $\text{gr} M = (M_0, \text{gr} \mathcal{O}_M)$ is the split supermanifold with the structure sheaf

$$\text{gr} \mathcal{O}_M = \bigoplus_{p \geq 0} (\text{gr} \mathcal{O}_M)_p, \quad \mathcal{J}_M^0 := \mathcal{O}_M, \quad (\text{gr} \mathcal{O}_M)_p := \mathcal{J}_M^p/\mathcal{J}_M^{p+1}.$$ 

In this case $(\text{gr} \mathcal{O}_M)_1$ is a locally free sheaf and there is a natural isomorphism of $\text{gr} \mathcal{O}_M$ onto $\bigwedge (\text{gr} \mathcal{O}_M)_1$. If $\psi = (\psi_0, \psi^*) : M \to N$ is a morphism, then $\text{gr}(\psi) = (\psi_0, \text{gr}(\psi^*)) : \text{gr} M \to \text{gr} N$ is defined by

$$\text{gr}(\psi^*)(f + \mathcal{J}_N^p) := \psi^*(f) + \mathcal{J}_M^p \text{ for } f \in (\mathcal{J}_N)^{p-1}.$$ 

Recall that by definition every morphism of supermanifolds is even and as a consequence sends $\mathcal{J}_N^p$ into $\mathcal{J}_M^p$.

### 3.4 Split Lie supergroups.

Let $G$ be a Lie supergroup with the supergroup morphisms $\mu$, $\iota$ and $\varepsilon$: the multiplication, the inversion and the identity morphism, respectively. In this section we assign three split Lie supergroups $G^1$, $G^2$ and $G^3$ to $G$ and we show that these split Lie supergroups are pairwise isomorphic.
(1) The construction of $G^1$ is very simple: we just apply functor $\text{gr}$ to $G$. Clearly, $G^1 := \text{gr} G$ is a split Lie supergroup with the supergroup morphisms $\text{gr}(\mu)$, $\text{gr}(\iota)$ and $\text{gr}(\varepsilon)$.

(2) Consider the super Harish-Chandra pair $(G_0, g^2)$, where $g^2$ is the following Lie superalgebra: $g^2$ and $g$ are isomorphic as vector superspaces and the Lie bracket in $g^2$ is defined by the following formula:

$$[X,Y]_{g^2} = \begin{cases} [X,Y]_g, & \text{if } X,Y \in g_0 \text{ or } X \in g_0 \text{ and } Y \in g_1; \\ 0, & \text{if } X,Y \in g_1. \end{cases}$$

(9)

Denote by $G^2$ the Lie supergroup corresponding to $(G_0, g^2)$.

(3) Consider the sheaf $O_{G^3} := \text{Hom}_C(\bigwedge g_1, F_{G_0})$. For the ringed space $G^3 := (G_0, O_{G^3})$ we can repeat the construction from Section 3.1. Indeed, this ringed space is clearly a supermanifold. Furthermore, the exterior algebra $\bigwedge g_1$ is also a Hopf algebra. Therefore, we can define on $G^3$ the multiplication, the inversion and the identity morphisms respectively by the following formulas:

$$(\mu^3)^*(f)(X \wedge Y)(g,h) = f(\text{Ad}(h^{-1})(X \wedge Y)(gh));$$

$$(\iota^3)^*(f)(X)(g) = f(\text{Ad}(g)(S'(X)))(g^{-1});$$

$$(\varepsilon^3)^*(f) = f(1)(e).$$

(10)

Here $X,Y \in \bigwedge g_1$, $f \in \text{Hom}_C(\bigwedge g_1, F_{G_0})$, $g, h \in G_0$ and $S'$ is the antipode map of the Hopf superalgebra $\bigwedge g_1$. Hence, $G^3 := (G_0, O_{G^3})$ is a Lie supergroup. Since

$$\text{Hom}_C(\bigwedge g_1, F_{G_0}) = \bigoplus_{p \geq 0} \text{Hom}_C(\bigwedge g_1, F_{G_0})$$

is $\mathbb{Z}$-graded and the morphisms (10) preserve this $\mathbb{Z}$-grading, we see that $G^3$ is a split Lie supergroup.

Later on we will need the explicit expression of left and right translations $l_g^r$ and $r_g^r$ in $G^3$:

$$(l_g^r)^*(f)(X)(h) = f(X)(gh); \quad (r_g^r)^*(f)(X)(h) = f(\text{Ad}(g^{-1})X)(hg),$$

(11)

where $f \in \text{Hom}_C(\bigwedge g_1, F_{G_0})$, $X \in \bigwedge g_1$ and $g, h \in G_0$.

In fact, all these split Lie supergroups are isomorphic. To show this we need the following lemma:

**Lemma 1.** Let $\mathfrak{k}$ be a Lie superalgebra and $X_i, Y_j \in \mathfrak{k}_1$, $i = 1, \ldots, r$, $j = 1, \ldots, s$ be any elements. Assume that $[X_i, Y_j] = 0$ for any $i, j$. Then we have

$$\gamma_\mathfrak{k}(X_1 \wedge \cdots \wedge X_r \wedge Y_1 \wedge \cdots \wedge Y_s) = \gamma_\mathfrak{k}(X_1 \wedge \cdots \wedge X_r) \cdot \gamma_\mathfrak{k}(Y_1 \wedge \cdots \wedge Y_s),$$

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where $\gamma_k$ is given by (1).

