SOME INSIGHTS ON BICATEGORIES OF FRACTIONS - I -

ASSOCIATORS, VERTICAL AND HORIZONTAL COMPOSITIONS

MATTEO TOMMASINI

Abstract. In this paper we investigate the construction of bicategories of fractions originally described by D. Pronk: given any bicategory $\mathscr{C}$ together with a suitable class of morphisms $\mathbf{W}$, one can construct a bicategory $\mathscr{C}[\mathbf{W}^{-1}]$, where all the morphisms of $\mathbf{W}$ are turned into internal equivalences, and that is universal with respect to this property. Most of the descriptions leading to such a construction were long and heavily based on the axiom of choice. In this paper we simplify considerably the constructions of associators, vertical and horizontal compositions in a bicategory of fractions, thus proving that the axiom of choice is not needed under certain conditions. The simplified description of associators and 2-compositions will also play a crucial role in the next papers of this series.

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Introduction

In 1996 Dorette Pronk introduced the notion of (*right*) *bicalculus of fractions* (see [Pr]), generalizing the concept of (right) calculus of fractions (described in 1967 by Pierre Gabriel and Michel Zisman, see [GZ]) from the framework of categories to that of bicategories. To be more precise, given any bicategory $\mathscr{C}$ and any class $\mathbf{W}$ of 1-morphisms in it, one considers the following set of axioms (where the 2-morphisms $\theta_\bullet$'s are the associators of $\mathscr{C}$):

(BF1) for every object A of $\mathscr{C}$, the 1-identity id_A belongs to $\mathbf{W}$;

Date: November 24, 2014.

2010 *Mathematics Subject Classification.* 18A05, 18A30, 22A22.

Key words and phrases. bicategories of fractions, bicalculus of fractions, pseudofunctors.

I would like to thank Dorette Pronk for several interesting discussions about her work on bicategories of fraction and for some useful suggestions about this series of papers. This research was performed at the Mathematics Research Unit of the University of Luxembourg, thanks to the grant 4773242 by Fonds National de la Recherche Luxembourg.

(BF2) $\mathbf{W}$ is closed under compositions;

(BF3) for every morphism $\mathrm{w} : A \to B$ in $\mathbf{W}$ and for every morphism $f : C \to B$, there are an object D, a morphism $\mathrm{w}' : D \to C$ in $\mathbf{W}$, a morphism $f' : D \to A$ and an invertible 2-morphism $\alpha : f \circ \mathrm{w}' \Rightarrow \mathrm{w} \circ f'$;

(BF4) (a) given any morphism $\mathrm{w} : B \to A$ in $\mathbf{W}$, any pair of morphisms $f^1, f^2 : C \to B$ and any 2-morphism $\alpha : \mathrm{w} \circ f^1 \Rightarrow \mathrm{w} \circ f^2$, there are an object D, a morphism $\mathrm{v} : D \to C$ in $\mathbf{W}$ and a 2-morphism $\beta : f^1 \circ \mathrm{v} \Rightarrow f^2 \circ \mathrm{v}$, such that $\alpha * i_{\mathrm{v}} = \theta_{\mathrm{w},f^2,\mathrm{v}} \odot (i_{\mathrm{w}} * \beta) \odot \theta^{-1}_{\mathrm{w},f^1,\mathrm{v}}$;

(b) if α in (a) is invertible, then so is β;

(c) if $(D', \mathrm{v}' : D' \to C, \beta' : f^1 \circ \mathrm{v}' \Rightarrow f^2 \circ \mathrm{v}')$ is another triple with the same properties of (D, v, β) in (a), then there are an object E, a pair of morphisms $\mathrm{u} : E \to D$, $\mathrm{u}' : E \to D'$ and an invertible 2-morphism $\zeta : \mathrm{v} \circ \mathrm{u} \Rightarrow \mathrm{v}' \circ \mathrm{u}'$, such that $\mathrm{v} \circ \mathrm{u}$ belongs to $\mathbf{W}$ and

$$\theta^{-1}_{f^2,\mathrm{v}',\mathrm{u}'} \odot \Big(\beta' * i_{\mathrm{u}'}\Big) \odot \theta_{f^1,\mathrm{v}',\mathrm{u}'} \odot \Big(i_{f^1} * \zeta\Big) =$$
$$= \Big(i_{f^2} * \zeta\Big) \odot \theta^{-1}_{f^2,\mathrm{v},\mathrm{u}} \odot \Big(\beta * i_{\mathrm{u}}\Big) \odot \theta_{f^1,\mathrm{v},\mathrm{u}}.$$

(BF5) if $\mathrm{w} : A \to B$ is a morphism in $\mathbf{W}$, $\mathrm{v} : A \to B$ is any morphism and if there is an invertible 2-morphism $\mathrm{v} \Rightarrow \mathrm{w}$, then also v belongs to $\mathbf{W}$.

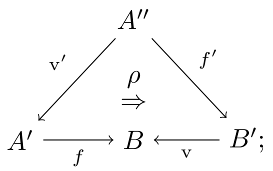
The pair $(\mathscr{C}, \mathbf{W})$ is said to *admit a (right) bicalculus of fractions* if all such conditions are satisfied (actually in [Pr, § 2.1] the first condition is slightly more restrictive, but it is not necessary, for all the constructions in that paper). Pronk proved that if this is the case, then there are a bicategory $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ (called (*right*) *bicategory of fractions*) and a pseudofunctor $\mathcal{U}_{\mathbf{W}} : \mathscr{C} \to \mathscr{C}\left[\mathbf{W}^{-1}\right]$, satisfying a universal property (see [Pr, Theorem 21]). For the description of the morphisms and 2-morphisms of $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ we refer directly to § 1.1 (this is also useful in order to understand all the notations used in the results below).

The aim of this paper is to considerably simplify most of the constructions leading to a bicategory of fractions in [Pr]. We recall that the original construction in [Pr, § 2.2, § 2.3 and Appendix] for associators and compositions of 2-morphisms depended on the following choices (both are in general non-unique since axioms (BF3) and (BF4) do not ensure uniqueness):

C($\mathbf{W}$): for every set of data in $\mathscr{C}$ as follows

$$A' \xrightarrow{\;f\;} B \xleftarrow{\;\mathrm{v}\;} B' \tag{0.1}$$

with v in $\mathbf{W}$, using axiom (BF3) we *choose* an object A'', a pair of morphisms v' in $\mathbf{W}$ and f' and an invertible 2-morphism ρ in $\mathscr{C}$, as follows:



(0.2)

D($\mathbf{W}$): given any morphism $\mathrm{w} : B \to A$ in $\mathbf{W}$, any pair of morphisms $f^1, f^2 : C \to B$ and any 2-morphism $\alpha : \mathrm{w} \circ f^1 \Rightarrow \mathrm{w} \circ f^2$, we *choose* a morphism $\mathrm{v} : D \to C$ in $\mathbf{W}$ and a 2-morphism $\beta : f^1 \circ \mathrm{v} \Rightarrow f^2 \circ \mathrm{v}$ as in (BF4a).

Having fixed all such choices, the description of associators and 2-compositions were very long and they did not allow not much freedom on some additional choices done at each step of the construction (see the explicit descriptions in the next pages). Different sets of choices (C($\mathbf{W}$),D($\mathbf{W}$)) apparently lead to different bicategories of fractions (with the same objects, morphisms and 2-morphisms, but different compositions and associators). Such different bicategories are a priori only weakly equivalent (by the already mentioned [Pr, Theorem 21]).

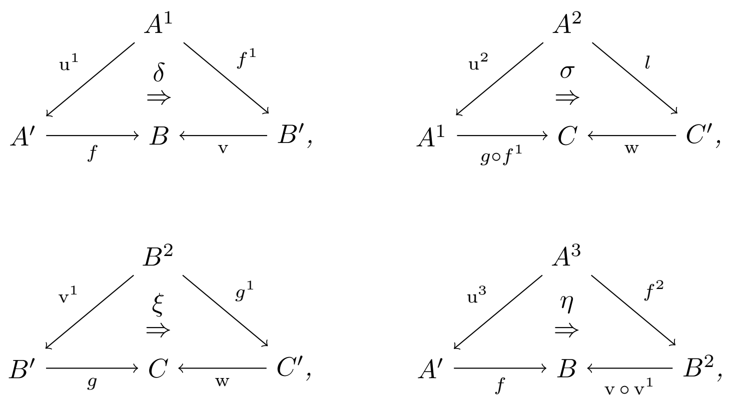
In the present paper we will prove that actually *choices* D($\mathbf{W}$) *are not necessary.* This will be a consequence of the following 4 propositions, that are of independent interest because they allow to simplify considerably the constructions of associators and vertical and horizontal compositions in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$. Indeed in [Pr] each intermediate datum (F1) – (F10) mentioned in the next propositions should be obtained as a result of a (sometimes long) procedure involving the mentioned choices C($\mathbf{W}$) and D($\mathbf{W}$). The next 4 propositions show that actually each such datum can vary in a much wide range, thus allowing more flexibility in the computations of associators and 2-compositions whenever it is necessary (the resulting constructions are still long, but considerably shorter than the original ones in [Pr]). This will serve as a key ingredient in some proofs in the next 2 papers of this series ([T1] and [T2]).

We remark that the choices (F1) – (F10) below actually vary in a a non-empty set: for (F1) – (F3) we refer to Remark 2.1, for the remaining data this is an easy consequence either of the axioms (BF) mentioned above, or of Lemmas 1.2 and 1.3 in § 1.1 below. In the next 4 propositions we assume all the time that $(\mathscr{C},\mathbf{W})$ is a pair satisfying conditions (BF).

Proposition 0.1. (associators of $\mathscr{C}\left[\mathbf{W}^{-1}\right]$) *Let us fix any triple of* 1*-morphisms in* $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ *as follows:*

$$\begin{aligned}
\underline{f} &:= \Big(A \xleftarrow{\;\mathrm{u}\;} A' \xrightarrow{\;f\;} B\Big),\\
\underline{g} &:= \Big(B \xleftarrow{\;\mathrm{v}\;} B' \xrightarrow{\;g\;} C\Big),\\
\underline{h} &:= \Big(C \xleftarrow{\;\mathrm{w}\;} C' \xrightarrow{\;h\;} D\Big).
\end{aligned} \tag{0.3}$$

Let us suppose that the fixed choices C($\mathbf{W}$) *give data as in the upper parts of the following diagrams* (*starting from the ones on the left*)*, with* u^1*,* u^2*,* v^1 *and* u^3 *in* $\mathbf{W}$ *and* δ*,* σ*,* ξ *and* η *invertible:*



(0.4)

so that by [Pr, § 2.2] *one has*

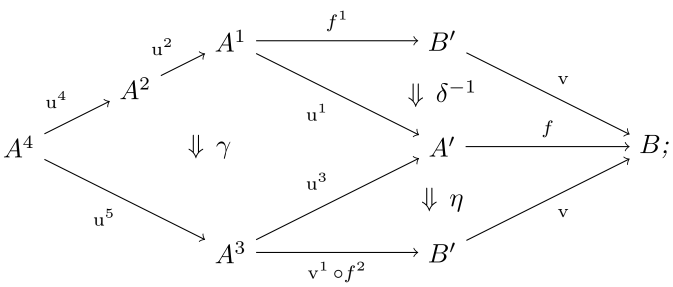
$$\underline{h} \circ \big(\underline{g} \circ \underline{f}\big) = \Big(A \xleftarrow{(\mathrm{u} \circ \mathrm{u}^1) \circ \mathrm{u}^2} A^2 \xrightarrow{h \circ l} D\Big), \tag{0.5}$$

$$\big(\underline{h} \circ \underline{g}\big) \circ \underline{f} = \Big(A \xleftarrow{\mathrm{u} \circ \mathrm{u}^3} A^3 \xrightarrow{(h \circ g^1) \circ f^2} D\Big). \tag{0.6}$$

Then let us fix any *set of choices as follows:*

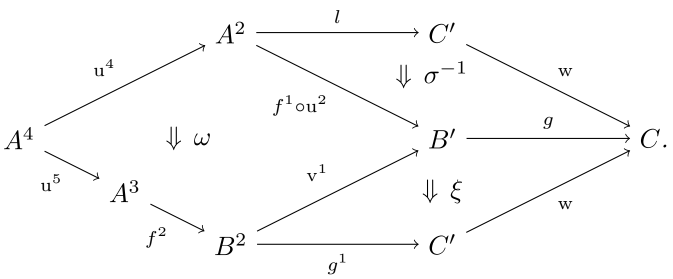
(*F1*) *an object A^4, a morphism $\mathrm{u}^4 : A^4 \rightarrow A^2$ in $\mathbf{W}$, a morphism $\mathrm{u}^5 : A^4 \rightarrow A^3$ and an invertible 2-morphism $\gamma : \mathrm{u}^1 \circ (\mathrm{u}^2 \circ \mathrm{u}^4) \Rightarrow \mathrm{u}^3 \circ \mathrm{u}^5$;*

(*F2*) *an invertible 2-morphism $\omega : f^1 \circ (\mathrm{u}^2 \circ \mathrm{u}^4) \Rightarrow \mathrm{v}^1 \circ (f^2 \circ \mathrm{u}^5)$, such that $i_{\mathrm{v}} * \omega$ coincides with the following composition (associators of $\mathscr{C}$ omitted for simplicity):*



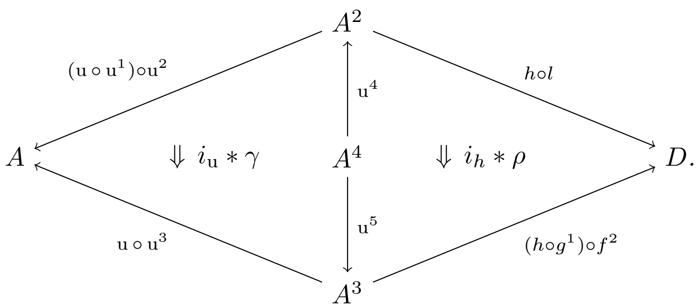
(0.7)

(*F3*) *an invertible 2-morphism $\rho : l \circ \mathrm{u}^4 \Rightarrow g^1 \circ (f^2 \circ \mathrm{u}^5)$, such that $i_{\mathrm{w}} * \rho$ coincides with the following composition (associators of $\mathscr{C}$ omitted):*



(0.8)

Then the associator $\Theta^{\mathscr{C},\mathbf{W}}_{\underline{h},\underline{g},\underline{f}}$ from (0.5) *to* (0.6) *is given by the class of the following diagram (associators of $\mathscr{C}$ omitted):*



(0.9)

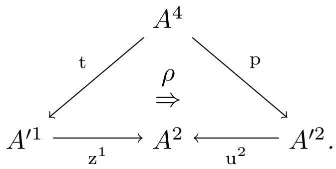
Therefore, $\Theta^{\mathscr{C},\mathbf{W}}_{\underline{h},\underline{g},\underline{f}}$ depends only on the 4 choices of type $\mathrm{C}(\mathbf{W})$ giving the 4 diagrams in (0.4); in particular, each associator of $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ *does not depend on choices* $\mathrm{D}(\mathbf{W})$.

