

Optimal control for rough differential equations

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Dedicated to Ludwig Arnold on occasion of his 70th birthday

Abstract

In this note, we consider an optimal control problem associated to a differential equation driven by a Hölder continuous function g of index $\beta > 1/2$. We split our study in two cases. If the coefficient of dg_t does not depend on the control process, we prove an existence theorem for a slightly generalized control problem, that is we obtain a literal extension of the corresponding situation for ordinary differential equations. If the coefficient of dg_t depends on the control process, we also prove an existence theorem but here we are obliged to restrict the set of controls to sufficiently regular functions.

Key words: Optimal control - Rough differential equations - Young integral - Doss-Sussmann's method.

1 Introduction

In this note, we consider an optimal control problem associated to the following differential equation driven by a Hölder continuous function $g : [0, T] \rightarrow \mathbb{R}$:

$$x_t^u = x_0^u + \int_0^t \sigma(s, u_s, x_s^u) dg_s + \int_0^t b(s, u_s, x_s^u) ds, \quad t \in [0, T]. \quad (1)$$

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Here, the control process $u : [0, T] \rightarrow \mathbb{R}$ belongs to a set of admissible controls \mathcal{U} and the Hölder coefficient β of g belongs in $(1/2, 1)$, so that it is possible to choose the Young integral [11] for integration with respect to dg_t in (1).

The control problem considered in the present paper can be formulated in the following way.

Problem: “A cost functional $J : \mathcal{U} \rightarrow \mathbb{R}$ being given, is it possible to prove (2) the existence of $u^* \in \mathcal{U}$ realizing $\inf_{u \in \mathcal{U}} J(u)$?”

As usual, the bigger \mathcal{U} is, the more difficult it is to answer this question. A general methodology is to look for conditions ensuring that \mathcal{U} is compact for a certain topology under which J is continuous.

Differential equations of the type (1) (without the control process u) have been intensively studied in the recent years, in particular with respect to possible applications for fractional Brownian motion (see, *e.g.*, [6, 8, 9, 10] and the references therein). To obtain solutions to (1) requires in general regularity on the coefficients (see Theorem 3 below). Thus, we split our study in two cases.

1. If the coefficient of dg_t does *not* depend on the control process, we are able to extend the situation known for ordinary differential equations, and to prove an existence theorem for a slightly generalized control problem where the controls are in fact randomized : see Corollary 1 and Proposition 4. In fact, we use the so-called ‘compactification methods’, which have been developed during the 1960’s for deterministic control problems (see [4], [12]), and during the 1970’s for the stochastic control problem (see [3], [2]).
2. If the coefficient of dg_t *does* depend on the control function, the situation is much more intricate, and this obliges us to severely restrict the set of controls to sufficiently regular functions. A challenging question would be to relax this hypothesis, but this would require a reasonable notion of a solution for a differential equation with a weaker regularity than Hölderian. This seems not to be already available in the literature, up to our best knowledge.

The paper is organized as follows. In section 2, we present some preliminary results. In section 3, we study the optimal control problem in the case where σ does not depend on u . The case where σ depends on u is considered in section 4.

2 Preliminaries

For $T \in (0, \infty)$, we note $C^0([0, T])$ the set of continuous functions $f : [0, T] \rightarrow \mathbb{R}$. If $\mu \in (0, 1)$ and $T \in (0, \infty)$, we note $\mathcal{C}^\mu([0, T])$ the set of functions $g : [0, T] \rightarrow \mathbb{R}$ such

that

$$\sup_{0 \leq s < t \leq T} \frac{|g(t) - g(s)|}{|t - s|^\mu} < +\infty.$$

If there is no ambiguity, we prefer the notation \mathcal{C}^μ instead of $\mathcal{C}^\mu([0, T])$. The set \mathcal{C}^μ is a Banach space when it is endowed with the following norm:

$$|g|_{\infty, \mu} := \sup_{0 \leq t \leq T} |g(t)| + \sup_{0 \leq s < t \leq T} \frac{|g(t) - g(s)|}{|t - s|^\mu}.$$

We also set, for $a, b \in [0, T]$ and $g \in \mathcal{C}^\mu$:

$$|g|_{\mu, [a, b]} = \sup_{a \leq s < t \leq b} \frac{|g(t) - g(s)|}{|t - s|^\mu}, \quad |g|_{\infty, [a, b]} = \sup_{a \leq t \leq b} |g(t)|$$

and

$$|g|_{\infty, \mu, [a, b]} = |g|_{\infty, [a, b]} + |g|_{\mu, [a, b]}.$$

When $a = 0$ and $b = T$ we simply note $|g|_\mu$, $|g|_\infty$ and $|g|_{\infty, \mu}$ instead of $|g|_{\mu, [0, T]}$, $|g|_{\infty, [0, T]}$ and $|g|_{\infty, \mu, [0, T]}$, respectively, if there is no risk of ambiguity.