Proof. A direct calculation. □

Proposition 2. We have $G^1 \simeq G^2 \simeq G^3$ in the category of Lie supergroups.

Proof. (a) The statement $G^1 \simeq G^2$ was proven in [15], Theorem 3.

(b) Let us show that $G^2 \simeq G^3$. Applying Lemma 1 to $g_2$ and to any elements $X_i, Y_j \in g_1$, we see that in this case $\gamma_{g_2}$ is not only isomorphism of super coalgebras but of Hopf superalgebras. In other words, the isomorphism

$$\Phi_{g_2} : \text{Hom}_{U(g_2)}(U(g^2), F_{g_2}) \to \text{Hom}_C(\bigwedge g_1, F_{g_2})$$

is an isomorphism of Lie supergroups. □

4 Split grading operators

Let again $M$ be a supermanifold, $\text{gr} M$ be the corresponding split supermanifold and $J$ be the sheaf of ideals generated by odd elements of $O_M$. We denote by $T = \text{Der} O_M$ and by $\text{gr} T = \text{Der}(\text{gr} O_M)$ the tangent sheaf of $M$ and of $\text{gr} M$, respectively. The sheaf $T$ is naturally $\mathbb{Z}_2$-graded and the sheaf $\text{gr} T$ is naturally $\mathbb{Z}$-graded: the gradings are induced by the $\mathbb{Z}_2$ and $\mathbb{Z}$-grading of $O_M$ and of $\text{gr} O_M$, respectively. In other words, we have the decomposition:

$$T = T_0 \oplus T_1, \quad \text{gr} T = \bigoplus_{p \geq -1} (\text{gr} T)_p.$$

The sheaves $T$ and $\text{gr} T$ are related: this relation can be expressed by the following exact sequence:

$$0 \longrightarrow T_{(2)0} \longrightarrow T_0 \overset{\alpha}{\longrightarrow} (\text{gr} T)_0 \longrightarrow 0,$$

where

$$T_{(2)0} = \{v \in T_0 \mid v(O_M) \subset J^2\}.$$

The morphism $\alpha$ in (12) is the composition of the natural morphism $T_0 \to T_0/T_{(2)0}$ and the isomorphism $T_0/T_{(2)0} \to (\text{gr} T)_0$ that is given by

$$[w] \mapsto \tilde{w}, \quad \tilde{w}(f + J^{p+1}) := w(f) + J^{p+1},$$

where $w \in T_0$, $[w]$ is the image of $w$ in $T_0/T_{(2)0}$ and $f \in J^p$.

Assume that the sheaf $O_M$ is $\mathbb{Z}$-graded, i.e. $O_M = \bigoplus_{p} (O_M)_p$. Then we have the map $w : O_M \to O_M$ defined by $w(f) = pf$, where $f \in (O_M)_p$. Such maps are called grading operators on $M$. 

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Definition 4. We call a grading operator $w$ on $\mathcal{M}$ a split grading operator if it corresponds to a split grading of $\mathcal{O}_M$, see Definition 1. In fact any split grading operator $w$ on $\mathcal{M}$ is an even vector field on $\mathcal{M}$. Indeed, $w$ is linear, it preserves the parity in $\mathcal{O}_M$ and for $f \in (\mathcal{O}_M)_p$ and $g \in (\mathcal{O}_M)_q$ we have:

$$w(fg) = (p+q)fg = (pf)g + f(qg) = w(f)g + fw(g).$$

Note that $fg \in (\mathcal{O}_M)_{p+q}$.

By definition the sheaf $\text{gr} \mathcal{O}_M$ is $\mathbb{Z}$-graded. Denote by $a$ the corresponding split grading operator.

Lemma 2. 1. A supermanifold $\mathcal{M}$ is split if and only if the vector field $a$ is contained in $\text{Im} \ H^0(\alpha)$, where

$$H^0(\alpha) : H^0(M_0, \mathcal{T}_0) \to H^0(M_0, (\text{gr} \mathcal{T})_0).$$

(We applied the functor $H^0(M_0, -)$ to the sequence (12). We write $H^0(\alpha)$ instead of $H^0(M_0, \alpha)$ for notational simplicity.)

2. If $w$ is a split grading operator on $\mathcal{M}$, then any other split grading operator on $\mathcal{M}$ has the form $w + \chi$, where $\chi \in H^0(M_0, \mathcal{T}(2))$.