Proposition 0.2. (vertical compositions) *Let us fix any pair of objects A, B, any triple of morphisms $\underline{f}^m := (A^m, \mathrm{w}^m, f^m) : A \to B$ for $m = 1, 2, 3$ and any pair of 2-morphisms in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$*

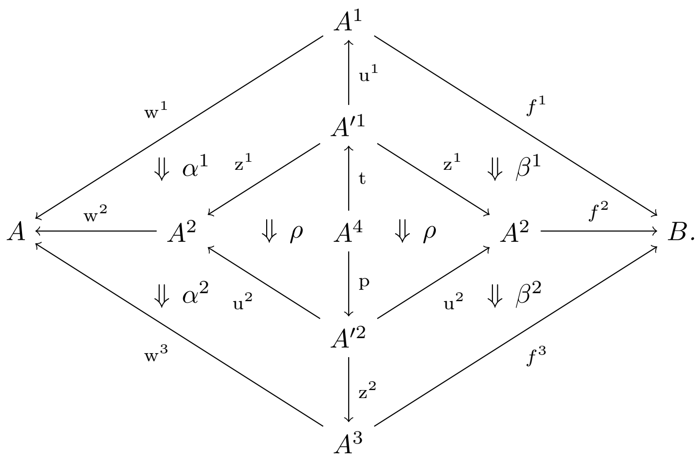
$$\Gamma^m := \Big[A'^m, \mathrm{u}^m, \mathrm{z}^m, \alpha^m, \beta^m\Big] : \underline{f}^m \Longrightarrow \underline{f}^{m+1} \quad \textit{for} \quad m = 1, 2.$$

Then let us choose any *data as follows:*

(F4) an object A^4, a morphism t in $\mathbf{W}$, a morphism p and an invertible 2-morphism ρ as below:



Then $\Gamma^2 \odot \Gamma^1$ is represented by the following diagram:



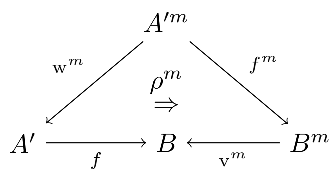
(0.10)

In particular, *vertical compositions in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ do not depend on choices* $\mathrm{C}(\mathbf{W})$ *and* $\mathrm{D}(\mathbf{W})$.

Proposition 0.3. (horizontal compositions with 1-morphisms on the left) *Let us fix any morphism $\underline{f} := (A', \mathrm{w}, f) : A \to B$, any pair of morphisms $\underline{g}^m := (B^m, \mathrm{v}^m, g^m) : B \to C$ for $m = 1, 2$, any 2-morphism*

$$\Delta := \Big[B^3, \mathrm{u}^1, \mathrm{u}^2, \alpha, \beta\Big] : \underline{g}^1 \Longrightarrow \underline{g}^2$$

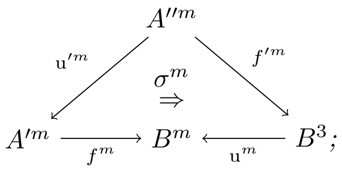
in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ and let us suppose that for each $m = 1, 2$, choices $\mathrm{C}(\mathbf{W})$ *give data as in the upper part of the following diagram, with w^m in $\mathbf{W}$ and ρ^m invertible*



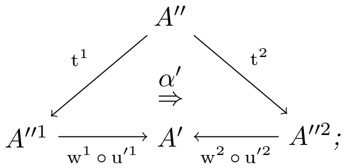
(0.11)

(*so that by* [Pr, § 2.2] *we have* $\underline{g^m} \circ \underline{f} = (A'^m, \mathrm{w} \circ \mathrm{w}^m, g^m \circ f^m)$ *for* $m = 1, 2$). *Then let us fix* any *set of choices as follows:*

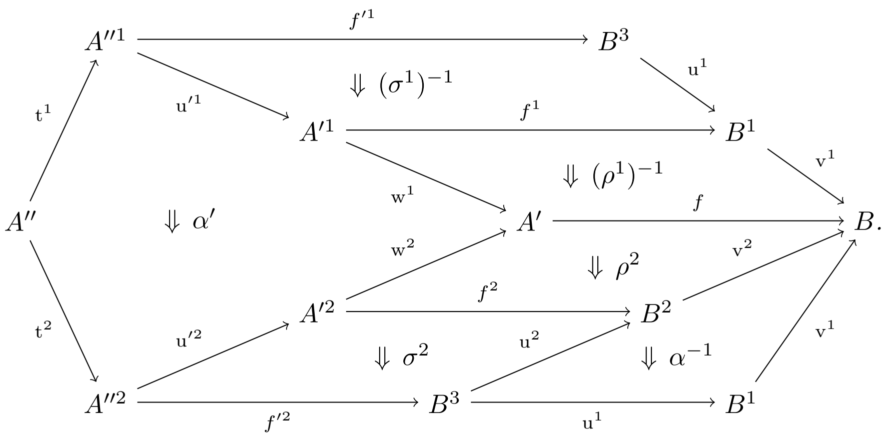
($F5$) *for each* $m = 1, 2$, *we choose data as in the upper part of the following diagram, with* u'^m *in* $\mathbf{W}$ *and* σ^m *invertible:*



($F6$) *we choose any set of data as in the upper part of the following diagram, with* t^1 *in* $\mathbf{W}$ *and* α' *invertible:*

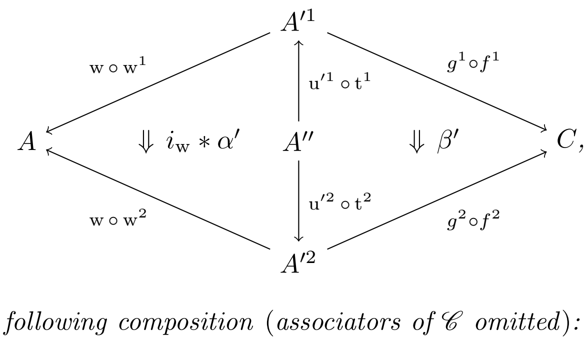


($F7$) *we choose any invertible* 2*-morphism* $\delta : f'^1 \circ \mathrm{t}^1 \Rightarrow f'^2 \circ \mathrm{t}^2$, *such that* $i_{\mathrm{v}^1 \circ \mathrm{u}^1} * \delta$ *coincides with the following composition* (*associators of* $\mathscr{C}$ *omitted*)*:*



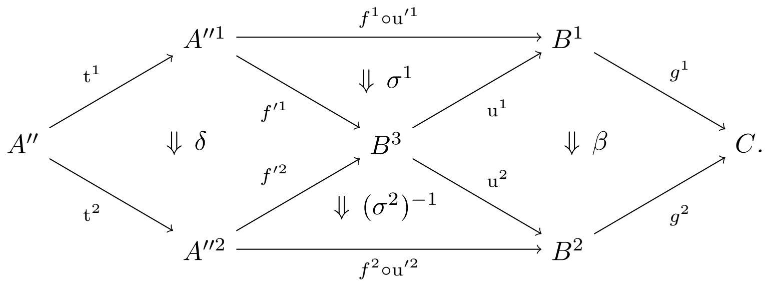
(0.12)

Then $\Delta * i_{\underline{f}}$ *is represented by the following diagram* (*associators of* $\mathscr{C}$ *omitted*)



(0.13)

where β' *is the following composition* (*associators of* $\mathscr{C}$ *omitted*)*:*

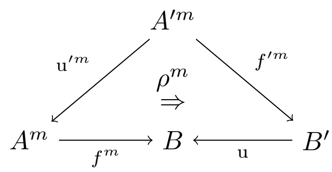


Therefore, each composition of the form $\Delta * i_{\underline{f}}$ depends only on the 2 choices $\mathrm{C}(\mathbf{W})$ giving diagram (0.11) for $m = 1, 2$; in particular, *it does not depend on choices* $\mathrm{D}(\mathbf{W})$.

Proposition 0.4. (horizontal compositions with 1-morphisms on the right) *Let us fix any morphism* $\underline{g} = (B', \mathrm{u}, g) : B \to C$*, any pair of morphisms* $\underline{f}^m := (A^m, \mathrm{w}^m, f^m) : A \to B$ *for* $m = 1, 2$*, any 2-morphism*

$$\Gamma := \Big[A^3, \mathrm{v}^1, \mathrm{v}^2, \alpha, \beta\Big] : \underline{f}^1 \Longrightarrow \underline{f}^2$$

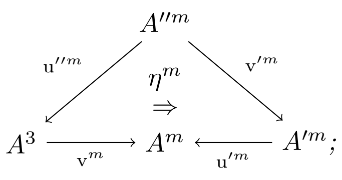
in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ *and let us suppose that for each* $m = 1, 2$*, choices* $\mathrm{C}(\mathbf{W})$ *give data as in the upper part of the following diagram, with* u'^m *in* $\mathbf{W}$ *and* ρ^m *invertible*



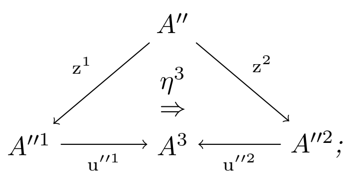
(0.14)

(so that by [Pr, § 2.2] *we have* $\underline{g} \circ \underline{f}^m = (A'^m, \mathrm{w}^m \circ \mathrm{u}'^m, g \circ f'^m)$ *for* $m = 1, 2$*). Then let us fix* any *set of choices as follows:*

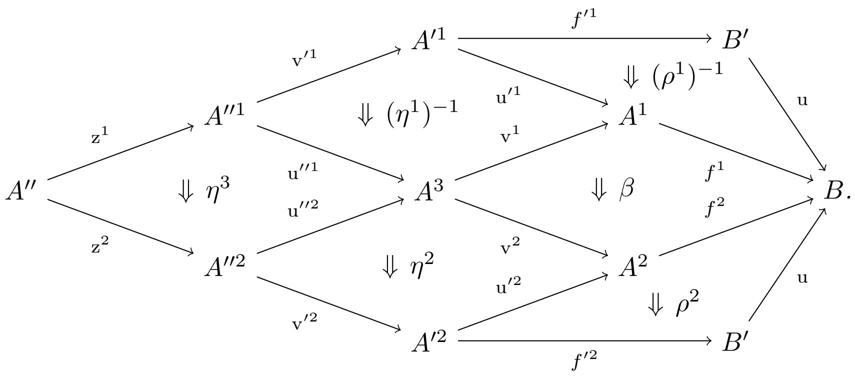
(*F8*) *for each* $m = 1, 2$*, we choose data as in the upper part of the following diagram, with* u''^m *in* $\mathbf{W}$ *and* η^m *invertible:*



(*F9*) *we choose data as in the upper part of the following diagram, with* z^1 *in* $\mathbf{W}$ *and* η^3 *invertible:*

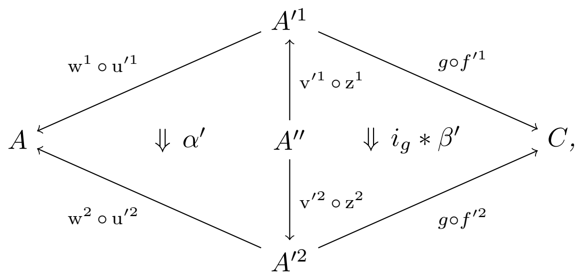


(*F10*) *we choose any 2-morphism* $\beta' : f'^1 \circ (\mathrm{v}'^1 \circ \mathrm{z}^1) \Rightarrow f'^2 \circ (\mathrm{v}'^2 \circ \mathrm{z}^2)$*, such that* $i_{\mathrm{u}} * \beta'$ *coincides with the following composition (associators of* $\mathscr{C}$ *omitted):*



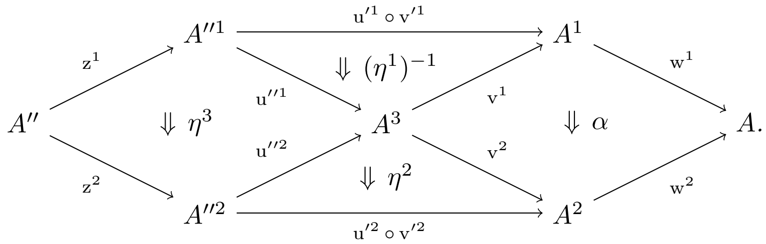
(0.15)

Then $i_{\underline{g}} * \Gamma$ is represented by the following diagram (associators of $\mathscr{C}$ omitted):



(0.16)

where α' is the following composition (associators of $\mathscr{C}$ omitted):



So each composition of the form $i_{\underline{g}} * \Gamma$ depends only on the 2 choices $\mathrm{C}(\mathbf{W})$ giving diagram (0.14) for $m = 1, 2$; in particular, *it does not depend on choices* $\mathrm{D}(\mathbf{W})$.

Since each horizontal composition in any bicategory can be obtained as a suitable combination of vertical compositions and compositions of the form $\Delta * i_{\underline{f}}$ and $i_{\underline{g}} * \Gamma$, then Propositions 0.2, 0.3 and 0.4 prove immediately that *horizontal compositions in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ do not depend on choices* $\mathrm{D}(\mathbf{W})$. This together with Propositions 0.1 and 0.2 implies at once that:

Theorem 0.5. (the structure of $\mathscr{C}\left[\mathbf{W}^{-1}\right]$) *Let us fix any pair $(\mathscr{C}, \mathbf{W})$ satisfying conditions* (BF)*. Then the construction of $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ depends only on choices* $\mathrm{C}(\mathbf{W})$*, i.e. different sets of choices* $\mathrm{D}(\mathbf{W})$ *(for choices* $\mathrm{C}(\mathbf{W})$ *fixed) give the* same *bicategory of fractions, instead of only equivalent ones.*

In particular, we get:

Corollary 0.6. *Let us suppose that for each pair (f, v) with v in $\mathbf{W}$ as in* (0.1) *there is a* unique choice *of $(A'', \mathrm{v}', f', \rho)$ as in* $\mathrm{C}(\mathbf{W})$*. Then the construction of*

$\mathscr{C}\left[\mathbf{W}^{-1}\right]$ does not depend on the axiom of choice. The same result holds if "unique choice" above is replaced by "canonical choice".

As a side result of the technical lemmas used in this paper, we will also prove the following 2 useful statements.