Let $f : \mathbb{R} \rightarrow \mathbb{R} \in \mathcal{C}^\alpha$ and $g : \mathbb{R} \rightarrow \mathbb{R} \in \mathcal{C}^\beta$ with $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta > 1$. Then, for any $s, t \in [0, T]$, the Young integral [11] $\int_s^t f dg$ exists and we have (see, for instance, [5, Proposition 3]):

$$\left| \int_s^t f dg \right| \leq |f|_{\infty, [0, T]} |g|_{\beta, [0, T]} |t - s|^\beta + (2^{\alpha+\beta} - 2)^{-1} |f|_{\alpha, [0, T]} |g|_{\beta, [0, T]} |t - s|^{\alpha+\beta}. \quad (3)$$

Moreover, when $y : [0, T] \rightarrow \mathbb{R} \in \mathcal{C}^1$ and when $\phi : \mathbb{R}^2 \rightarrow \mathbb{R} \in \mathcal{C}^{1,2}$ is such that $r \mapsto \partial_1 \phi(g_r, y_r)$ is Hölder continuous of order $\beta > 1/2$, then, for any $s, t \in [0, T]$:

$$\phi(g_t, y_t) = \phi(g_s, y_s) + \int_s^t \partial_1 \phi(g_r, y_r) dg_r + \int_s^t \partial_2 \phi(g_r, y_r) y_r' dr. \quad (4)$$

3 First case: when σ does not depend of u

In the sequel, we fix $x_0 \in \mathbb{R}$, $\beta \in (1/2, 1)$, $T \in (0, \infty)$ and $g \in \mathcal{C}^\beta = \mathcal{C}^\beta([0, T])$. We assume moreover that $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is $\mathcal{C}^{1,2}$ with bounded derivatives and that $b : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is bounded and globally Lipschitz, uniformly in $x \in \mathbb{R}$ with respect to $(t, u) \in [0, T] \times \mathbb{R}$.

Theorem 1 *For any measurable control $u : [0, T] \rightarrow \mathbb{R}$, the integral equation*

$$x_t^u = x_0^u + \int_0^t \sigma(r, x_r^u) dg_r + \int_0^t b(r, x_r^u, u_r) dr, \quad t \in [0, T] \quad (5)$$

admits a unique solution $x^u \in \mathcal{C}^0([0, T])$.

Proof of Theorem 1.

- We first prove Theorem 1 in the autonomous case, that is when $\sigma(t, x) = \sigma(x)$ and $b(t, x, u) = b(x, u)$. In other words, we consider

$$x_t^u = x_0^u + \int_0^t \sigma(x_r^u) dg_r + \int_0^t b(x_r^u, u_r) dr, \quad t \in [0, T] \quad (6)$$

instead of (5). At this level, we need a preliminary lemma:

Lemma 1 *Assume that $h : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is such that, for any $R > 0$, there exists $c_R > 0$ verifying*

$$\forall (r, g, u, y, z) \in [0, T] \times [-R, R] \times \mathbb{R}^3 : \quad |h(r, g, u, y) - h(r, g, u, z)| \leq c_R |y - z|, \quad (7)$$

and assume moreover that $u : [0, T] \rightarrow \mathbb{R}$ is a measurable function. Then the integral equation

$$y_t = y_0 + \int_0^t h(r, g_r, u_r, y_r) dr, \quad t \in [0, T] \quad (8)$$

admits a unique solution $y \in C^0([0, T])$.

Proof of Lemma 1. We only sketch the proof, the arguments used being classical.