Proof. 1. The statement of the lemma can be deduced from the following observation made by Koszul in [4, Lemma 1.1 and Section 3]. Let $A$ be a commutative superalgebra over $\mathbb{C}$ and $\mathfrak{m}$ be a nilpotent ideal in $A$. An even derivation $w$ of $A$ is called adapted to the filtration

$$A \supset \mathfrak{m} \supset \mathfrak{m}^2 \ldots$$

if

$$(w - r \text{id})(\mathfrak{m}^r) \subset \mathfrak{m}^{r+1} \text{ for any } r \geq 0.$$

Denote by $D^\text{ad}_\mathfrak{m}$ the set of all derivations adapted to $\mathfrak{m}$. In [4, Lemma 1.1] it was shown that $D^\text{ad}_\mathfrak{m}$ is not empty if and only if the filtration of $A$ is splittable. Moreover, if $w \in D^\text{ad}_\mathfrak{m}$, then the corresponding splitting of $A$ is given by eigenspaces of the derivation $w$: $A = \bigoplus_i A_i$, where $A_i$ is the eigenspace of $w$ with the eigenvalue $i$, and $\mathfrak{m}^r = A_r \oplus \mathfrak{m}^{r+1}$ for all $r \geq 0$.

We apply Koszul’s observation to the sheaf of superalgebras $\mathcal{O}_M$ and its subsheaf of ideals $\mathcal{J}$. The set $D^\text{ad}_\mathcal{J}$ is in this case the set of global derivations of $\mathcal{O}_M$ adapted to the filtration

$$\mathcal{O}_M \supset \mathcal{J} \supset \mathcal{J}^2 \supset \ldots.$$  

Clearly, $D^\text{ad}_\mathcal{J}$ is not empty if and only if $a$ is contained in $\text{Im} \ H^0(\alpha)$. (Actually, $H^0(\alpha)(D^\text{ad}_\mathcal{J}) = a$.) Furthermore, if the supermanifold $\mathcal{M}$ is split, i.e.
we have a split grading $\mathcal{O}_M = \bigoplus_{p \geq 0} (\mathcal{O}_M)_p$, then $\mathcal{F}^q = \bigoplus_{p \geq q} (\mathcal{O}_M)_p$ and $\mathcal{F}^q = (\mathcal{O}_M)_q \oplus \mathcal{F}^{q+1}$. Hence, the split grading determines the splitting of the filtration (13) and the corresponding split grading operator belongs to $D^{ad}_{\mathcal{F}}$.

Conversely, if there exists $w \in D^{ad}_{\mathcal{F}}$, then we can decompose the sheaf $\mathcal{O}_M$ into eigenspaces $(\mathcal{O}_M)_q := \{ f \in \mathcal{O}_M | w(f) = qf \}$.

In this case the sheaves $\bigoplus_p (\mathcal{O}_M)_p$ and $\text{gr} \mathcal{O}_M$ are isomorphic as $\mathbb{Z}$-graded sheaves of superalgebras since $\mathcal{F}^q = (\mathcal{O}_M)_q \oplus \mathcal{F}^{q+1}$. Hence, the supermanifold is split.

2. Applying the left-exact functor $H^0(\mathcal{M}_0, -)$ to (12), we get the following exact sequence:

$$0 \rightarrow H^0(\mathcal{M}_0, \mathcal{T}(2)\bar{0}) \rightarrow H^0(\mathcal{M}_0, \mathcal{T}_0) \rightarrow H^0(\mathcal{M}_0, \text{gr} \mathcal{T}_0).$$

If $w_1, w_2$ are two split grading operators on $\mathcal{M}$, then

$$H^0(\alpha)(w_1) = H^0(\alpha)(w_2) = a,$$

according to the part 1. Therefore, $w_1 - w_2 \in H^0(\mathcal{M}_0, \mathcal{T}(2)\bar{0})$. The result follows.

Example 2. Consider the supermanifold $G_0 \backslash G$. Its structure sheaf is isomorphic to $\bigwedge (g_1)$ (compare with Example 1). Denote by $(\varepsilon_i)$ the system of odd (global) coordinates on $G_0 \backslash G$. An example of a split grading operator on the Lie supergroup $G$ is $\sum \varepsilon^i X_i$. Here $(X_i)$ is a basis of odd left invariant vector fields on $G$ such that $X_i(\varepsilon^j)(e) = \delta_i^j$. We may produce other examples if we use right invariant vector fields or odd (global) coordinates on $G_0 \backslash G_0$.

By Lemma 2, any split grading operator on a Lie supergroup $G$ is given by $\sum \varepsilon^i X_i + \chi$, where $\chi \in H^0(G_0, \mathcal{T}(2)\bar{0})$ is any vector field on $G$.

5 Compatible split gradings on $G \backslash H$

5.1 Compatible gradings on $G \backslash H$.

Let $G$ be a Lie supergroup and $\mathcal{M} = G \backslash H$ be a homogeneous supermanifold. As above we denote by $\pi : G \rightarrow G \backslash H$ the natural projection.

Definition 5. A split grading of the sheaf $\mathcal{O}_G = \bigoplus_p (\mathcal{O}_G)_p$ is called compatible with the inclusion $\mathcal{O}_M \subset (\pi_0)_*(\mathcal{O}_G)$ if the following holds:

$$f \in \mathcal{O}_M \Rightarrow f_p \in \mathcal{O}_M$$

for all $p$.