Proposition 0.7. (comparison of 2-morphisms in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$) *Let us fix any pair $(\mathscr{C}, \mathbf{W})$ satisfying conditions* (BF)*, any pair of objects A, B, any pair of morphisms $\underline{f}^m := (A^m, \mathrm{w}^m, f^m) : A \to B$ for $m = 1, 2$ and any pair of 2-morphisms $\Gamma^1, \Gamma^2 : \underline{f}^1 \Rightarrow \underline{f}^2$ in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$. Then there are an object A^3, a pair of morphisms $\mathrm{v}^m : A^3 \to A^m$ for $m = 1, 2$, an invertible 2-morphism $\alpha : \mathrm{w}^1 \circ \mathrm{v}^1 \Rightarrow \mathrm{w}^2 \circ \mathrm{v}^2$ and a pair of 2-morphisms $\gamma^1, \gamma^2 : f^1 \circ \mathrm{v}^1 \Longrightarrow f^2 \circ \mathrm{v}^2$, such that $\mathrm{w}^1 \circ \mathrm{v}^1$ belongs to $\mathbf{W}$ and*

$$\Gamma^m = \left[A^3, \mathrm{v}^1, \mathrm{v}^2, \alpha, \gamma^m\right] \quad \textit{for} \quad m = 1, 2 \tag{0.17}$$

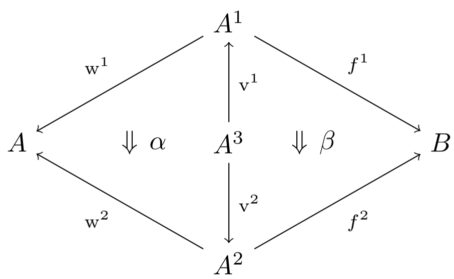
(*in other terms, given any pair of 2-morphisms with the same source and target in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$, they differ at most by one term*)*. Moreover, given any pair of 2-morphisms $\Gamma^1, \Gamma^2 : \underline{f}^1 \Rightarrow \underline{f}^2$ as in* (0.17)*, the following facts are equivalent:*

(*i*) *$\Gamma^1 = \Gamma^2$;*
(*ii*) *there are an object A^4 and a morphism $\mathrm{z} : A^4 \to A^3$, such that $(\mathrm{w}^1 \circ \mathrm{v}^1) \circ \mathrm{z}$ belongs to $\mathbf{W}$ and $\gamma^1 * i_{\mathrm{z}} = \gamma^2 * i_{\mathrm{z}}$.*

The description above simplifies considerably the comparison of 2-morphisms in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$: just compare it with the original comparison in [Pr, § 2.3].

Proposition 0.8. (invertibility of 2-morphisms in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$) *Let us fix any pair $(\mathscr{C}, \mathbf{W})$ satisfying conditions* (BF)*, any pair of morphisms $(A^m, \mathrm{w}^m, f^m) : A \to B$ in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ for $m = 1, 2$ and any 2-morphism $\Gamma : (A^1, \mathrm{w}^1, f^1) \Rightarrow (A^2, \mathrm{w}^2, f^2)$. Then the following facts are equivalent:*

(*i*) *Γ is invertible in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$;*
(*ii*) *Γ has a representative*



(0.18)

such that β is invertible in $\mathscr{C}$;

(*iii*) *given any representative* (0.18) *for Γ, there are an object A^4 and a morphism $\mathrm{u} : A^4 \to A^3$, such that*
- *$(\mathrm{w}^1 \circ \mathrm{v}^1) \circ \mathrm{u}$ belongs to $\mathbf{W}$,*
- *$\beta * i_{\mathrm{u}}$ is invertible in $\mathscr{C}$.*

Note that in (0.18) α is always invertible and $\mathrm{w}^1 \circ \mathrm{v}^1$ always belongs to $\mathbf{W}$ by definition of 2-morphism in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$, see [Pr, § 2.3].

We are going to apply all the results mentioned so far in the next 2 papers of this series, where we will investigate the problem of constructing pseudofunctors (and equivalences) between right bicategories of fractions.

1. Notations and basic facts

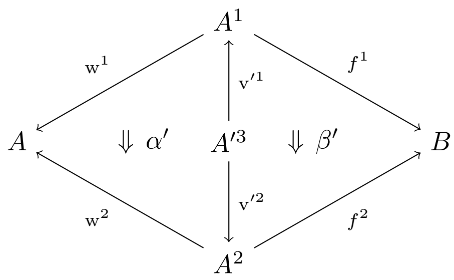
We mainly refer to [L] and [PW, § 1] for a general overview on bicategories, pseudofunctors (i.e. homomorphisms of bicategories), Lax natural transformations and modifications. Given any bicategory $\mathscr{C}$, we denote its objects by $A, B, \cdots$, its morphisms by $f, g, \cdots$ and its 2-morphisms by $\alpha, \beta, \cdots$. Given any triple of morphisms $f : A \rightarrow B$, $g : B \rightarrow C$, $h : C \rightarrow D$ in $\mathscr{C}$, we denote by $\theta_{h,g,f}$ the associator $h \circ (g \circ f) \Rightarrow (h \circ g) \circ f$ that is part of the structure of the bicategory $\mathscr{C}$. We denote by $\pi_f : f \circ \mathrm{id}_A \Rightarrow f$ and $\upsilon_f : \mathrm{id}_B \circ f \Rightarrow f$ the right and left unitors for $\mathscr{C}$ relative to any morphism f as above.

1.1. **Morphisms and 2-morphisms in a bicategory of fractions.** We recall (see [Pr, § 2.2]) that the objects of $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ are the same as those of $\mathscr{C}$. A morphism from A to B in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ is any triple (A', w, f), where A' is an object of $\mathscr{C}$, $\mathrm{w} : A' \rightarrow A$ is an element of $\mathbf{W}$ and $f : A' \rightarrow B$ is a morphism of $\mathscr{C}$. Given any pair of morphisms from A to B and from B to C in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ as follows

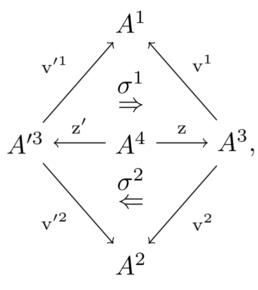
$$\underline{f} := \Big(A \xleftarrow{\;\mathrm{w}\;} A' \xrightarrow{\;f\;} B\Big) \qquad \text{and} \qquad \underline{g} := \Big(B \xleftarrow{\;\mathrm{v}\;} B' \xrightarrow{\;g\;} C\Big)$$

(with both w and v in $\mathbf{W}$), following [Pr, § 2.2] one has to use choices $\mathrm{C}(\mathbf{W})$ for the pair (f, v) in order to get data as in (0.2) and then set $\underline{g} \circ \underline{f} := (A'', \mathrm{w} \circ \mathrm{v}', g \circ f')$.

Given any pair of objects A, B and any pair of morphisms $(A^m, \mathrm{w}^m, f^m) : A \rightarrow B$ for $m = 1, 2$ in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$, a 2-morphism from (A^1, w^1, f^1) to (A^2, w^2, f^2) is an equivalence class of data $(A^3, \mathrm{v}^1, \mathrm{v}^2, \alpha, \beta)$ in $\mathscr{C}$ as in (0.18) such that $\mathrm{w}^1 \circ \mathrm{v}^1$ belongs to $\mathbf{W}$ and such that α is invertible in $\mathscr{C}$ (in [Pr, § 2.3] it is also required that $\mathrm{w}^2 \circ \mathrm{v}^2$ belongs to $\mathbf{W}$, but this follows from (BF5)). Any other set of data



(such that $\mathrm{w}^1 \circ \mathrm{v}'^1$ belongs to $\mathbf{W}$ and α' is invertible) represents the same 2-morphism in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ if and only if there is a set of data $(A^4, \mathrm{z}, \mathrm{z}', \sigma^1, \sigma^2)$ in $\mathscr{C}$ as in the following diagram



such that $(\mathrm{w}^1 \circ \mathrm{v}^1) \circ \mathrm{z}$ belongs to $\mathbf{W}$, σ^1 and σ^2 are both invertible,

$$\Big(i_{\mathrm{w}^2} * \sigma^2\Big) \odot \theta^{-1}_{\mathrm{w}^2,\mathrm{v}^2,\mathrm{z}} \odot \Big(\alpha * i_{\mathrm{z}}\Big) \odot \theta_{\mathrm{w}^1,\mathrm{v}^1,\mathrm{z}} \odot \Big(i_{\mathrm{w}^1} * \sigma^1\Big) = \\ = \theta^{-1}_{\mathrm{w}^2,\mathrm{v}'^2,\mathrm{z}'} \odot \Big(\alpha' * i_{\mathrm{z}'}\Big) \odot \theta_{\mathrm{w}^1,\mathrm{v}'^1,\mathrm{z}'}$$

and

$$\Big(i_{f^2} * \sigma^2\Big) \odot \theta^{-1}_{f^2,\mathrm{v}^2,\mathrm{z}} \odot \Big(\beta * i_{\mathrm{z}}\Big) \odot \theta_{f^1,\mathrm{v}^1,\mathrm{z}} \odot \Big(i_{f^1} * \sigma^1\Big) = \\ = \theta^{-1}_{f^2,\mathrm{v}'^2,\mathrm{z}'} \odot \Big(\beta' * i_{\mathrm{z}'}\Big) \odot \theta_{f^1,\mathrm{v}'^1,\mathrm{z}'}.$$

For symmetric reasons, in [Pr, § 2.3] it is also required that $(\mathrm{w}^1 \circ \mathrm{v}'^1) \circ \mathrm{z}'$ belongs to $\mathbf{W}$, but this follows from (BF5), using the invertible 2-morphism:

$$\theta_{\mathrm{w}^1,\mathrm{v}^1,\mathrm{z}} \odot \Big(i_{\mathrm{w}^1} * \sigma^1\Big) \odot \theta^{-1}_{\mathrm{w}^1,\mathrm{v}'^1,\mathrm{z}'} : (\mathrm{w}^1 \circ \mathrm{v}'^1) \circ \mathrm{z}' \Longrightarrow (\mathrm{w}^1 \circ \mathrm{v}^1) \circ \mathrm{z},$$

so we will always omit this unnecessary technical condition. We denote by

$$\Big[A^3, \mathrm{v}^1, \mathrm{v}^2, \alpha, \beta\Big] : \Big(A^1, \mathrm{w}^1, f^1\Big) \Longrightarrow \Big(A^2, \mathrm{w}^2, f^2\Big) \tag{1.1}$$

the class of any data as in (0.18). We will denote morphisms of $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ as $\underline{f}, \underline{g}, \cdots$ and 2-morphisms by $\Gamma, \Delta, \cdots$; in particular $\Theta^{\mathscr{C},\mathbf{W}}_{\bullet}$ will denote any associator of $\mathscr{C}\left[\mathbf{W}^{-1}\right]$. Note that even if $\mathscr{C}$ is a 2-category, in general $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ is only a bicategory (with trivial right and left unitors, but non-trivial associators if the choices in $\mathrm{C}(\mathbf{W})$ are non-unique).

In the following pages we will often use this easy lemma (see the Appendix for a proof).

Lemma 1.1. *Let us fix any pair* $(\mathscr{C}, \mathbf{W})$ *satisfying conditions* (BF). *Let us fix any morphism* $\mathrm{w} : B \to A$ *in* $\mathbf{W}$, *any pair of morphisms* $f^1, f^2 : C \to B$ *and any pair of* 2*-morphisms* $\gamma, \gamma' : f^1 \Rightarrow f^2$, *such that* $i_{\mathrm{w}} * \gamma = i_{\mathrm{w}} * \gamma'$. *Then there are an object* D *and a morphism* $\mathrm{u} : D \to C$ *in* $\mathbf{W}$, *such that* $\gamma * i_{\mathrm{u}} = \gamma' * i_{\mathrm{u}}$.

The next 2 lemmas prove that if conditions (BF) hold, then conditions (BF3), (BF4a) and (BF4b) hold under less restrictive conditions on the morphism w (see the Appendix for the proofs). To be more precise, instead of imposing that w belongs to $\mathbf{W}$, it is sufficient to impose that $\mathrm{z} \circ \mathrm{w}$ belongs to $\mathbf{W}$ for some morphism z in $\mathbf{W}$ (as a special case, one gets back again (BF3), (BF4a) and (BF4b) when we choose z as a 1-identity).

Lemma 1.2. *Let us fix any pair* $(\mathscr{C}, \mathbf{W})$ *satisfying conditions* (BF). *Let us choose any quadruple of objects* A, B, B', C *and any quadruple of morphisms* $\mathrm{w} : A \to B$, $\mathrm{z} : B \to B'$ *and* $f : C \to B$, *such that both* z *and* $\mathrm{z} \circ \mathrm{w}$ *belong to* $\mathbf{W}$. *Then there are an object* D, *a morphism* $\mathrm{w}' : D \to C$ *in* $\mathbf{W}$, *a morphism* $f' : D \to A$ *and an invertible* 2*-morphism* $\alpha : f \circ \mathrm{w}' \Rightarrow \mathrm{w} \circ f'$.

Lemma 1.3. *Let us fix any pair* $(\mathscr{C}, \mathbf{W})$ *satisfying conditions* (BF). *Let us choose any quadruple of objects* A, A', B, C *and any triple of morphisms* $\mathrm{w} : B \to A$, $\mathrm{z} : A \to A'$, $f^1, f^2 : C \to B$, *such that both* z *and* $\mathrm{z} \circ \mathrm{w}$ *belong to* $\mathbf{W}$. *Moreover, let us fix any* 2*-morphism* $\alpha : \mathrm{w} \circ f^1 \Rightarrow \mathrm{w} \circ f^2$. *Then there are an object* D, *a morphism* $\mathrm{v} : D \to C$ *in* $\mathbf{W}$ *and a* 2*-morphism* $\beta : f^1 \circ \mathrm{v} \Rightarrow f^2 \circ \mathrm{v}$, *such that* $\alpha * i_{\mathrm{v}} = \theta_{\mathrm{w},f^2,\mathrm{v}} \odot (i_{\mathrm{w}} * \beta) \odot \theta^{-1}_{\mathrm{w},f^1,\mathrm{v}}$. *Moreover, if* α *is invertible, so is* β.

2. The associators of a bicategory of fractions

For simplicity of exposition, in this and in the next sections all the proof will be given assuming all the time that $\mathscr{C}$ is a 2-category. The general case when $\mathscr{C}$ is a bicategory follows the same ideas, adding associators and unitors whenever it is necessary and using the coherence conditions on the bicategory $\mathscr{C}$. You can have a glimpse of how to deal with the general case by looking at the proofs of Lemmas 1.1, 1.2 and 1.3 in the Appendix.

In the initial part of this section we will give a proof of Proposition 0.1. Then we will give some interesting corollaries of this result.