Existence. Let us define (y^n) recursively by $y^0(t) \equiv y_0$ and

$$y^{n+1}(t) = y_0 + \int_0^t h(r, g(r), u(r), y^n(r)) dr, \quad t \in [0, T].$$

Since g is continuous, there exists $R > 0$ such that $g([0, T]) \subset [-R, R]$. Thus, using the hypothesis made on h , it is classical to prove that $|y^{n+1} - y^n|_\infty \leq \frac{c_R^n}{n!}$. In particular, the sequence (y^n) is Cauchy and the limit y is a solution to (8).

Uniqueness. Let y and z be two solutions of (8). Then, for any $t \in [0, T]$, we easily have

$$|y - z|_{\infty, [0, t]} \leq c_R \int_0^t |y - z|_{\infty, [0, r]} dr$$

and we can conclude that $y = z$ using Gronwall's lemma. □

We now apply the Doss-Sussmann's method in order to finish the proof of Theorem 1 in the autonomous case. First, we denote by ϕ the unique solution to

$$\frac{\partial \phi}{\partial g}(g, y) = \sigma \circ \phi(g, y), \quad \forall g, y \in \mathbb{R} \text{ and } \phi(0, y) = y, \quad \forall y \in \mathbb{R}. \quad (9)$$

The hypothesis made on σ ensures that ϕ is well-defined. We also have, for $g, y \in \mathbb{R}$:

$$\frac{\partial \phi}{\partial y}(g, y) = \exp \left(\int_0^g \sigma'(\phi(h, y)) dh \right).$$

Define $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$f(g, u, y) = \frac{b(\phi(g, y), u)}{\frac{\partial \phi}{\partial y}(g, y)} = b(\phi(g, y), u) \exp \left(- \int_0^g \sigma'(\phi(\ell, y)) d\ell \right).$$

The hypothesis made on b and σ ensures that $h : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $h(r, g, u, y) = f(g, u, y)$ verifies (7). Thus, there exist a unique y solution to (8). Using the change of variable formula (4), it is now immediate to prove that $x_t^u = \phi(g_t, y_t)$ is a solution to (6). For the uniqueness, it suffices to adapt the proof contained in [1], page 103, to our context.

- Since the general case is similar to the previous case, we only sketch the proof. Here, we have to consider ϕ given by

$$\frac{\partial \phi}{\partial g}(r, g, y) = \sigma(r, \phi(r, g, y)), \quad \forall (r, g, y) \in [0, T] \times \mathbb{R}^2 \quad (10)$$

with initial conditions

$$\phi(r, 0, y) = y, \quad \forall (r, y) \in [0, T] \times \mathbb{R}$$

instead of (9). Moreover, $y : [0, T] \rightarrow \mathbb{R}$ is, in the case, defined as the unique solution to (8) with h given by

$$h(r, g, u, y) = \frac{b(r, \phi(r, g, y), u) - \frac{\partial \phi}{\partial r}(r, g, y)}{\frac{\partial \phi}{\partial y}(r, g, y)},$$

see also [1], page 116. Finally, the unique solution to (5) is given by

$$x_t^u = \phi(t, g_t, y_t).$$

□

In order to make use of a compactification method, it is necessary to enlarge the set of controls by considering *relaxed controls*.

Definition 1 *A relaxed control is a measure q over $U \times [0, T]$ such that the projection of q on $[0, T]$ is the Lebesgue measure. We denote by \mathcal{V} the set of relaxed controls.*

A relaxed control q can be decomposed with a measurable kernel: $q(da, dt) = q_t(da)dt$ where $t \mapsto q_t$ is a measurable function from \mathbb{R}^+ to the set of probability measures on U . There is a natural embedding of (non-relaxed) controls in the set of relaxed controls: q is a non-relaxed control if at each time t , q_t concentrates on a single point u_t . In other words, the control $(u_t)_{t \in [0, T]}$ corresponds to the relaxed control $\delta_{u_t}dt$ where δ_x denotes the Dirac measure at x . We denote by \mathcal{V}' the set of non-relaxed controls.

The main result that we shall need is the following immediate consequence of the vague topology.

Proposition 1 *Suppose U is a compact subset of \mathbb{R}^n . The set \mathcal{V} of relaxed controls equipped with the vague topology is compact.*

From now on, we shall suppose that the set U is compact. A solution to equation (5) associated to a relaxed control q is obtained in the following extension of Theorem 1.