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where \( f = \sum f_p \) and \( f_p \in (\pi_0)_*((\mathcal{O}_G)_p) \).

Let us take any split grading operator \( w \) on \( G \). Clearly, the corresponding split grading of \( \mathcal{O}_G \) is compatible with \( \mathcal{O}_M \) if and only if \( w(\mathcal{O}_M) \subset \mathcal{O}_M \). It is not clear from Definition 5 that the compatible grading

\[
(\mathcal{O}_M)_p = \mathcal{O}_M \cap (\pi_0)_*((\mathcal{O}_G)_p) \tag{14}
\]

of \( \mathcal{O}_M \), if it exists, is a split grading of \( \mathcal{O}_M \). However, the following proposition holds:

**Proposition 3.** Assume that we have the \( \mathbb{Z} \)-grading:

\[
\mathcal{O}_M = \bigoplus_{p \geq 0} (\mathcal{O}_M)_p,
\]

where \((\mathcal{O}_M)_p \) are as in (14). Then this grading is a split grading.

**Proof.** The idea of the proof is to apply Lemma 2 to the grading operator \( w' := w|_{\mathcal{O}_M} \) on \( M \). Denote by \( J_M \) and by \( J_G \) the sheaves of ideals generated by odd elements of \( \mathcal{O}_M \) and \( \mathcal{O}_G \), respectively. Our aim is to show that

\[
w'(f) + J_{p+1}^M = pf + J_{p+1}^G;
\]

where \( f \in J^p_M \). In other words, we want to show that \( H^0(\alpha)(w') \) is a split grading operator for the grading of \( \text{gr} \mathcal{O}_M \). (We use notations of Lemma 2.) We have:

\[
(\text{gr} \pi)^*(w'(f) + J_{p+1}^M) = w(f) + J_{p+1}^G = pf + J_{p+1}^G;
\]

\[
(\text{gr} \pi)^*(pf + J_{p+1}^M) = pf + J_{p+1}^G.
\]

Since the map \( (\text{gr} \pi)^* \) is injective, we get, \( w'(f) + J_{p+1}^M = pf + J_{p+1}^M \). \( \square \)

## 5.2 \( \mathcal{H} \)-invariant split grading operators.

First of all let us consider the situation when a split grading operator \( w \) on \( G \) is invariant with respect to a Lie subsupergroup \( \mathcal{H} \). In terms of super Harish-Chandra pairs this means:

\[
\begin{align*}
    r_h^* \circ w &= w \circ r_h^*, \quad \text{for all} \quad h \in \mathcal{H}_0; \\
    [Y, w] &= 0, \quad \text{for all} \quad Y \in \mathfrak{h}_1.
\end{align*}
\tag{15}
\]

Here \((\mathcal{H}_0, \mathfrak{h})\) is the super Harish-Chandra pair of \( \mathcal{H} \), \( r_h \) is the right translation and \( Y \) is an odd left invariant vector field.
Proposition 4. Assume that \( w \) is an \( \mathcal{H} \)-invariant split grading operator on \( G \), i.e. equations (15) hold. Then \( \mathcal{H} \) is an ordinary Lie group.

Proof. The idea of the proof is to show that the Lie superalgebra \( \mathfrak{h} \) of \( \mathcal{H} \) has the trivial odd part: \( \mathfrak{h}_{\overline{1}} = \{0\} \).

In Example 2 we saw that any split grading operator on \( G \) is given by \( w = \sum \varepsilon^i X_i + \chi \). If \( Z \) is a vector field on \( G \), denote by \( Z_e \in T_e(G) \) the corresponding tangent vector at the identity \( e \in G_0 \). Consider the second equation in (15). At the point \( e \), we have

\[
[Y, w]_e = \left( \sum_i Y(\varepsilon^i)X_i - \sum_i \varepsilon^i Y \circ X_i - \sum_i \varepsilon^i X_i \circ Y + [Y, \chi] \right)_e = 0
\]

for any \( Y \in \mathfrak{h}_{\overline{1}} \). Furthermore,

\[
\left( \sum_i \varepsilon^i Y \circ X_i - \sum_i \varepsilon^i X_i \circ Y \right)_e = 0 \quad \text{and} \quad [Y, \chi]_e = 0,
\]

because \( \varepsilon^i(e) = 0 \) and because \( \chi \in H^0(M_0, T(2)_{\overline{0}}) \). Therefore,

\[
[Y, w]_e = \sum_i Y(\varepsilon^i)(e)(X_i)_e = 0
\]

The tangent vectors \( (X_i)_e \) form a basis in \( T_e(G)_{\overline{1}} \), hence \( Y(\varepsilon^i)(e) = 0 \) for all \( i \). The last statement is equivalent to \( Y_e = 0 \). Since \( Y \) is a left invariant vector field, we get \( Y = 0 \). The proof is complete. □

Remark. It is well known that the supermanifold \( G/\mathcal{H} \), where \( \mathcal{H} \) is an ordinary Lie group, is split (see [5] or [14]). Therefore, the case of \( \mathcal{H} \)-invariant split grading operators does not lead to new examples of homogeneous split supermanifolds.