Proof of Proposition 0.1. Following [Pr, Appendix], one gets immediately a set of data satisfying conditions (F1) – (F3), inducing the desired associator as in (0.9). So in order to prove the claim, it is sufficient to prove that given any 2 different sets of choices of data as in (F1) – (F3), the 2-morphism of $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ induced by the first set of choices coincides with the 2-morphism induced by the second set of choices. So let us fix any other set of data satisfying the same conditions, as follows:

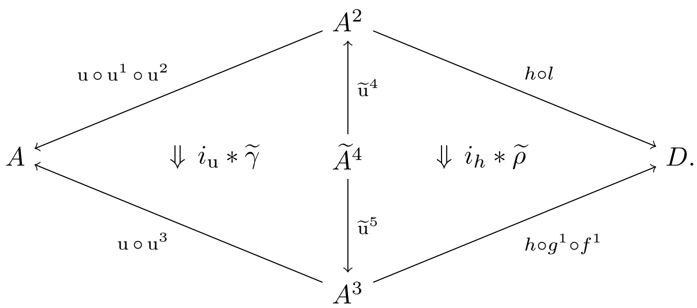
- an object $\widetilde{A}^4$, a morphism $\widetilde{\mathrm{u}}^4 : \widetilde{A}^4 \to A^2$ in $\mathbf{W}$, a morphism $\widetilde{\mathrm{u}}^5 : \widetilde{A}^4 \to A^3$ and an invertible 2-morphism $\widetilde{\gamma} : \mathrm{u}^1 \circ \mathrm{u}^2 \circ \widetilde{\mathrm{u}}^4 \Rightarrow \mathrm{u}^3 \circ \widetilde{\mathrm{u}}^5$;
- an invertible 2-morphism $\widetilde{\omega} : f^1 \circ \mathrm{u}^2 \circ \widetilde{\mathrm{u}}^4 \Rightarrow \mathrm{v}^1 \circ f^2 \circ \widetilde{\mathrm{u}}^5$, such that

$$i_{\mathrm{v}} * \widetilde{\omega} = \Big(\eta * i_{\widetilde{\mathrm{u}}^5}\Big) \odot \Big(i_f * \widetilde{\gamma}\Big) \odot \Big(\delta^{-1} * i_{\mathrm{u}^2 \circ \widetilde{\mathrm{u}}^4}\Big); \tag{2.1}$$

- an invertible 2-morphism $\widetilde{\rho} : l \circ \widetilde{\mathrm{u}}^6 \Rightarrow g^1 \circ f^2 \circ \widetilde{\mathrm{u}}^5$, such that

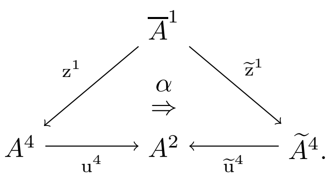
$$i_{\mathrm{w}} * \widetilde{\rho} = \Big(\xi * i_{f^2 \circ \widetilde{\mathrm{u}}^5}\Big) \odot \Big(i_g * \widetilde{\omega}\Big) \odot \Big(\sigma^{-1} * i_{\widetilde{\mathrm{u}}^4}\Big). \tag{2.2}$$

Then proving the claim is equivalent to proving that the class of (0.9) coincides with the class of



(2.3)

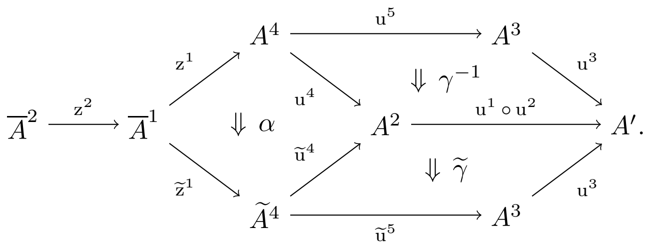
First of all, we use axiom (BF3) in order to get data as in the upper part of the following diagram, with z^1 in $\mathbf{W}$ and α invertible:



Since u^3 belongs to $\mathbf{W}$ by hypothesis, then using (BF4a) and (BF4b) there are an object $\overline{A}^2$, a morphism $\mathrm{z}^2 : \overline{A}^2 \to \overline{A}^1$ in $\mathbf{W}$ and an invertible 2-morphism

$$\varepsilon :\ \mathrm{u}^5 \circ \mathrm{z}^1 \circ \mathrm{z}^2 \Longrightarrow \widetilde{\mathrm{u}}^5 \circ \widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2,$$

such that $i_{\mathrm{u}^3} * \varepsilon$ coincides with the following composition:



This implies that

$$\gamma * i_{\mathrm{z}^1 \circ \mathrm{z}^2} = \Big(i_{\mathrm{u}^3} * \varepsilon^{-1}\Big) \odot \Big(\widetilde{\gamma} * i_{\widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2}\Big) \odot \Big(i_{\mathrm{u}^1 \circ \mathrm{u}^2} * \alpha * i_{\mathrm{z}^2}\Big). \tag{2.4}$$

So we get:

$$\begin{gathered}
i_{\mathrm{v}} * \Big(\omega * i_{\mathrm{z}^1 \circ \mathrm{z}^2}\Big) \stackrel{(0.7)}{=} \Big(\eta * i_{\mathrm{u}^5 \circ \mathrm{z}^1 \circ \mathrm{z}^2}\Big) \odot \\
\odot \Big(i_f * \gamma * i_{\mathrm{z}^1 \circ \mathrm{z}^2}\Big) \odot \Big(\delta^{-1} * i_{\mathrm{u}^2 \circ \mathrm{u}^4 \circ \mathrm{z}^1 \circ \mathrm{z}^2}\Big) \stackrel{(2.4)}{=} \\
\stackrel{(2.4)}{=} \Big(\eta * i_{\mathrm{u}^5 \circ \mathrm{z}^1 \circ \mathrm{z}^2}\Big) \odot \Big(i_{f \circ \mathrm{u}^3} \circ \varepsilon^{-1}\Big) \odot \Big(i_f * \widetilde{\gamma} * i_{\widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2}\Big) \odot \\
\odot \Big(i_{f \circ \mathrm{u}^1 \circ \mathrm{u}^2} * \alpha * i_{\mathrm{z}^2}\Big) \odot \Big(\delta^{-1} * i_{\mathrm{u}^2 \circ \mathrm{u}^4 \circ \mathrm{z}^1 \circ \mathrm{z}^2}\Big) \stackrel{(*)}{=} \\
\stackrel{(*)}{=} \Big(i_{\mathrm{v} \circ \mathrm{v}^1 \circ f^2} * \varepsilon^{-1}\Big) \odot \Big(\eta * i_{\widetilde{\mathrm{u}}^5 \circ \widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2}\Big) \odot \Big(i_f * \widetilde{\gamma} * i_{\widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2}\Big) \odot \\
\odot \Big(\delta^{-1} * i_{\mathrm{u}^2 \circ \widetilde{\mathrm{u}}^4 \circ \widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2}\Big) \odot \Big(i_{\mathrm{v} \circ f^1 \circ \mathrm{u}^2} * \alpha * i_{\mathrm{z}^2}\Big) \stackrel{(2.1)}{=} \\
\stackrel{(2.1)}{=} \Big(i_{\mathrm{v} \circ \mathrm{v}^1 \circ f^2} * \varepsilon^{-1}\Big) \odot \Big(i_{\mathrm{v}} * \widetilde{\omega} * i_{\widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2}\Big) \odot \Big(i_{\mathrm{v} \circ f^1 \circ \mathrm{u}^2} * \alpha * i_{\mathrm{z}^2}\Big) = \\
= i_{\mathrm{v}} * \Big(\Big(i_{\mathrm{v}^1 \circ f^2} * \varepsilon^{-1}\Big) \odot \Big(\widetilde{\omega} * i_{\widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2}\Big) \odot \Big(i_{f^1 \circ \mathrm{u}^2} * \alpha * i_{\mathrm{z}^2}\Big)\Big),
\end{gathered} \tag{2.5}$$

where $(*)$ is given by applying the interchange law twice. Using Lemma 1.1 and (2.5), there are an object $\overline{A}^3$ and a morphism $\mathrm{z}^3 : \overline{A}^3 \to \overline{A}^2$ in $\mathbf{W}$, such that

$$\begin{gathered}
\omega * i_{\mathrm{z}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3} = \\
= \Big(i_{\mathrm{v}^1 \circ f^2} * \varepsilon^{-1} * i_{\mathrm{z}^3}\Big) \odot \Big(\widetilde{\omega} * i_{\widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3}\Big) \odot \Big(i_{f^1 \circ \mathrm{u}^2} * \alpha * i_{\mathrm{z}^2 \circ \mathrm{z}^3}\Big).
\end{gathered} \tag{2.6}$$

Then we get:

$$\begin{gathered}
i_{\mathrm{w}} * \rho * i_{\mathrm{z}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3} \stackrel{(0.8),(2.6)}{=} \\
\stackrel{(0.8),(2.6)}{=} \Big(\xi * i_{f^2 \circ \mathrm{u}^5 \circ \mathrm{z}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3}\Big) \odot \Big(i_{g \circ \mathrm{v}^1 \circ f^2} * \varepsilon^{-1} * i_{\mathrm{z}^3}\Big) \odot \Big(i_g * \widetilde{\omega} * i_{\widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3}\Big) \odot \\
\odot \Big(i_{g \circ f^1 \circ \mathrm{u}^2} * \alpha * i_{\mathrm{z}^2 \circ \mathrm{z}^3}\Big) \odot \Big(\sigma^{-1} * i_{\mathrm{u}^4 \circ \mathrm{z}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3}\Big) \stackrel{(*)}{=} \\
\stackrel{(*)}{=} \Big(i_{\mathrm{w} \circ g^1 \circ f^2} * \varepsilon^{-1} * i_{\mathrm{z}^3}\Big) \odot \Big(\xi * i_{f^2 \circ \widetilde{\mathrm{u}}^5 \circ \widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3}\Big) \odot \Big(i_g * \widetilde{\omega} * i_{\widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3}\Big) \odot \\
\odot \Big(\sigma^{-1} * i_{\widetilde{\mathrm{u}}^4 \circ \widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3}\Big) \odot \Big(i_{\mathrm{w} \circ l} * \alpha * i_{\mathrm{z}^2 \circ \mathrm{z}^3}\Big) \stackrel{(2.2)}{=} \\
\stackrel{(2.2)}{=} \Big(i_{\mathrm{w} \circ g^1 \circ f^2} * \varepsilon^{-1} * i_{\mathrm{z}^3}\Big) \odot \Big(i_{\mathrm{w}} * \widetilde{\rho} * i_{\widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3}\Big) \odot \Big(i_{\mathrm{w} \circ l} * \alpha * i_{\mathrm{z}^2 \circ \mathrm{z}^3}\Big),
\end{gathered}$$

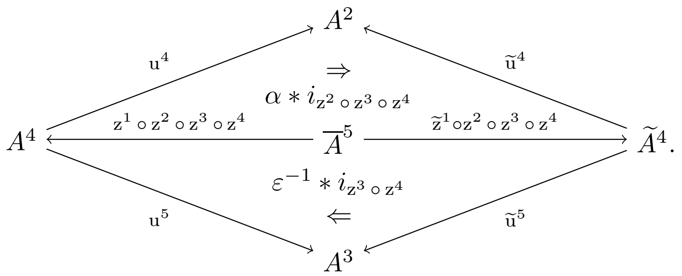
where $(*)$ is given by the interchange law. So by Lemma 1.1 there are an object $\overline{A}^4$ and a morphism $\mathrm{z}^4 : \overline{A}^4 \to \overline{A}^3$ in $\mathbf{W}$, such that

$$\rho * i_{\mathrm{z}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3 \circ \mathrm{z}^4} =$$
$$= \Big(i_{g^1 \circ f^2} * \Big(\varepsilon^{-1} * i_{\mathrm{z}^3 \circ \mathrm{z}^4}\Big)\Big) \odot \Big(\widetilde{\rho} * i_{\widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3 \circ \mathrm{z}^4}\Big) \odot \Big(i_l * \Big(\alpha * i_{\mathrm{z}^2 \circ \mathrm{z}^3 \circ \mathrm{z}^4}\Big)\Big). \quad (2.7)$$

Moreover, from (2.4) we have:

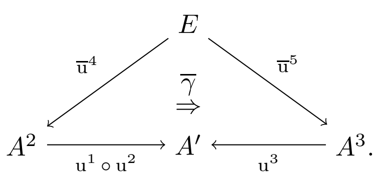
$$\gamma * i_{\mathrm{z}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3 \circ \mathrm{z}^4} =$$
$$= \Big(i_{\mathrm{u}^3} * \Big(\varepsilon^{-1} * i_{\mathrm{z}^3 \circ \mathrm{z}^4}\Big)\Big) \odot \Big(\widetilde{\gamma} * i_{\widetilde{\mathrm{z}}^1 \circ \mathrm{z}^2 \circ \mathrm{z}^3 \circ \mathrm{z}^4}\Big) \odot \Big(i_{\mathrm{u}^1 \circ \mathrm{u}^2} * \Big(\alpha * i_{\mathrm{z}^2 \circ \mathrm{z}^3 \circ \mathrm{z}^4}\Big)\Big). \quad (2.8)$$

Then using together identities (2.7) and (2.8), the following diagram and the definition of 2-morphism in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ (see § 1.1), we get that the class of (0.9) coincides with the class of (2.3).

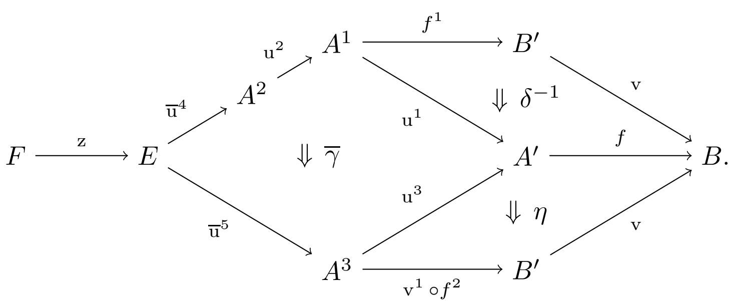


□

Remark 2.1. Apart from the procedure explained in [Pr, Appendix], when $\mathscr{C}$ is a 2-category a simple way for finding a tuple $(A^4, \mathrm{u}^4, \mathrm{u}^5, \gamma, \omega, \rho)$ as in (F1) – (F3) is given as follows (in the general case of a bicategory, simply add associators wherever it is necessary). First of all, we use axiom (BF3) in order to get data as in the upper part of the following diagram, with $\overline{\mathrm{u}}^4$ in $\mathbf{W}$ and $\overline{\gamma}$ invertible:



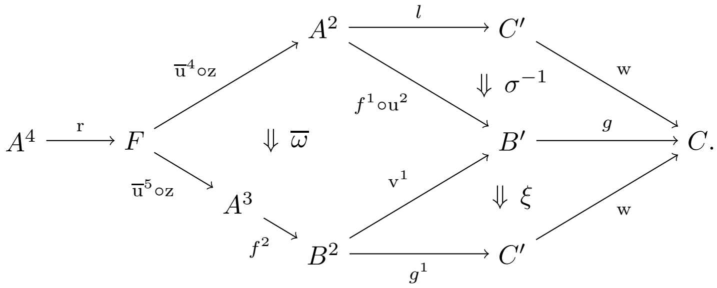
Then we use (BF4a) and (BF4b) in order to get an object F, a morphism $\mathrm{z} : F \to E$ in $\mathbf{W}$ and an invertible 2-morphism $\overline{\varpi} : f^1 \circ \mathrm{u}^2 \circ \overline{\mathrm{u}}^4 \circ \mathrm{z} \Rightarrow \mathrm{v}^1 \circ f^2 \circ \overline{\mathrm{u}}^5 \circ \mathrm{z}$, such that $i_{\mathrm{v}} * \overline{\varpi}$ coincides with the following composition:



Then we use again (BF4a) and (BF4b) in order to get an object A^4, a morphism $\mathrm{r}: A^4 \to F$ in $\mathbf{W}$ and an invertible 2-morphism

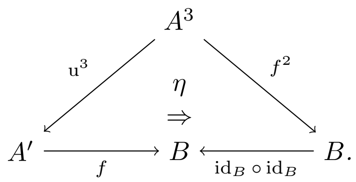
$$\rho:\ l \circ \overline{\mathrm{u}}^4 \circ \mathrm{z} \circ \mathrm{r} \Longrightarrow g^1 \circ f^2 \circ \overline{\mathrm{u}}^5 \circ \mathrm{z} \circ \mathrm{r},$$

such that $i_{\mathrm{w}} * \rho$ coincides with the following composition:



Then it suffices to define $\mathrm{u}^4 := \overline{\mathrm{u}}^4 \circ \mathrm{z} \circ \mathrm{r}$ (this morphism belongs to $\mathbf{W}$ by (BF2)), $\mathrm{u}^5 := \overline{\mathrm{u}}^5 \circ \mathrm{z} \circ \mathrm{r}$, $\gamma := \overline{\gamma} * i_{\mathrm{z} \circ \mathrm{r}}$ and $\omega := \overline{\omega} * i_{\mathrm{r}}$.