Theorem 2 *Let $q \in \mathcal{V}$ be a relaxed control. There exists a unique solution $x^q \in C^0([0, T])$ of the equation*

$$x_t^q = x_0^q + \int_0^t \sigma(r, x_r^q) dg_r + \int_0^t \int_U b(r, x_r^q, a) q_r(da) dr. \quad (11)$$

Moreover, $q \mapsto x^q$ is continuous from \mathcal{V} to $C^0([0, T])$.

Proof. Denote by ϕ the unique solution to (10). Set

$$h(r, g, q, y) = \frac{\int_U b(r, \phi(r, g, y), a) q_r(da) - \frac{\partial \phi}{\partial r}(r, g, y)}{\frac{\partial \phi}{\partial y}(r, g, y)}. \quad (12)$$

Clearly, due to the hypotheses on b and σ , $\forall (r, g, q, y, z) \in [0, T] \times [-R, R] \times \mathcal{V} \times \mathbb{R} \times \mathbb{R}$,

$$|h(r, g, q, y) - h(r, g, q, z)| \leq c_R |y - z|.$$

Therefore, the integral equation (8) admits a unique solution $y \in C^0([0, T])$, see Lemma 1. Then, one may check that $x_t^q = \phi(t, g_t, y_t)$ is a solution to (11). Uniqueness is obtained as before.

Suppose now that q^n is a sequence in \mathcal{V} , converging to $q \in \mathcal{V}$ and let y^n be the solution of (8) associated to $h = h(r, g, q^n, y)$ given by (12). Using the hypotheses on b , we now prove that y^n converges to y in $C^0([0, T])$. Indeed,

$$\begin{aligned}
|y_t - y_t^n| &= \left| \int_0^t [h(s, g_s, q, y_s) - h(s, g_s, q^n, y_s^n)] ds \right| \\
&\leq \left| \int_0^t h(s, g_s, q, y_s) ds - \int_0^t h(s, g_s, q^n, y_s) ds \right| \\
&\quad + \left| \int_0^t h(s, g_s, q^n, y_s) ds - \int_0^t h(s, g_s, q^n, y_s^n) ds \right| \\
&\leq \left| \int_0^t \int_U \frac{b(s, \phi(s, g_s, y_s), a)}{\partial \phi / \partial y(s, g_s, y_s)} q_s^n(da) ds - \int_0^t \int_U \frac{b(s, \phi(s, g_s, y_s), a)}{\partial \phi / \partial y(s, g_s, y_s)} q_s(da) ds \right| \\
&\quad + c_R \int_0^t |y_s - y_s^n| ds.
\end{aligned}$$

In the last expression, the first term tends to 0 due to the vague convergence of $q_s^n(da)ds$ to $q_s(da)ds$, and the continuity and boundedness hypotheses on b . It results therefore from Gronwall's lemma that $|y - y^n|_\infty$ tends to 0. Finally, as the solution x^q (resp. x^{q^n}) of (11) associated to q (resp. q^n) is given by $x_t = \phi(t, g_t, y_t)$ (resp. $x_t^n = \phi(t, g_t, y_t^n)$), one easily deduces that $|x - x^n|_\infty$ tends to 0. \square

Consider now a cost in integral form: for u_t a given control taking values in U , we set

$$J(u) = \int_0^T \ell(r, x_r^u, u_r) dr$$

where $\ell : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is bounded and continuous. This definition can be immediately extended to the case of relaxed controls: if q is a relaxed control from \mathcal{V} , then

$$J(q) = \int_0^T \int_U \ell(r, x_r^q, a) q_r(da) dr.$$

Using the continuity property of Theorem 2, and the hypotheses on ℓ , one obtains the following Proposition.

Proposition 2 *Under the hypotheses of the present section, the application $q \mapsto J(q)$ is continuous on \mathcal{V} .*

The set \mathcal{V} being compact, one immediately deduces the following existence result.

Corollary 1 *Under the prevailing hypotheses, there exists $q^* \in \mathcal{V}$ such that*

$$J(q^*) = \inf_{q \in \mathcal{V}} J(q).$$

We conclude the present section by proving that one has not enlarged too much the control problem by considering relaxed controls. More precisely, we now prove that the optimal cost (i.e. the infimum of the cost functional) over the relaxed and non-relaxed controls is the same. This result is obtained, as in the case of ordinary differential equations, by means of approximation of relaxed controls by *step constant* relaxed controls, and then by non-relaxed controls via the so-called *chattering lemma*, a method originally introduced in [4]. Here, we only sketch these two steps.