5.3 \( G_0 \)-left invariant split grading operators.

Consider now a more general situation, when a split grading operator \( w \) leaves \( \mathcal{O}_M \) invariant. Let \( f \in \mathcal{O}_M \). Then \( w(f) \in \mathcal{O}_M \) if and only if

\[
r_h^*(w(f)) = w(f) \quad \text{and} \quad Y(w(f)) = 0
\]

for \( h \in \mathcal{H}_0 \) and \( Y \in \mathfrak{h}_{\overline{1}} \). These conditions are equivalent to the following ones:

\[
(r_h^* \circ w \circ (r_h^{-1})^* - w)|_{\mathcal{O}_M} = 0; \quad [Y, w]|_{\mathcal{O}_M} = 0. \quad (16)
\]

Recall that \( r_h^{-1} = r_{h^{-1}} \).
It seems to us that the system (16) is hard to solve in general. Consider now a special type of split grading operators, called $G_0$-left invariant grading operators.

**Definition 6.** A split grading of $O_G$ is called $G_0$-left invariant if it is invariant with respect to left translations. In other words, from $f \in (O_G)_p$ it follows that $l_g^*(f) \in (O_G)_p$ for all $g \in G_0$.

It is easy to see that a split grading of $O_G$ is $G_0$-left invariant if and only if the corresponding split grading operator $w$ is invariant with respect to left translations: $l_g^* \circ w = w \circ l_g^*$, $g \in G_0$. For example, the split grading operator $\sum \epsilon_i X_i$ constructed in Example 2 is a $G_0$-left invariant split grading operator, because $\epsilon_i$ are $G_0$-left invariant functions and $X_i$ are left invariant vector fields. In this section we will describe all such operators.

In Section 3.4 we have seen that the supermanifold $(G_0, \text{Hom}_C(\bigwedge \bar{g}_1, F_G))$ is a Lie supergroup isomorphic to $gr G$. We need the following lemma:

**Lemma 3.** The map

$$\Phi_g : O_G \rightarrow \text{Hom}_C(\bigwedge \bar{g}_1, F_G),$$

$$f \mapsto f \circ \gamma_g$$

from Section 3.1 is invariant with respect to left and right translations.

**Proof.** For any $h \in G_0$, denote by $r'_h$ and $l'_h$ the right and the left translation in the Lie supergroup $G^3 = (G_0, \text{Hom}_C(\bigwedge \bar{g}_1, F_G))$, respectively. (See, (11)) Let us show that

$$(r'_h)^* \circ \Phi_g = \Phi_g \circ r^*_h. \quad (17)$$

Let us take $Z \in \bigwedge \bar{g}_1$ and $g, h \in G_0$. Using (4) we have

$$[(r'_h)^* \circ \Phi_g](f)(Z)(g) = \Phi_g(f)(\text{Ad}(h^{-1})(Z))(gh) =$$

$$f(\gamma_g(\text{Ad}(h^{-1})(Z)))(gh) = f(\text{Ad}(h^{-1})(\gamma_g(Z)))(gh) =$$

$$r^*_h(f)(\gamma_g(Z))(g) = [\Phi_g \circ r^*_h](f)(Z)(g).$$

Similarly, we get

$$(l'_h)^* \circ \Phi_g = \Phi_g \circ l^*_h.$$

□

The following observation is known to experts, but we cannot find it in the literature:

**Lemma 4.** The space of $G_0$-left invariant vector fields $H^0(G_0, T)^{G_0}$ on a Lie supergroup $G$ is isomorphic to $H^0(\text{pt}, O_{G_0/G}) \otimes \mathfrak{g}$. The isomorphism is given by:

$$f \otimes Z \mapsto fZ,$$
where \( f \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0/\mathcal{G}}) \) and \( Z \in \mathfrak{g} \).

**Proof.** Clearly, the map \( F \) is injective and its image is contained in the vector space \( H^0(\mathcal{G}_0, \mathcal{T})^\mathcal{G}_0 \). Let us show that any vector field \( v \) in \( H^0(\mathcal{G}_0, \mathcal{T})^\mathcal{G}_0 \) is contained in \( \text{Im}(F) \).

Let \( (X_i) \) and \( (Z_j) \) be a basis of odd and even left invariant (with respect to the supergroup \( \mathcal{G} \)) vector fields on \( \mathcal{G} \), respectively. Assume that

\[
v = \sum f^i X_i + \sum g^j Z_j,
\]

where \( f^i, g^j \in H^0(\mathcal{G}_0, \mathcal{O}_\mathcal{G}) \), be the decomposition of \( v \) with respect to this basis. We have:

\[
l^*_g \circ v = \sum l^*_g(f^i)l^*_g \circ X_i + \sum l^*_g(g^j)l^*_g \circ Z_j = \sum l^*_g(f^i)X_i \circ l^*_g + \sum l^*_g(g^j)Z_j \circ l^*_g = v \circ l^*_g.
\]