Corollary 2.2. *Let us fix any pair* $(\mathscr{C}, \mathbf{W})$ *satisfying conditions* (BF), *any triple of morphisms* $\underline{f}, \underline{g}, \underline{h}$ *as in* (0.3) *and let us suppose that* $B = B'$, $C = C'$, $\mathrm{v} = \mathrm{id}_B$ *and* $\mathrm{w} = \mathrm{id}_C$. *Moreover, let us suppose that choices* $\mathrm{C}(\mathbf{W})$ *give data as in the upper part of the following diagram, with* u^3 *in* $\mathbf{W}$ *and* η *invertible:*



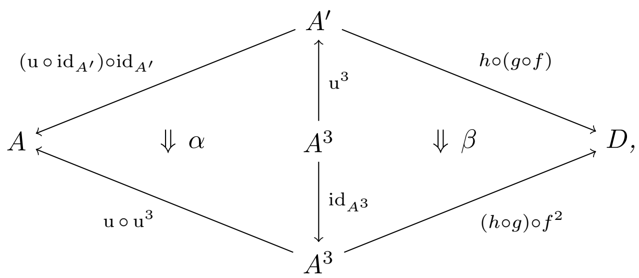
(2.9)

Then

$$\underline{h} \circ \big(\underline{g} \circ \underline{f}\big) = \Big(A \xleftarrow{(\mathrm{u} \circ \mathrm{id}_{A'}) \circ \mathrm{id}_{A'}} A' \xrightarrow{h \circ (g \circ f)} D\Big), \tag{2.10}$$

$$\big(\underline{h} \circ \underline{g}\big) \circ \underline{f} = \Big(A \xleftarrow{\mathrm{u} \circ \mathrm{u}^3} A^3 \xrightarrow{(h \circ g) \circ f^2} D\Big) \tag{2.11}$$

and the associator $\Theta^{\mathscr{C},\mathbf{W}}_{\underline{h},\underline{g},\underline{h}}$ *from* (2.10) *to* (2.11) *is given by the class of the following diagram:*



where

$$\alpha := \pi^{-1}_{\mathrm{u}\circ\mathrm{u}^3} \odot \Big(\pi_{\mathrm{u}} * i_{\mathrm{u}^3}\Big) \odot \Big(\Big(\pi_{\mathrm{u}} * i_{\mathrm{id}_{A'}}\Big) * i_{\mathrm{u}^3}\Big)$$

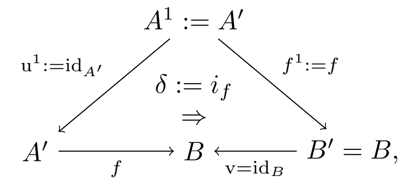
and

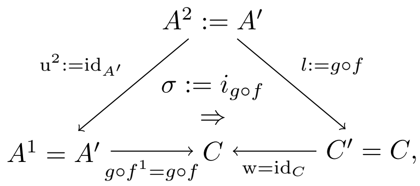
$$\beta := \pi^{-1}_{(h\circ g)\circ f^2} \odot \Big(i_{h\circ g} * \Big(\upsilon_{f^2} \odot \Big(\upsilon_{\mathrm{id}_B} * i_{f^2}\Big) \odot \eta\Big)\Big) \odot \theta^{-1}_{h\circ g, f, \mathrm{u}^3} \odot \Big(\theta_{h,g,f} * i_{\mathrm{u}^3}\Big)$$

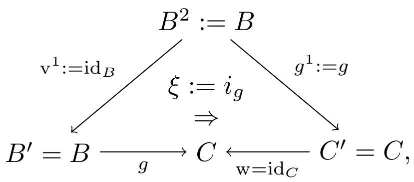
(for the notations concerning the associators $\theta_\bullet$ and the unitors $\pi_\bullet$ and $\upsilon_\bullet$, we refer to § 1).

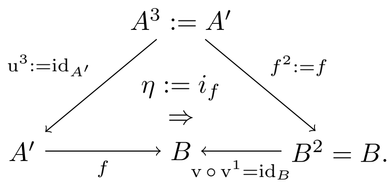
Remark 2.3. In particular, if $\mathscr{C}$ is a 2-category, then the associators $\theta_\bullet$ and the unitors $\pi_\bullet$ and $\upsilon_\bullet$ are all trivial; moreover, $\mathrm{id}_B \circ \mathrm{id}_B = \mathrm{id}_B$. We recall that the fixed choices $\mathrm{C}(\mathbf{W})$ imposed by Pronk on any pair (f, v) assume a very simple form in the case when either f or v are an identity (see [Pr, pag. 256]). So the quadruple $(A^3, \mathrm{u}^3, f^2, \eta)$ coincides with $(A', \mathrm{id}_{A'}, f, i_f)$. So when $\mathscr{C}$ is a 2-category, (2.10) and (2.11) coincide and the associator above is the 2-identity of such morphism.

Proof of Corollary 2.2. As usual, for simplicity of exposition we give the proof only in the special case when $\mathscr{C}$ is a 2-category. By the already mentioned [Pr, pag. 256], since both v and w are identities, one get that the 4 diagrams of (0.4) (chosen from left to right) assume this simple form:





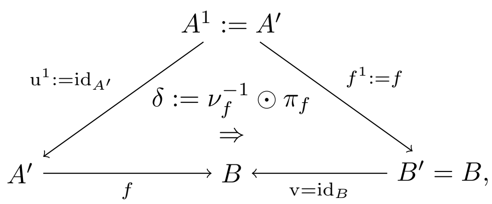




Then identities (2.10) and (2.11) follow at once from (0.5) and (0.6). In order to compute the associator, according to Proposition 0.1 we have to choose a set of data as in (F1) – (F3). For that, we choose:

- $A^4 := A'$, $\mathrm{u}^4 := \mathrm{id}_{A'}$, $\mathrm{u}^5 := \mathrm{id}_{A'}$ and $\gamma := i_{\mathrm{id}_{A'}}$;
- $\omega := i_f$;
- $\rho := i_{g\circ f}$.

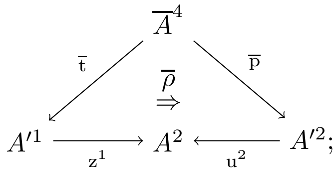
Then the claim follows. In the general case when $\mathscr{C}$ is a bicategory, the first diagram above is given by



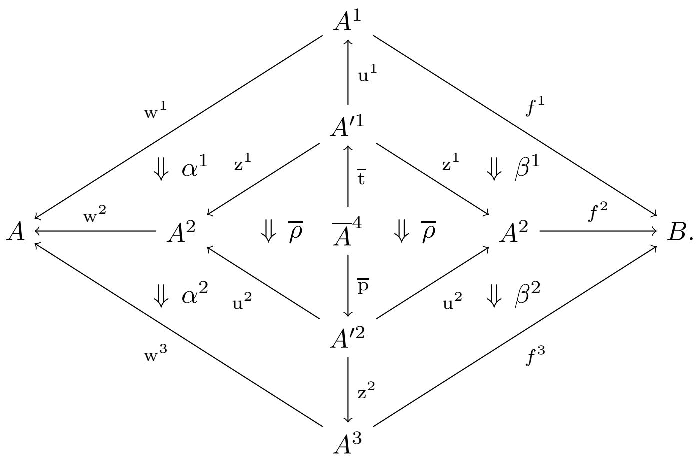
and analogously for the second diagram and the third one. The fourth diagram must be replaced by (2.9). The data of (F1) – (F3) above have to be changed according to this. □

3. Vertical compositions

Proof of Proposition 0.2. Following [Pr, pag. 258], the composition $\Gamma^2 \odot \Gamma^1$ has to be computed as follows: firstly one has to fix data as in the upper part of the following diagram, with $\overline{\mathrm{t}}$ in $\mathbf{W}$ and $\overline{\rho}$ invertible:



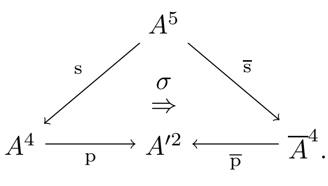
these data are induced by choices $\mathrm{C}(\mathbf{W})$ and $\mathrm{D}(\mathbf{W})$ similarly to the construction given in the proof of Lemma 1.2 (since both w^2 and $\mathrm{w}^2 \circ \mathrm{v}^2$ belong to $\mathbf{W}$), but we don't need to describe how this is done explicitly. Then by [Pr], $\Gamma^2 \odot \Gamma^1$ is represented by the following diagram:



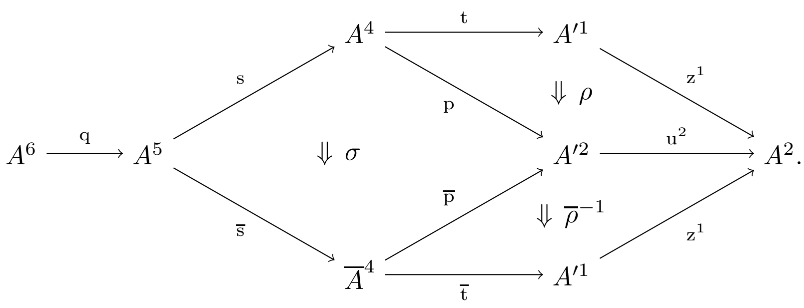
(3.1)

So we need only to prove that (3.1) and (0.10) belong to the same equivalence class.

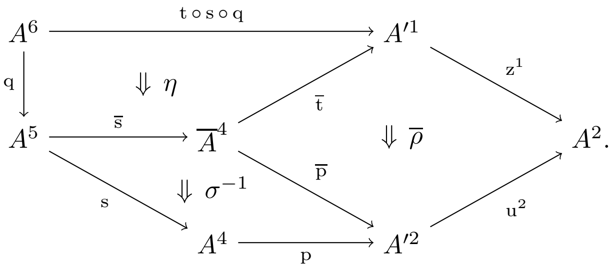
By hypothesis, Γ^1 is a 2-morphism in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$, hence $\mathrm{w}^1 \circ \mathrm{u}^1$ belongs to $\mathbf{W}$ so by (BF5) applied to $(\alpha^1)^{-1}$ we get that $\mathrm{w}^2 \circ \mathrm{z}^1$ belongs to $\mathbf{W}$. Therefore, $\mathrm{w}^2 \circ \mathrm{z}^1 \circ \overline{\mathrm{t}}$ belongs to $\mathbf{W}$. By (BF5) applied to $i_{\mathrm{w}^2} * \overline{\rho}^{\,-1}$, this implies that $\mathrm{w}^2 \circ \mathrm{u}^2 \circ \overline{\mathrm{p}}$ belongs to $\mathbf{W}$. Moreover, by hypothesis Γ^2 belongs to $\mathscr{C}\left[\mathbf{W}^{-1}\right]$, so $\mathrm{w}^2 \circ \mathrm{u}^2$ belongs to $\mathbf{W}$. Therefore, we can apply Lemma 1.2 for $\mathrm{w} := \overline{\mathrm{p}}$. So there are data as in the upper part of the following diagram, with s in $\mathbf{W}$ and σ invertible.