First step : $q \in \mathcal{V}$ is approximated by relaxed controls of the form

$$\sum_{j=0}^{N-1} \sum_{i=1}^k q_i^j \delta_{a_i}(da) \mathbf{1}_{[t_j, t_{j+1}[}$$

where $0 = t_0 < t_1 < \dots < t_N = T$, a_1, \dots, a_k are elements in U , and for each $j = 0, \dots, N-1$, q_1^j, \dots, q_k^j are non-negative real numbers such that $\sum_{i=1}^k q_i^j = 1$. This is a straightforward consequence of approximation of the measurable function $t \mapsto q_t$ by a step function and of approximation of a probability measure μ on U by point measures of the form $\sum_{i=1}^m \mu_i \delta_{a_i}$.

Second step: Recall the chattering lemma (see [4], Theorem 1)

Proposition 3 *Let a_1, \dots, a_k be in U and q_1, \dots, q_k be non-negative real numbers such that $\sum_{i=1}^k q_i = 1$. Let f be a bounded continuous function from $[s, t] \times U$ to \mathbb{R} . Then, for $\varepsilon > 0$ given, there exists a measurable partition V_1, \dots, V_k of $[s, t]$ such that*

$$\left| \int_s^t \sum_{i=1}^k q_i f(r, a_i) dr - \sum_{i=1}^k \int_{V_i} f(r, a_i) dr \right| < \varepsilon.$$

The previous result implies that any step-relaxed control $(\sum_{i=1}^k q_i \delta_{a_i}(da)) dr$ is the limit in \mathcal{V} (equipped with the vague topology of measures on $[0, T] \times U$) of a sequence of controls of the form $\sum_{i=1}^k \mathbf{1}_{V_i}(r) \delta_{a_i}(da) dr$: these controls belong to \mathcal{V}' , the set of non-relaxed controls. Since, from the first step, we already know that any $q \in \mathcal{V}$ is the limit of step-relaxed controls, we deduce that, for any $q \in \mathcal{V}$, there exists a sequence of (non-relaxed) controls q^n which converges to q in the vague topology of $[0, T] \times U$. Therefore, since the function J is continuous over \mathcal{V} , its infimum on \mathcal{V}' is smaller than the infimum on \mathcal{V} . Since the other inequality is obviously satisfied, we obtain the following comparison result.

Proposition 4 *Under the hypotheses of the present section, $\inf_{q \in \mathcal{V}} J(q) = \inf_{q \in \mathcal{V}'} J(q)$.*

4 Second case: when σ depends of u

As already mentioned in the introduction, the case when u enters the coefficient of dg_t seems to be much more complicated as we do not have a reasonable way to integrate functions, which are only measurable, with respect to function, which has at most Hölderian regularity. Therefore we need to restrict very strongly our admissible controls set.

In the sequel, we fix $x_0 \in \mathbb{R}$, $\beta \in (1/2, 1)$, $T \in (0, \infty)$, $g \in \mathcal{C}^\beta = \mathcal{C}^\beta([0, T])$, $\sigma : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R} \in \mathcal{C}^{1,2,2}$ with bounded derivatives and $b : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ globally Lipschitz continuous.

Theorem 3 *For any control $u \in \mathcal{C}^\mu = \mathcal{C}^\mu([0, T])$ with $1 - \beta < \mu < \beta$, the integral equation*

$$x_t^u = x_0^u + \int_0^t \sigma(r, x_r^u, u_r) dg_r + \int_0^t b(r, x_r^u, u_r) dr, \quad t \in [0, T] \quad (13)$$

admits a unique solution $x^u \in \mathcal{C}^\mu$. Moreover, the application $T^\mu : \mathcal{C}^\mu \rightarrow \mathcal{C}^\mu$ defined by $T^\mu(u) = x^u$ is continuous.