Therefore, \( l^*_g(f^i) = f^i \) and \( l^*_g(g^j) = g^j \) for all \( g \in \mathcal{G}_0 \). In other words, \( f^i, g^j \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0/\mathcal{G}}) \). The proof is complete. \( \square \)

The Lie supergroup \( \mathcal{G} \) acts on the vector superspace \( H^0(\mathcal{G}_0, \mathcal{T})^\mathcal{G}_0 \). This action we can describe in terms of the corresponding super Harish-Chandra pair \( (\mathcal{G}_0, \mathfrak{g}) \) in the following way:

\[
g \mapsto (X \mapsto r^*_g \circ X \circ (r^{-1}_g)^*), \quad Y \mapsto (X \mapsto [Y, X]), \quad (18)
\]

where \( g \in \mathcal{G}_0 \), \( X \in H^0(\mathcal{G}_0, \mathcal{T})^\mathcal{G}_0 \) and \( Y \in \mathfrak{g} \). Note that this action is well-defined because \( \mathcal{G} \)-left and right actions on \( H^0(\mathcal{G}_0, \mathcal{T}) \) commute. The Lie supergroup \( \mathcal{G} \) acts also on the vector superspace \( H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0/\mathcal{G}}) \otimes \mathfrak{g} \). This action is given by right translations \( r^*_g \) on \( H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0/\mathcal{G}}) \) and by the formulas (18) on \( \mathfrak{g} \) if we assume that \( X \in \mathfrak{g} \). Clearly, the isomorphism \( F \) from Lemma 4 is equivariant. From now on we will identify \( H^0(\mathcal{G}_0, \mathcal{T})^\mathcal{G}_0 \) and \( H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0/\mathcal{G}}) \otimes \mathfrak{g} \) via isomorphism \( F \) from Lemma 4.

If \( \mathcal{H} \) is a Lie subsupergroup of \( \mathcal{G} \) and \( \mathfrak{h} = \text{Lie} \mathcal{H} \) then \( \mathfrak{g}/\mathfrak{h} \) is an \( \mathcal{H} \)-module.

**Lemma 5.** Let us take a \( \mathcal{G}_0 \)-left invariant split grading operator \( w \). The vector field \( w \) satisfies (16) if and only if

\[
\bar{w} \in (H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0/\mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h})^{\mathcal{H}}, \quad (19)
\]

where \( \bar{w} \) is the image of \( w \) by the natural mapping

\[
H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0/\mathcal{G}}) \otimes \mathfrak{g} \to H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0/\mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h}.
\]

**Proof.** Let \( \bar{w} \in (H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0/\mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h})^{\mathcal{H}} \). It follows that

\[
r^*_h \circ w \circ (r^{-1}_h)^* - w \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0/\mathcal{G}}) \otimes \mathfrak{h}, \quad h \in \mathcal{H}_0,
\]

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and 

\[ [Y, w] \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}}) \otimes \mathfrak{h}, \ Y \in \mathfrak{h}. \]

Hence, the conditions (16) are satisfied.

On the other hand, if the conditions (16) are satisfied, then the vector fields \( r^*_h \circ w \circ (r^{-1}_h)^* - w \) and \([Y, w]\) are vertical with respect to the projection \( \pi : \mathcal{G} \to \mathcal{G}/\mathcal{H} \). Therefore, \( r^*_h \circ w \circ (r^{-1}_h)^* - w \) and \([Y, w]\) belong to the superspace \( H^0(\text{pt}, \mathcal{O}_{\mathcal{G}}) \otimes \mathfrak{h} \). It is equivalent to conditions (19). □

Now our aim is to describe the space \( (H^0(\text{pt}, \mathcal{O}_{\mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h})^{\mathcal{H}_0} \). We have seen in Proposition 1 that the superspace \( H^0(\text{pt}, \mathcal{O}_{\mathcal{G}}) \) is isomorphic to \( \bigwedge (\mathfrak{g}_1) \otimes \mathfrak{g} \) as \( \mathcal{G}_0 \)-modules.

Proposition 5. a. We have

\[
H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0}) \otimes \mathfrak{g} \simeq \bigwedge (\mathfrak{g}_1) \otimes \mathfrak{g} \quad \text{as } \mathcal{G}_0 \text{-modules,}
\]
\[
H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0}) \otimes \mathfrak{g}/\mathfrak{h} \simeq \bigwedge (\mathfrak{g}_1) \otimes \mathfrak{g}/\mathfrak{h} \quad \text{as } \mathcal{H}_0 \text{-modules,}
\]

where the action of \( \mathcal{G}_0 \) on \( \bigwedge (\mathfrak{g}_1) \) is standard.

b. There exists a \( \mathcal{G}_0 \)-left and right invariant split grading operator on \( \mathcal{G} \).