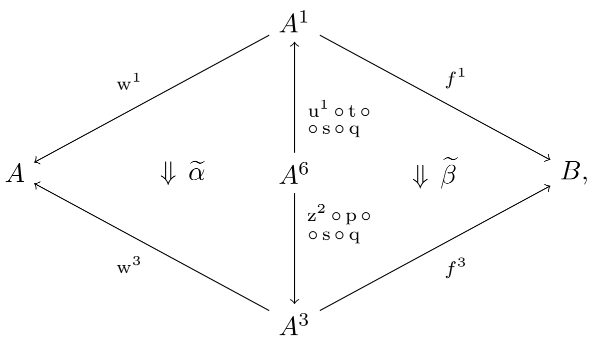
We have already remarked that $\mathrm{w}^2 \circ \mathrm{z}^1$ belongs to $\mathbf{W}$; moreover w^2 belongs to $\mathbf{W}$ since $\underline{f}^2$ is a morphism in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$. So by Lemma 1.3 for $\mathrm{w} := \mathrm{z}^1$, there are an object A^6, a morphism $\mathrm{q} : A^6 \to A^5$ in $\mathbf{W}$ and an invertible 2-morphism $\eta : \mathrm{t} \circ \mathrm{s} \circ \mathrm{q} \Rightarrow \overline{\mathrm{t}} \circ \overline{\mathrm{s}} \circ \mathrm{q}$, such that $i_{\mathrm{z}^1} * \eta$ coincides with the following composition:



Therefore, $\rho * i_{\mathrm{s} \circ \mathrm{q}}$ coincides with the following composition:

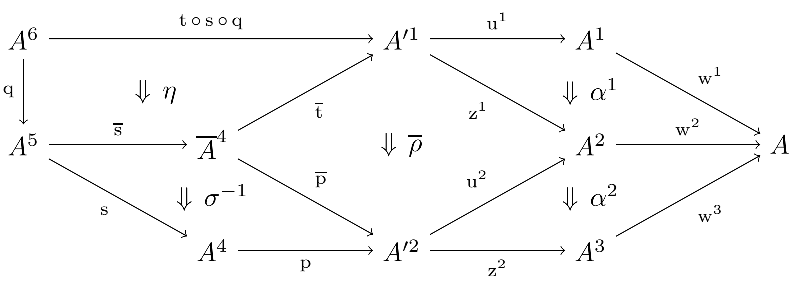


Then the class of (0.10) coincides with the class of the following diagram

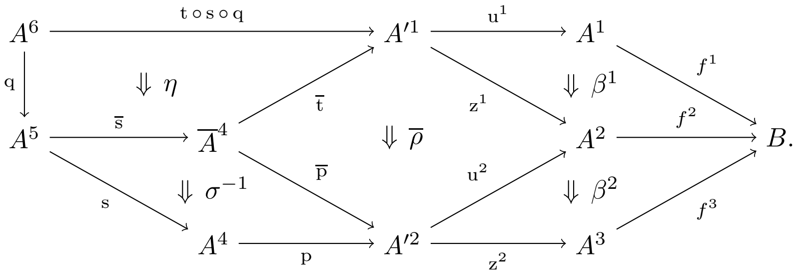


(3.2)

where $\widetilde{\alpha}$ is the following composition



and $\widetilde{\beta}$ is the following composition:

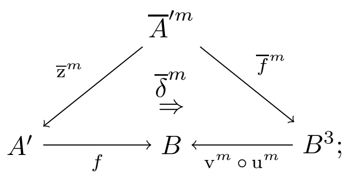


Then using the description of 2-morphisms in § 1.1, it is easy to see that the class of (3.2) (hence also the class of (0.10)) coincides with the class of (3.1). ☐

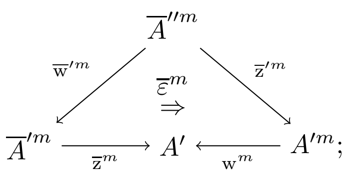
4. Horizontal compositions with 1-morphisms on the left

Proof of Proposition 0.3. By [Pr, pagg. 259–261], one has to compute $\Delta * i_{\underline{f}}$ as follows:

(i) for each $m = 1, 2$, we use choices $\mathrm{C}(\mathbf{W})$ in order to get data as in the upper part of the following diagram, with $\overline{\mathrm{z}}^m$ in $\mathbf{W}$ and $\overline{\delta}^m$ invertible:



(ii) for each $m = 1, 2$, we use again choices $\mathrm{C}(\mathbf{W})$ in order to get data as in the upper part of the following diagram, with $\overline{\mathrm{w}}'^m$ in $\mathbf{W}$ and $\overline{\varepsilon}^m$ invertible:

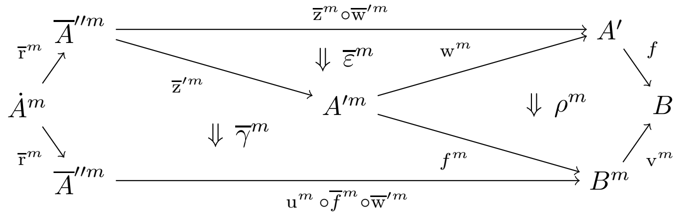


(4.1)

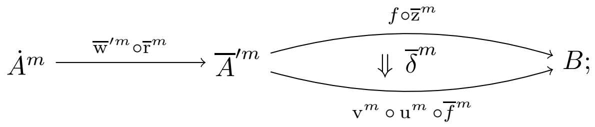
(iii) we use a long procedure (that we don't need to explain explicitly for the purposes of this proof) in order to get for each $m = 1, 2$ an object $\dot{A}^m$, a morphism $\overline{\mathrm{r}}^m : \dot{A}^m \to \overline{A}''^m$ in $\mathbf{W}$ and an invertible 2-morphism

$$\overline{\gamma}^m : f^m \circ \overline{\mathrm{z}}'^m \circ \overline{\mathrm{r}}^m \Longrightarrow \mathrm{u}^m \circ \overline{f}^m \circ \overline{\mathrm{w}}'^m \circ \overline{\mathrm{r}}^m,$$

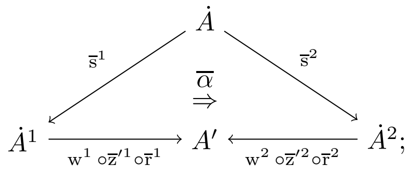
such that the following composition



coincides with



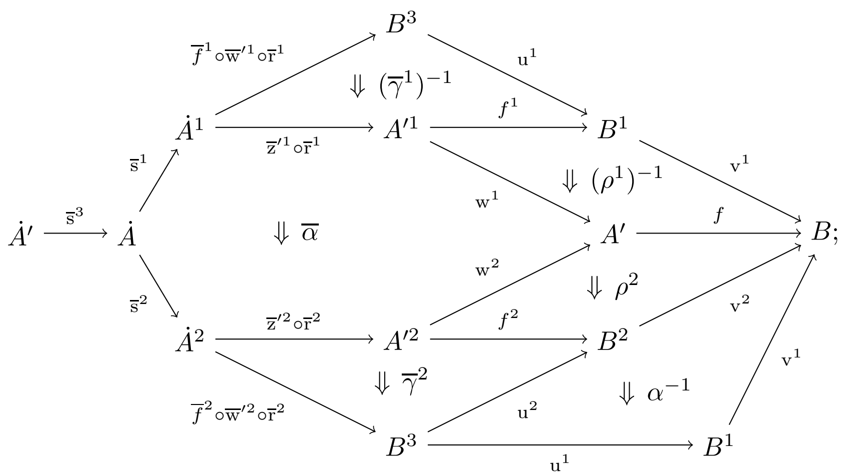
(iv) we use choices $\mathrm{C}(\mathbf{W})$ in order to get data as in the upper part of the following diagram, with $\overline{\mathrm{s}}^1$ in $\mathbf{W}$ and $\overline{\alpha}$ invertible (note that here $\mathrm{w}^2 \circ \overline{\mathrm{z}}'^2 \circ \overline{\mathrm{r}}^2$ belongs to $\mathbf{W}$ by (BF2) and (BF5) applied to $(\overline{\varepsilon}^2)^{-1} * i_{\overline{\mathrm{r}}^2}$):



(v) we use choices $\mathrm{D}(\mathbf{W})$ in order to get an object $\dot{A}'$, a morphism $\overline{\mathrm{s}}^3 : \dot{A}' \to \dot{A}$ in $\mathbf{W}$ and an invertible 2-morphism

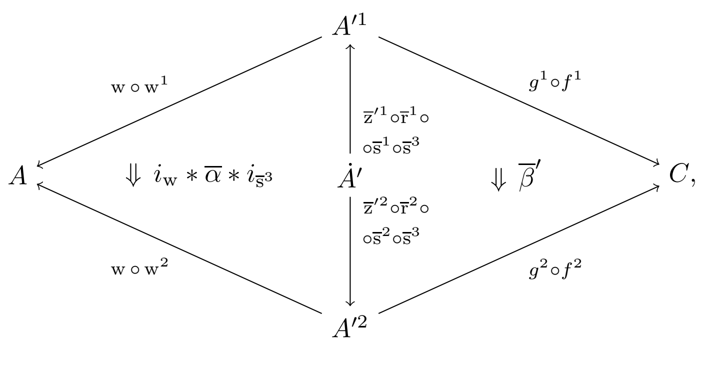
$$\overline{\eta} : \overline{f}^1 \circ \overline{\mathrm{w}}'^1 \circ \overline{\mathrm{r}}^1 \circ \overline{\mathrm{s}}^1 \circ \overline{\mathrm{s}}^3 \Longrightarrow \overline{f}^2 \circ \overline{\mathrm{w}}'^2 \circ \overline{\mathrm{r}}^2 \circ \overline{\mathrm{s}}^2 \circ \overline{\mathrm{s}}^3,$$

such that $i_{\mathrm{v}^1 \circ \mathrm{u}^1} * \overline{\eta}$ coincides with the following composition:



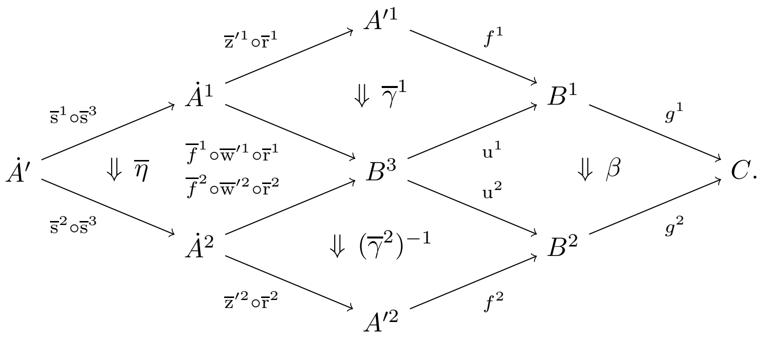
(4.2)

(vi) then according to [Pr, pagg. 259–261], one defines $\Delta * i_{\underline{f}}$ as the class of the following diagram:



(4.3)

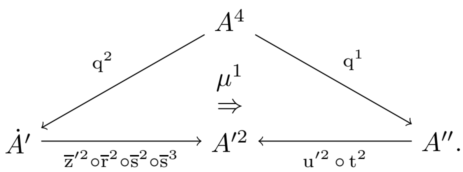
where $\overline{\beta}'$ is the following composition:



(4.4)

Now the claim of this proposition is equivalent to the claim that (0.13) and (4.3) are in the same equivalence class for each set of choices (F5) – (F7); we are going to prove this in the next lines.

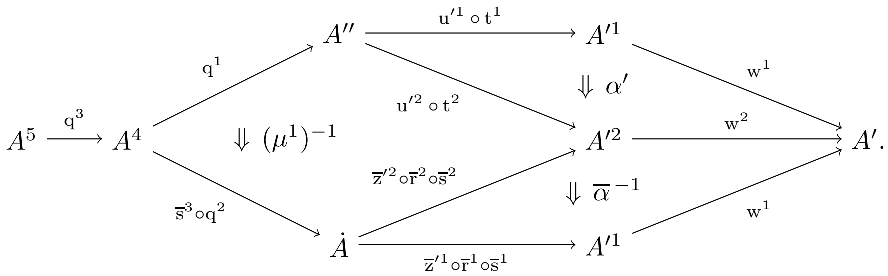
By hypothesis, $\mathrm{w}^1, \mathrm{u}'^1$ and t^1 belong to $\mathbf{W}$, hence also $\mathrm{w}^1 \circ \mathrm{u}'^1 \circ \mathrm{t}^1$ belongs to $\mathbf{W}$ by (BF2), so by (BF5) for $(\alpha')^{-1}$ we conclude that also $\mathrm{w}^2 \circ \mathrm{u}'^2 \circ \mathrm{t}^2$ belongs to $\mathbf{W}$. Moreover, w^2 belongs to $\mathbf{W}$ by hypothesis, so by Lemma 1.2 for $\mathrm{w} := \mathrm{u}'^2 \circ \mathrm{t}^2$, there are data as in the upper part of the following diagram, with q^2 in $\mathbf{W}$ and μ^1 invertible:



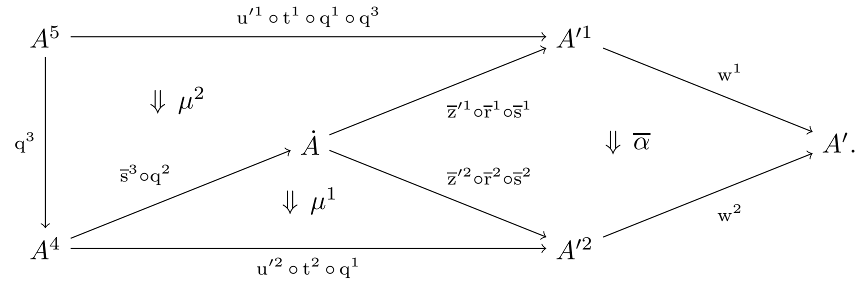
Then we use (BF4a) and (BF4b) in order to get an object A^5, a morphism $\mathrm{q}^3 : A^5 \rightarrow A^4$ in $\mathbf{W}$ and an invertible 2-morphism

$$\mu^2 : \mathrm{u}'^1 \circ \mathrm{t}^1 \circ \mathrm{q}^1 \circ \mathrm{q}^3 \Longrightarrow \overline{\mathrm{z}}'^1 \circ \overline{\mathrm{r}}^1 \circ \overline{\mathrm{s}}^1 \circ \overline{\mathrm{s}}^3 \circ \mathrm{q}^2 \circ \mathrm{q}^3,$$

such that $i_{\mathrm{w}^1} * \mu^2$ coincides with the following composition:

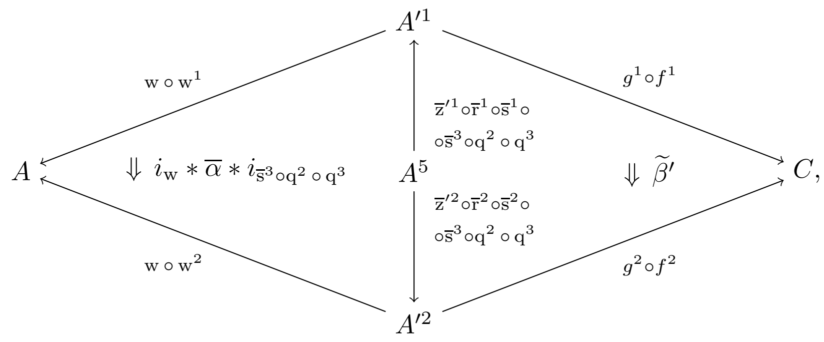


This implies that $\alpha' * i_{\mathrm{q}^1 \circ \mathrm{q}^3}$ coincides with the following composition:



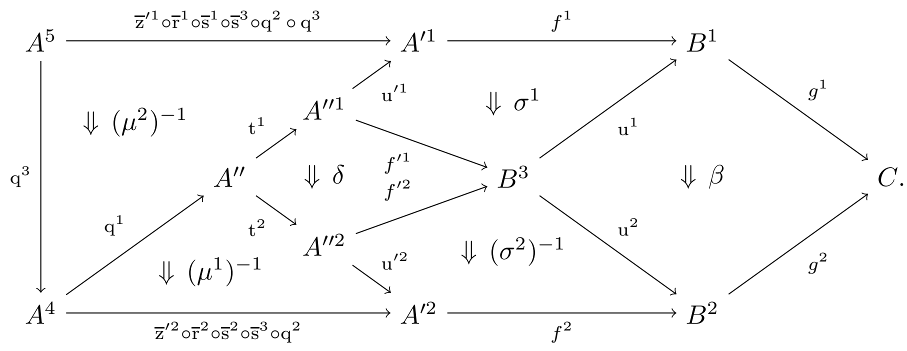
(4.5)