Actually, in equations like (13), the drift term $\int_0^t b(r, x_r^u, u_r) dr$ is usually harmless, but causes some cumbersome notations. Thus, for sake of simplicity, we will only make the proof of Theorem 3 in the case where $b \equiv 0$. Moreover, still for sake of simplicity, we will also assume that we are in the autonomous case, that is $\sigma(r, x, u) = \sigma(x, u)$. The proof in the general case is similar.

We introduce $\Gamma^\mu : \mathcal{C}^\mu \times \mathcal{C}^\mu \rightarrow \mathcal{C}^\beta \subset \mathcal{C}^\mu$ defined by $\Gamma^\mu(x, u) = \hat{x}^u$ where

$$\hat{x}_t^u = x_0 + \int_0^t \sigma(x_r, u_r) dg_r, \quad t \in [0, T].$$

We give several lemmata:

Lemma 2 *For any $x, y, u, v \in \mathcal{C}^\mu$, we have*

$$\begin{aligned} |\sigma(x, u) - \sigma(x, v)| &\leq |\sigma'|_\infty (|x - y|_\mu + |u - v|_\mu) \\ &\quad + |\sigma''|_\infty (|x|_\mu + |y|_\mu + |u|_\mu + |v|_\mu) (|x - y|_\infty + |u - v|_\infty). \end{aligned}$$

Proof. We have

$$\sigma(x_t, u_t) - \sigma(y_t, v_t) = (x_t - y_t)I^1(t) + (u_t - v_t)I^2(t),$$

for

$$I^i(t) = \int_0^1 \partial_i \sigma(rx_t + (1-r)y_t, ru_t + (1-r)v_t) dr, \quad i = 1, 2.$$

Thus,

$$\begin{aligned} & |\sigma(x_t, u_t) - \sigma(y_t, v_t) - \sigma(x_s, u_s) + \sigma(y_s, v_s)| \\ = & |(x_t - y_t - x_s + y_s)I_1(t) + (x_s - y_s)(I_1(t) - I_1(s)) \\ & + (u_t - v_t - u_s + v_s)I_2(t) + (u_s - v_s)(I_2(t) - I_2(s))| \\ \leq & |x - y|_\mu |\sigma'|_\infty |t - s|^\mu + |x - y|_\infty |\sigma''|_\infty (|x|_\mu + |y|_\mu + |u|_\mu + |v|_\mu) |t - s|^\mu \\ & + |u - v|_\mu |\sigma'|_\infty |t - s|^\mu + |u - v|_\infty |\sigma''|_\infty (|x|_\mu + |y|_\mu + |u|_\mu + |v|_\mu) |t - s|^\mu. \end{aligned}$$

The desired conclusion follows. \square

Lemma 3 For any $x, y, u, v \in \mathcal{C}^\mu$, we have

$$\begin{aligned} |\hat{x}^u - \hat{y}^v|_\infty & \leq |\sigma'|_\infty (|x - y|_\infty + |u - v|_\infty) |g|_\beta T^\beta \\ & + (2^{\beta+\mu} - 2)^{-1} |g|_\beta |\sigma'|_\infty (|x - y|_\mu + |u - v|_\mu) T^{\beta+\mu} \\ & + (2^{\beta+\mu} - 2)^{-1} |g|_\beta |\sigma''|_\infty (|x|_\mu + |y|_\mu + |u|_\mu + |v|_\mu) (|x - y|_\infty + |u - v|_\infty) T^{\beta+\mu}. \end{aligned}$$

Proof. We have

$$\begin{aligned} |\hat{x}_t^u - \hat{y}_t^v| & = \left| \int_0^t (\sigma(x_r, u_r) - \sigma(y_r, v_r)) dg_r \right| \\ & \leq |\sigma(x, u) - \sigma(y, v)|_\infty |g|_\beta T^\beta \\ & \quad (2^{\beta+\mu} - 2)^{-1} |\sigma(x, u) - \sigma(y, v)|_\mu |g|_\beta T^{\beta+\mu}. \end{aligned}$$