Proof. a. We have to show that there exists an \( \mathcal{G}_0 \)-equivariant isomorphism

\[
H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0}) \xrightarrow{\beta} \bigwedge \mathfrak{g}_1.
\]

Then the map \( \beta \otimes \text{id} \) will provide the required isomorphism of \( \mathcal{G}_0 \)-modules. Consider the Lie supergroup

\[
\mathcal{G}^3 = (\mathcal{G}_0, \text{Hom}_\mathbb{C}(\bigwedge \mathfrak{g}_1, \mathcal{F}_{\mathcal{G}_0}))
\]

from Section 3.4. It follows from (4) that

\[
H^0(\mathcal{G}_0, \mathcal{O}_{\mathcal{G}_0}) = \text{Hom}_\mathbb{C}(\bigwedge \mathfrak{g}_1, \mathbb{C}) = (\bigwedge \mathfrak{g}_1)^*.
\]

Note that the action of \( \mathcal{G}_0 \) on \( (\bigwedge \mathfrak{g}_1)^* \) by right translations in \( \mathcal{G}^3 \), denoted by \( (r'_g)^* \), coincides with the standard action of \( \mathcal{G}_0 \) on \( (\bigwedge \mathfrak{g}_1)^* \). Indeed, let us take

\[
f \in H^0(\mathcal{G}_0, \mathcal{O}_{\mathcal{G}^3})^{\mathcal{G}_0} = (\bigwedge \mathfrak{g}_1)^*.
\]

By (11), we have:

\[
(r'_g)^*(f)(X)(e) = (r'_g)^*(f)(X)(h) = f(\text{Ad}(g^{-1})X)(hg) = f(\text{Ad}(g^{-1})X)(e).
\]
Here $g, h \in G_0$, $X \in \bigwedge g_1$ and $e \in G_0$ is the identity. It remains to note that by Lemma 3, the map $\Phi_g$ induces the equivariant isomorphism between the superspaces of left invariants $H^0(\text{pt}, O_{G_0 \backslash G})$ and $(\bigwedge g_1)^*$.

b. We need to show that in the vector space

$$(\bigwedge (g_1^* \otimes g))^{G_0} = (\bigwedge (g_1^* \otimes g_0))^{G_0} \oplus (\bigwedge (g_1^* \otimes g_1))^{G_0}$$

there exists points corresponding to split grading operators. This space always possesses a $G_0$-invariant, precisely, the identity operator $\text{id} \in g_1^* \otimes g_1$. The pre-image of $\beta^{-1}(\text{id}) \in H^0(\text{pt}, O_{G_0 \backslash G}) \otimes g$ has the form $\sum \varepsilon^i X_i$ for some choice of local coordinates such that $X_i(\varepsilon^j)(e) = \delta_i^j$, see Example 2. We have seen that such vector fields correspond to $G_0$-left invariant split grading operators on $G$. □

Denote by $T_G$ the tangent sheaf of a Lie group $G$ and by $\overline{v}$ is the image of $v$ by the natural mapping $H^0(\text{pt}, O_{G_0 \backslash G}) \otimes g \to H^0(\text{pt}, O_{G_0 \backslash G}) \otimes g/h$.

The result of our study is:

**Theorem 1.** The following conditions are equivalent:

a. A homogeneous supermanifold $M = G/H$ admits a $G_0$-left invariant split grading that is induced by a grading of $O_G$ and the inclusion $O_M \subset (\pi_0)_*(O_G)$.

b. There exists a $G_0$-left invariant vector field $\chi \in H^0(G_0, (T_G)(2g))$ such that

$$\overline{\chi} \in \left(\bigwedge (g_1^* \otimes g/h)\right)^{G_0}$$

and such that for $w = \beta^{-1}(\text{id}) + \chi$, where $\beta^{-1}(\text{id}) = \sum \varepsilon^i X_i$ is from the proof of Proposition 5.b, we have

$$[Y, w] \in H^0(\text{pt}, O_{G_0 \backslash G}) \otimes h, \ Y \in h_1.$$ (21)

### 6 An application

As above let $G$ be a Lie supergroup and $H$ be a Lie subsupergroup, $g$ and $h$ be the Lie superalgebras of $G$ and $H$, respectively, and $M := G/H$. Consider the map

$$\rho : g_0 \to H^0(\text{pt}, T_{G_0 \backslash G})$$
induced by the action of $G_0$ on $M$. (Here $T_{G_0\backslash G}$ is the sheaf of vector fields on $G_0\backslash G$.) Let us describe its kernel. For $X \in \mathfrak{g}_0$ and $f \in H^0(pt, \mathcal{O}_{G_0\backslash G})$, we have:

$$X(f)(Y)(e) = \left. \frac{d}{dt} \right|_{t=0} f(\text{Ad}(\exp(-tX))Y)(\exp(tX)) = \left. \frac{d}{dt} \right|_{t=0} f(\text{Ad}(\exp(-tX))Y)(e),$$

where $Y = Y_1 \cdots Y_r$, $Y_i \in \mathfrak{g}_i$ and $t$ is an even parameter. A vector field $X$ is in $\text{Ker} \rho$ if and only if $X(f)(Y)(e) = 0$ for all $f$ and $Y$. Hence,

$$\text{Ker} \rho = \text{Ker}(\text{ad} \mid \mathfrak{g}_1),$$

where $\text{ad}$ is the adjoint representation of $\mathfrak{g}_0$ in $\mathfrak{g}$.