So using again the description of 2-morphisms in § 1.1, the class of (0.13) coincides with the class of the following diagram



(4.6)

where $\widetilde{\beta}'$ is the following composition:

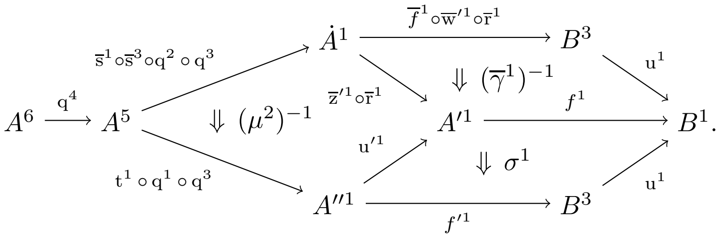


(4.7)

By hypothesis, Δ is 2-morphism in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$, so $\mathrm{v}^1 \circ \mathrm{u}^1$ belongs to $\mathbf{W}$; moreover v^1 belongs to $\mathbf{W}$ because $\underline{g}^1$ is a morphism in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$. So by Lemma 1.3 there are an object A^6, a morphism $\mathrm{q}^4 : A^6 \to A^5$ in $\mathbf{W}$ and an invertible 2-morphism

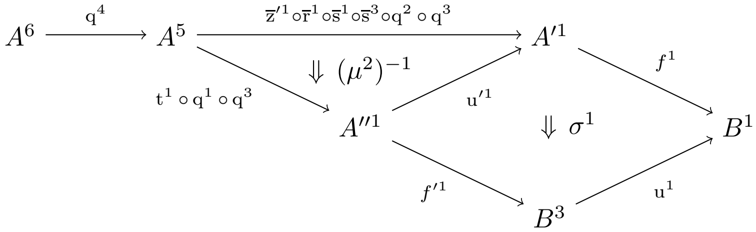
$$\eta^1 : \overline{f}^1 \circ \overline{\mathrm{w}}'^1 \circ \overline{\mathrm{r}}^1 \circ \overline{\mathrm{s}}^1 \circ \overline{\mathrm{s}}^3 \circ \mathrm{q}^2 \circ \mathrm{q}^3 \circ \mathrm{q}^4 \Longrightarrow f'^1 \circ \mathrm{t}^1 \circ \mathrm{q}^1 \circ \mathrm{q}^3 \circ \mathrm{q}^4,$$

such that $i_{\mathrm{u}^1} * \eta^1$ coincides with the following composition:

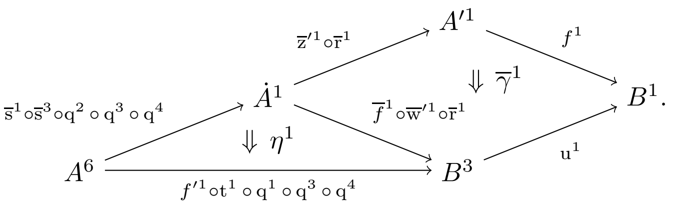


(4.8)

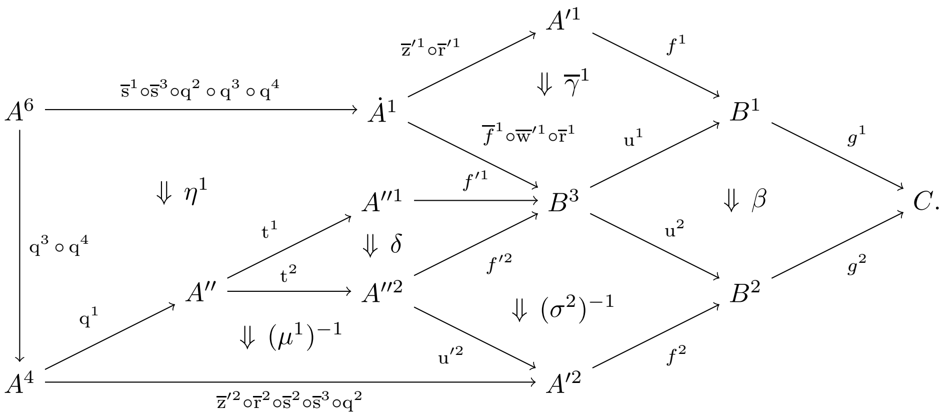
This implies that the following composition



coincides with the following one:



Therefore, using (4.7) we get that $\widetilde{\beta}' * i_{\mathrm{q}^4}$ coincides with the following composition:

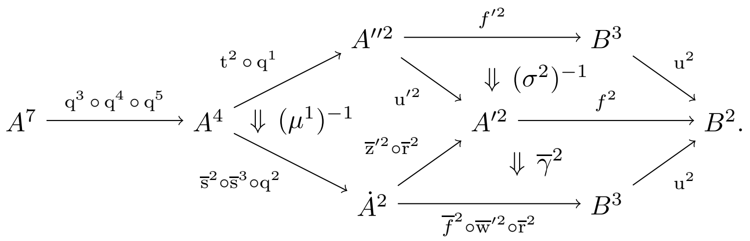


(4.9)

We have already remarked that $\mathrm{v}^1 \circ \mathrm{u}^1$ belongs to $\mathbf{W}$; so by (BF5) applied to α^{-1} we get that also $\mathrm{v}^2 \circ \mathrm{u}^2$ belongs to $\mathbf{W}$; moreover v^2 belongs to $\mathbf{W}$ because $\underline{g}^2$ is a morphism in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$. So by Lemma 1.3 there are an object A^7, a morphism $\mathrm{q}^5 : A^7 \to A^6$ in $\mathbf{W}$ and an invertible 2-morphism

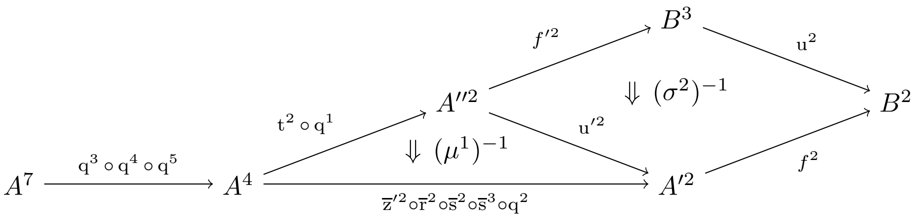
$$\eta^2 : f'^2 \circ \mathrm{t}^2 \circ \mathrm{q}^1 \circ \mathrm{q}^3 \circ \mathrm{q}^4 \circ \mathrm{q}^5 \Longrightarrow \overline{f}^2 \circ \overline{\mathrm{w}}'^2 \circ \overline{\mathrm{r}}^2 \circ \overline{\mathrm{s}}^2 \circ \overline{\mathrm{s}}^3 \circ \mathrm{q}^2 \circ \mathrm{q}^3 \circ \mathrm{q}^4 \circ \mathrm{q}^5,$$

such that $i_{\mathrm{u}^2} * \eta^2$ coincides with the following composition

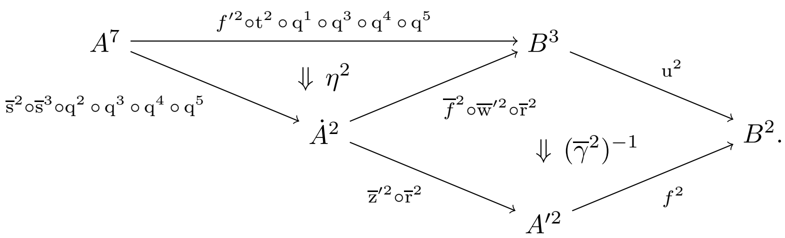


(4.10)

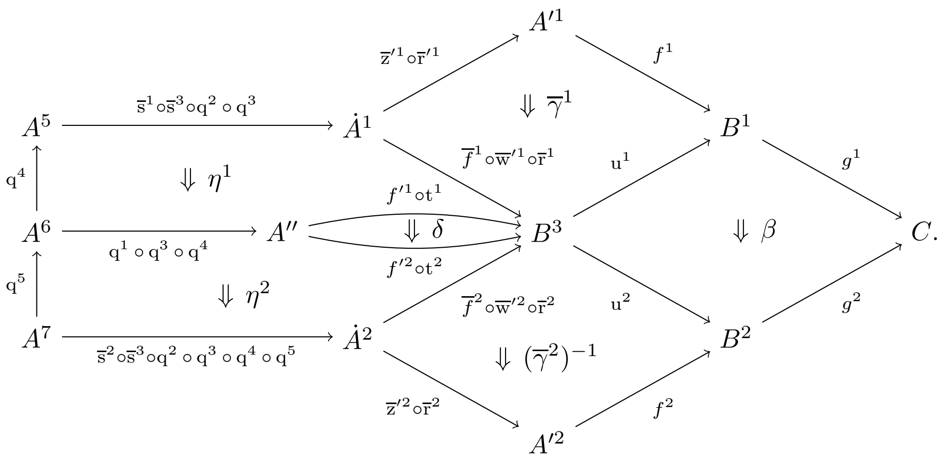
This implies that the following composition



coincides with the following one:



Therefore, using (4.9) we get that $\widetilde{\beta}' * i_{q^4 \circ q^5}$ coincides with the following composition:



(4.11)

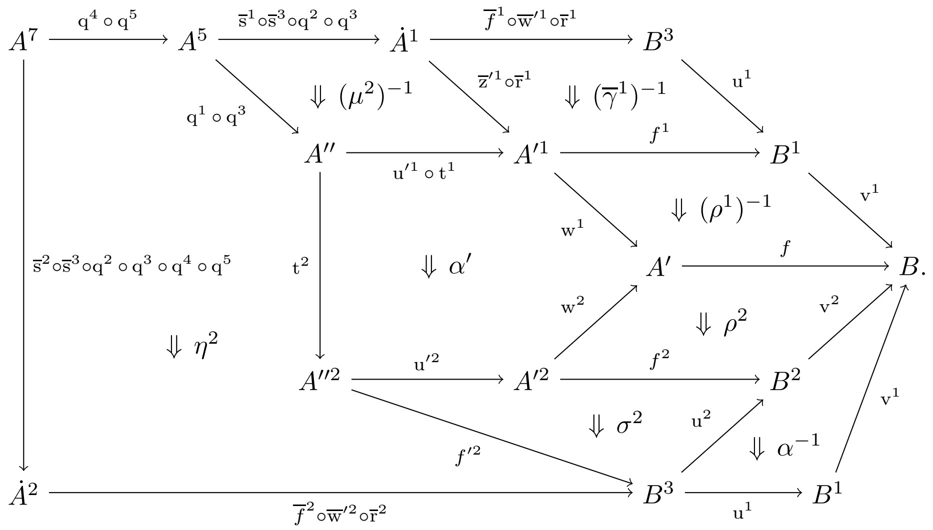
Now we compute

$$i_{v^1 \circ u^1} * \Big(\eta^2 \odot \Big(\delta * i_{q^1 \circ q^3 \circ q^4 \circ q^5}\Big) \odot \Big(\eta^1 * i_{q^5}\Big)\Big). \tag{4.12}$$

In order to do that:

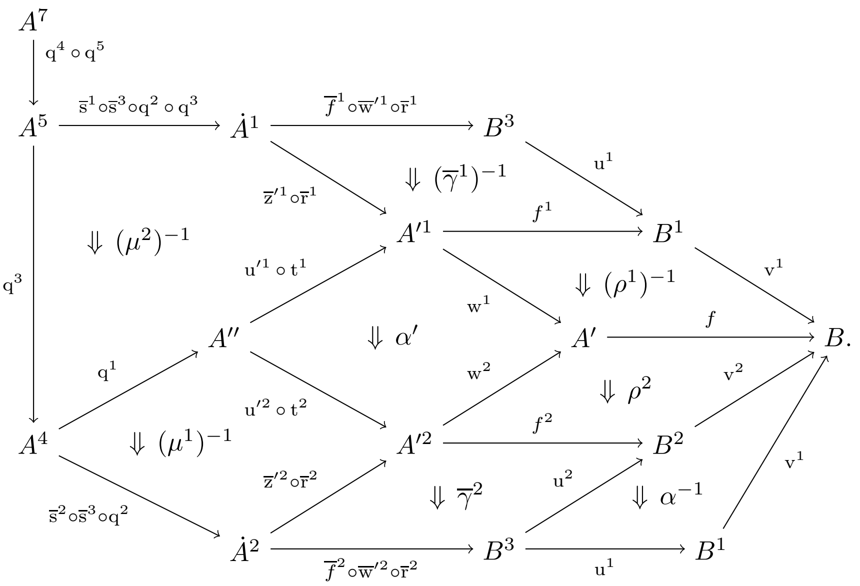
- we replace $i_{\mathrm{u}^1} * \eta^1$ with diagram (4.8);
- we replace $i_{\mathrm{v}^1 \circ \mathrm{u}^1} * \delta$ with diagram (0.12);
- we simplify σ^1 (from (4.8)) with its inverse (from (0.12)).