Combined with the previous lemma, we obtain the desired conclusion. \square

Lemma 4 For any $x, y, u, v \in \mathcal{C}^\mu$, we have

$$\begin{aligned} |\hat{x}^u - \hat{y}^v|_\mu & \leq |\sigma'|_\infty (|x - y|_\infty + |u - v|_\infty) |g|_\beta T^{\beta-\mu} \\ & + (2^{\beta+\mu} - 2)^{-1} |g|_\beta |\sigma'|_\infty (|x - y|_\mu + |u - v|_\mu) T^\beta \\ & + (2^{\beta+\mu} - 2)^{-1} |g|_\beta |\sigma''|_\infty (|x|_\mu + |y|_\mu + |u|_\mu + |v|_\mu) (|x - y|_\infty + |u - v|_\infty) T^\beta. \end{aligned}$$

Proof. We have

$$|\hat{x}_t^u - \hat{y}_t^v - \hat{x}_s^u + \hat{y}_s^v| = \left| \int_s^t (\sigma(x_r, u_r) - \sigma(y_r, v_r)) dg_r \right|.$$

We conclude as in the proof of Lemma 3.

□

We are now in position to make the proof of Theorem 3:

Proof of Theorem 3. We adapt the proof by Ruzmaikina [10] to our context, *i.e.* we take into account the control u . Since there is no major differences, we only sketch the proof.

1) *Existence of an invariant ball.* Fix $u \in \mathcal{C}^\mu$ and let us first consider an interval of the form $[0, \varepsilon]$. For $x \in \mathcal{C}^\mu$, we have, by lemmata 3 and 4:

$$|\hat{x}^u|_{\infty, \mu, [0, \varepsilon]} \leq c_{\sigma, g} (|x|_{\infty, \mu, [0, \varepsilon]} + |u|_{\infty, \mu, [0, T]}) |x|_{\infty, \mu, [0, \varepsilon]} \varepsilon^{\beta - \mu}, \quad (14)$$

for $c_{\sigma, g}$ a constant depending only on σ (and its derivatives) and g . Let us choose

$$\varepsilon = \left(\frac{1}{c_{\sigma, g}(1 + |u|_\mu)} \right)^{\frac{1}{\beta - \mu}}. \quad (15)$$

Then, observe that (14) implies that the unit ball $B = \{x \in \mathcal{C}^\mu : |x|_{\infty, \mu, [0, \varepsilon]} \leq 1\}$ is invariant by $\Gamma(\cdot, u)$.

2) *Fixed point argument.* Since we are now working in B , the fixed point argument for $\Gamma(\cdot, u)$ is a standard argument and is left to the reader. This leads to a unique solution to equation (13) (for $\sigma(r, x, u) = \sigma(x, u)$ and $b \equiv 0$) on a small interval $[0, \tau]$. One is then able to obtain the unique solution on an arbitrary interval $[0, k\tau]$ with $k \geq 1$ by concatenating solutions on $[j\tau, (j+1)\tau]$. Notice here that an important point, which allows us to use a constant step ε , is the fact that (15) does not depend on the initial condition.

3) *Continuity.* Once the existence of a unique solution x^u to (13) is proved, the continuity of this solution with respect to u can be deduced, by standard arguments, from lemmata 3 and 4 (first on $[0, \tau]$ and then on the whole interval $[0, T]$). □

Theorem 4 *Let μ, μ' be such that $1 - \beta < \mu < \mu' < \beta$. If \mathcal{U} is a bounded subset of \mathcal{C}^μ and if $J : \mathcal{U} \rightarrow \mathbb{R}$ is continuous for $|\cdot|_{\infty, \mu'}$, then the following control problem can be solved:*

$$\text{there exists } u^* \in \mathcal{U} \text{ realizing } \inf_{u \in \mathcal{U}} J(u).$$

Proof of Theorem 4. According to Lamperti [7], we know that \mathcal{U} is relatively compact in $\mathcal{C}^{\mu'}$. The continuity of T^μ in Theorem 3 implies that the set of all couples $(u, x^u) \in \mathcal{U} \times \mathcal{C}^\mu$ is relatively compact in $\mathcal{C}^{\mu'} \times \mathcal{C}^{\mu'}$. The desired conclusion follows by the continuity property of J . □

An example of a cost J satisfying the conditions of Theorem 4 is

$$J(u) = \int_0^T \ell(r, x_r^u, u_r) dr$$

with $\ell : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ verifying

$$\forall (r, x, y, u, v) \in [0, T] \times \mathbb{R}^4, \quad |\ell(r, x, u) - \ell(r, y, v)| \leq \text{cst} (|x - y| + |u - v|).$$

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