Furthermore, denote

$$A := \text{Ker} \left( G_0 \ni g \mapsto \text{Ad}(g) : G/H \to G/H \right);$$
$$a := \text{Ker} \left( \mathfrak{g} \ni X \mapsto H^0(G_0/H_0, T_{G/H} G_0) \right).$$

Here $\text{Ad}(g)$ is the automorphism of $G/H$ induced by the left translation $l_g$. The pair $(A, a)$ is a super Harish-Chandra pair. An action of $G$ on $M$ is called effective if the corresponding to $(A, a)$ Lie supergroup is trivial. As in the case of Lie groups any action of a Lie supergroup can be factored to be effective.

**Theorem 2.** Assume that the action of $G$ on $M$ is effective. If

$$[\mathfrak{g}_1, \mathfrak{h}_1] \subset \mathfrak{h}_0 \cap \text{Ker}(\text{ad} \mid \mathfrak{g}_1),$$

then $M$ is split.

**Proof.** Let us show that in this case the vector field $w = \sum \varepsilon^i X_i + 0 = \sum \varepsilon^i X_i$ from Proposition 5.b is a (left invariant) split grading operator on $M$ using Theorem 1.

The condition (20) is satisfied trivially, because $\chi = 0$. Let us check the condition (21). We have:

$$[Y, v] = \sum Y(\varepsilon^i)X_i - \sum \varepsilon^i[Y, X_i].$$

Since $[\mathfrak{g}_1, \mathfrak{h}_1] \subset \mathfrak{h}_0$, we get

$$\sum \varepsilon^i[Y, X_i] \in H^0(pt, \mathcal{O}_{G_0\backslash G}) \otimes \mathfrak{h}. $$

Hence, we have to show that

$$\sum Y(\varepsilon^i)X_i \in H^0(pt, \mathcal{O}_{G_0\backslash G}) \otimes \mathfrak{h}. $$

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Assume that $X_1, \ldots, X_k$ is a basis of $\mathfrak{h}$, $X_1, \ldots, X_k, X_{k+1}, \ldots, X_m$ is a basis of $\mathfrak{g}$, and $(\varepsilon^i)$ is the system of global odd $G_0$-left invariant coordinates corresponding to this basis such that $\sum \varepsilon^i X_i$ is as in Proposition 5.b. In particular, $\varepsilon^i(\gamma(X_j)) = \delta^i_j$, because $\sum (\varepsilon^i \circ \gamma) \otimes X_i$ is the identity operator in $\mathfrak{g}_1 \otimes \mathfrak{g}$. Let us take $Z \in \text{Ker} \rho$. Clearly, $Z(\varepsilon^i) = 0$ and $X_j(\varepsilon^i)$ is again a $G_0$-left invariant function on $G$. By (8), we also have:

$$\varepsilon^i(X_{i_1} \cdots Z \cdots X_{i_k}) = 0.$$ 

Furthermore, by definition of $\varepsilon^i$, we get that $\varepsilon^i \circ \gamma \in \mathfrak{g}_1^*$. Hence,

$$\varepsilon^i(\gamma(X_{i_1} \wedge \cdots \wedge X_{i_k})) = 0,$$

if $k > 1$. Summing up all these observations we see that

$$\varepsilon^i(\gamma(X) \cdot Y) = \varepsilon^i(\gamma(X \wedge Y)) + 0,$$

where $Y \in \mathfrak{h}$ and $X \in \bigwedge \mathfrak{g}$. Now we can conclude that

$$\sum Y(\varepsilon^i)X_i = -Y \in \mathfrak{h} \subset H^0(\text{pt}, \mathcal{O}_{G_0}G) \otimes \mathfrak{h}.$$ 

The proof is complete. $\square$

**Example 3.** Consider the super-grassmannian $\text{Gr}_{m|n,k|m,l}$. It is a $\text{GL}_{m|n}$-homogeneous space, see [9] for more details. Hence, $\text{Gr}_{m|n,k|m,l} \simeq \text{GL}_{m|n}/\mathcal{H}$ for a certain $\mathcal{H}$. (See, for example, [15].) It the case $k = 0$ or $k = m$, the following holds $[[\mathfrak{g}_{m|n}], \mathfrak{h}] = 0$. Therefore, by Theorem 2, the super-grassmannian is split.

In [9] it was shown that the super-grassmannian $\text{GL}_{m|n,k|m,l}$ is not split if and only if $0 < k < m$ and $0 < l < n$. (This fact also follows from results in [6] and [13] about non-projectivity of super-grassmannian.)

Finally, let us recall a result proved in [14]:

**Theorem 3.** If a complex homogeneous supermanifold $\mathcal{M}$ is split, then there is a Lie supergroup $G$ with $[\mathfrak{g}, \mathfrak{g}] = 0$, where $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \text{Lie} G$, such that $G$ acts on $\mathcal{M}$ transitively.

**References**


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