So we get that (4.12) coincides with the following composition:



(4.13)

Now we replace $i_{\mathrm{u}^2} * \eta^2$ in (4.13) with diagram (4.10) and we simplify σ^2 (from (4.13)) with its inverse (from (4.10)). So we get that (4.12) coincides with the following composition:



(4.14)

Now we replace $\alpha' * i_{\mathrm{q}^1 \circ \mathrm{q}^3}$ in (4.14) with (4.5) and we simplify μ^2 and μ^1. So we get that (4.14) (hence (4.12)) coincides with the composition of (4.2) with the 2-identity of $\mathrm{q}^2 \circ \mathrm{q}^3 \circ \mathrm{q}^4 \circ \mathrm{q}^5 : A^7 \to \dot{A}'$. In other terms, we have proved that

$$i_{\mathrm{v}^1\circ\mathrm{u}^1}*\left(\eta^2\odot\left(\delta*i_{\mathrm{q}^1\circ\mathrm{q}^3\circ\mathrm{q}^4\circ\mathrm{q}^5}\right)\odot\left(\eta^1*i_{\mathrm{q}^5}\right)\right)=i_{\mathrm{v}^1\circ\mathrm{u}^1}*\left(\overline{\eta}*i_{\mathrm{q}^2\circ\mathrm{q}^3\circ\mathrm{q}^4\circ\mathrm{q}^5}\right).$$

Using Lemma 1.1 we conclude that there are an object A^8 and a morphism $\mathrm{q}^6 : A^8 \to A^7$ in $\mathbf{W}$, such that

$$\left(\eta^2\odot\left(\delta*i_{\mathrm{q}^1\circ\mathrm{q}^3\circ\mathrm{q}^4\circ\mathrm{q}^5}\right)\odot\left(\eta^1*i_{\mathrm{q}^5}\right)\right)*i_{\mathrm{q}^6}=\overline{\eta}*i_{\mathrm{q}^2\circ\mathrm{q}^3\circ\mathrm{q}^4\circ\mathrm{q}^5\circ\mathrm{q}^6}.$$

By replacing this in (4.11) and comparing with (4.4), we get that

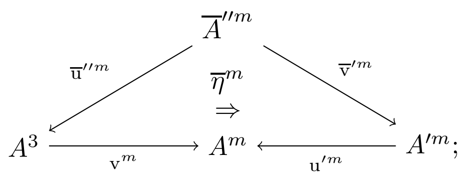
$$\widetilde{\beta}*i_{\mathrm{q}^4\circ\mathrm{q}^5\circ\mathrm{q}^6}=\overline{\beta}'*i_{\mathrm{q}^2\circ\mathrm{q}^3\circ\mathrm{q}^4\circ\mathrm{q}^5\circ\mathrm{q}^6}.$$

This implies easily that the class of (4.6) (hence also the class of (0.13) coincides with the class of (4.3), so we get the claim. □

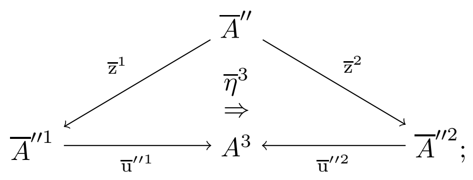
5. Horizontal compositions with 1-morphisms on the right

Proof of Proposition 0.4. According to [Pr, pag. 259], in order to compute $i_{\underline{g}} * \Gamma$ we have to proceed as follows:

(i) for each $m = 1, 2$, we use choices $\mathrm{C}(\mathbf{W})$ in order to get data as in the upper part of the following diagram, with $\overline{\mathrm{u}}''^m$ in $\mathbf{W}$ and $\overline{\eta}^m$ invertible:



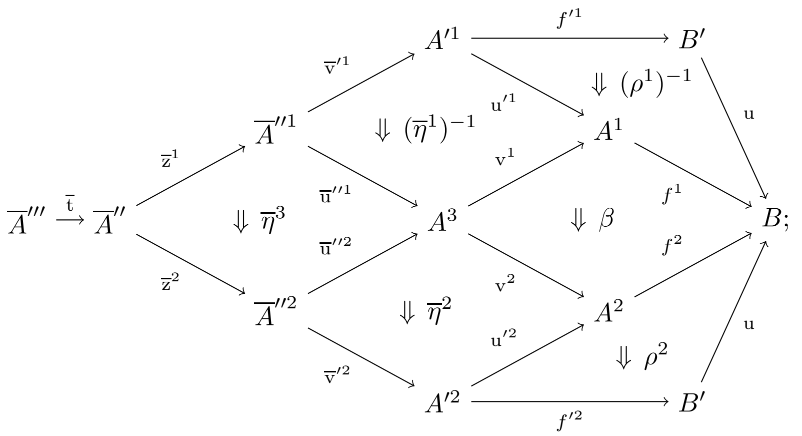
(ii) we use again choices $\mathrm{C}(\mathbf{W})$ in order to get data as in the upper part of the following diagram, with $\overline{\mathrm{z}}^1$ in $\mathbf{W}$ and $\overline{\eta}^3$ invertible:



(iii) we use choices $\mathrm{D}(\mathbf{W})$ in order to get an object $\overline{A}'''$, a morphism $\overline{\mathrm{t}} : \overline{A}''' \to \overline{A}''$ in $\mathbf{W}$ and a 2-morphism

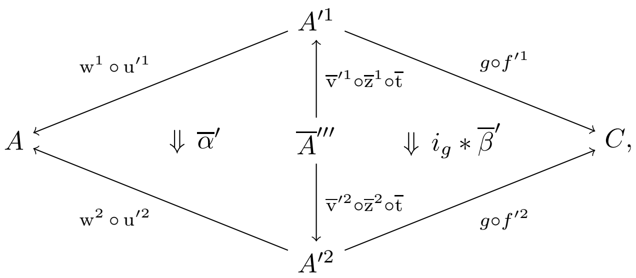
$$\overline{\beta}' :\; f'^1\circ\overline{\mathrm{v}}'^1\circ\overline{\mathrm{z}}^1\circ\overline{\mathrm{t}}\Longrightarrow f'^2\circ\overline{\mathrm{v}}'^2\circ\overline{\mathrm{z}}^2\circ\overline{\mathrm{t}},$$

such that $i_{\mathrm{u}} * \overline{\beta}'$ coincides with the following composition:



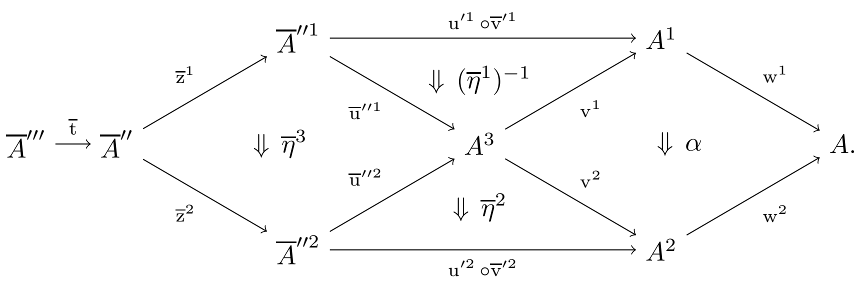
(5.1)

(iv) then following [Pr], one defines $i_{\underline{g}} * \Gamma$ as the class of the following diagram

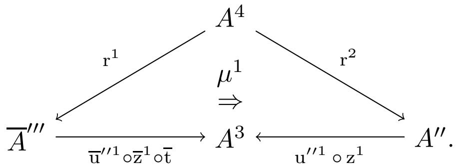


(5.2)

where $\overline{\alpha}'$ is the following composition:



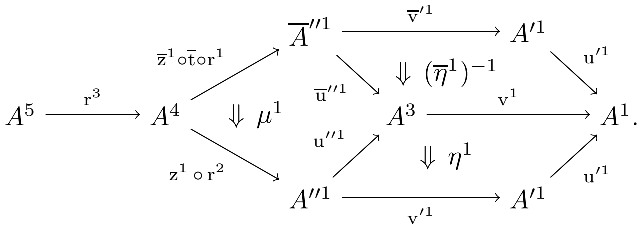
So the claim is equivalent to proving that the class of (0.16) coincides with the class of (5.2). In order to do that, we proceed as follows. By hypothesis, both u''^1 and z^1 belong to $\mathbf{W}$; so we use (BF2) and (BF3) in order to get data as in the upper part of the following diagram, with r^1 in $\mathbf{W}$ and μ^1 invertible:



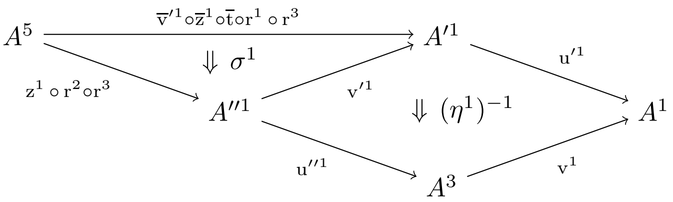
By hypothesis u'^1 belongs to $\mathbf{W}$, so we use (BF4a) and (BF4b) in order to get an object A^5, a morphism $r^3 : A^5 \to A^4$ in $\mathbf{W}$ and an invertible 2-morphism

$$\sigma^1 : \overline{\mathrm{v}}'^1 \circ \overline{\mathrm{z}}^1 \circ \overline{\mathrm{t}} \circ \mathrm{r}^1 \circ \mathrm{r}^3 \Longrightarrow \mathrm{v}'^1 \circ \mathrm{z}^1 \circ \mathrm{r}^2 \circ \mathrm{r}^3,$$

such that $i_{\mathrm{u}'^1} * \sigma^1$ coincides with the following composition:

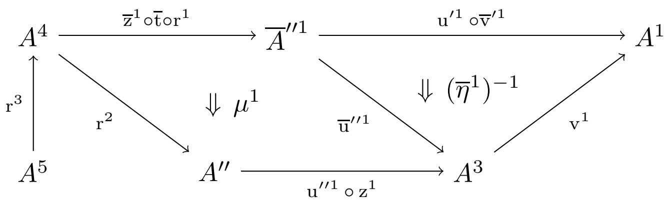


This implies that the following composition



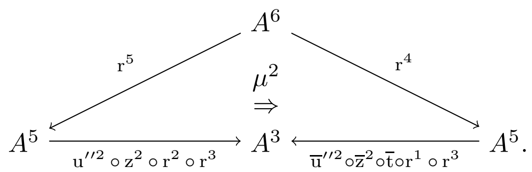
(5.3)

coincides with the following one:



(5.4)

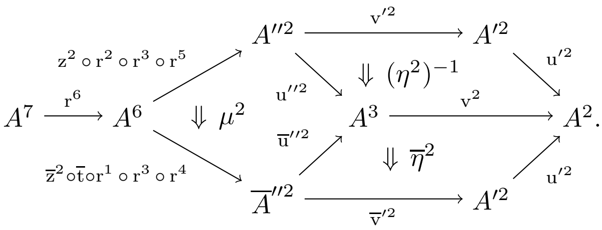
By construction, both $\overline{\mathrm{u}}''^1$ and $\overline{\mathrm{z}}^1$ belong to $\mathbf{W}$, so using (BF2) and (BF5) (for $(\overline{\eta}^3)^{-1}$), we get that $\overline{\mathrm{u}}''^2 \circ \overline{\mathrm{z}}^2$ belongs to $\mathbf{W}$. Moreover, by construction, $\overline{\mathrm{t}}$, r^1 and r^3 belong to $\mathbf{W}$. So using (BF2) and (BF3) we get data as in the upper part of the following diagram, with r^5 in $\mathbf{W}$ and μ^2 invertible:



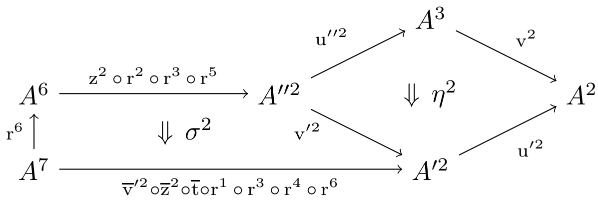
By hypothesis, u'^2 belongs to $\mathbf{W}$, so using (BF4a) and (BF4b) we get an object A^7, a morphism $\mathrm{r}^6 : A^7 \to A^6$ in $\mathbf{W}$ and an invertible 2-morphism

$$\sigma^2 : \mathrm{v}'^2 \circ \mathrm{z}^2 \circ \mathrm{r}^2 \circ \mathrm{r}^3 \circ \mathrm{r}^5 \circ \mathrm{r}^6 \Longrightarrow \overline{\mathrm{v}}'^2 \circ \overline{\mathrm{z}}^2 \circ \overline{\mathrm{t}} \circ \mathrm{r}^1 \circ \mathrm{r}^3 \circ \mathrm{r}^4 \circ \mathrm{r}^6,$$

such that $i_{\mathrm{u}'^2} * \sigma^2$ coincides with the following composition:

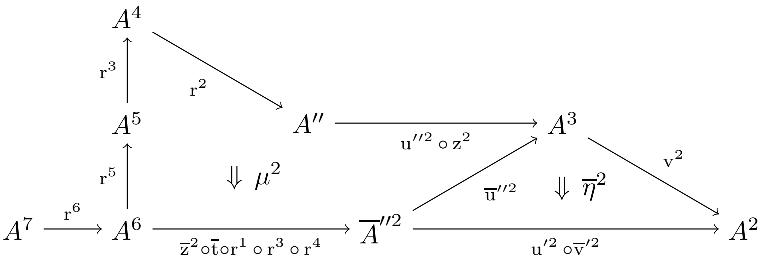


This implies that the following composition



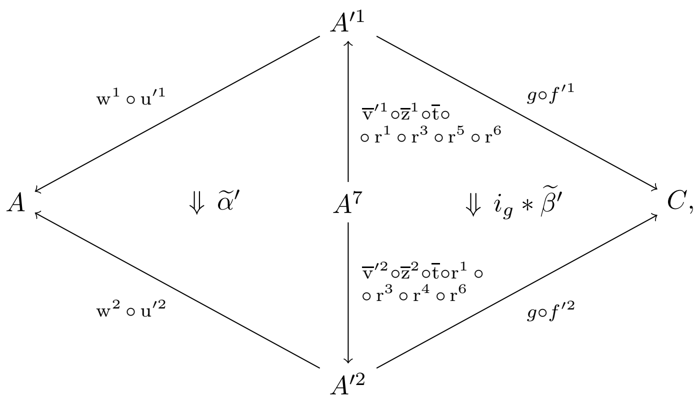
(5.5)

coincides with the following one:



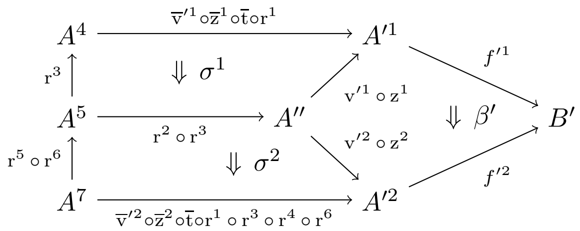
(5.6)

Then we get easily that the class of (0.16) coincides with the class of the following diagram:

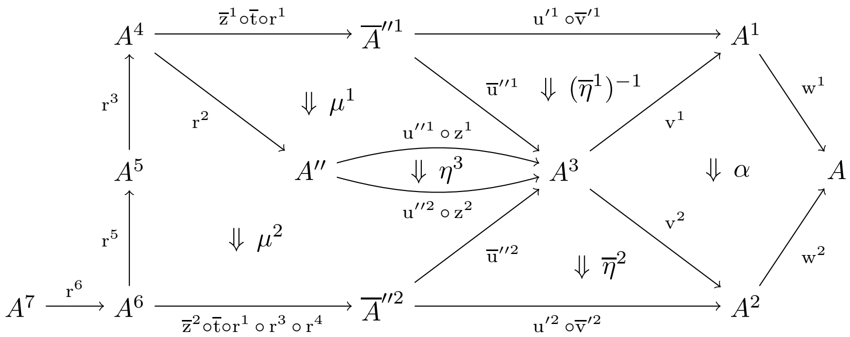


(5.7)

where $\widetilde{\beta}'$ is the following composition:



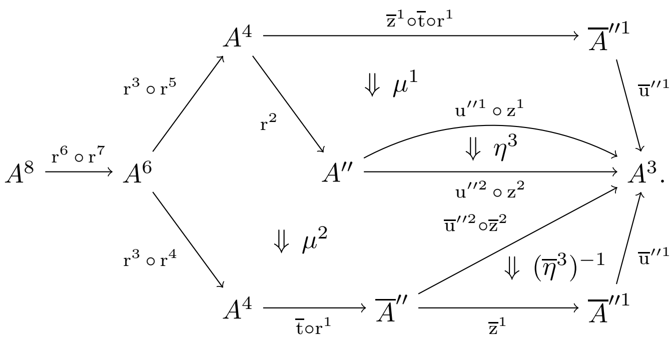
and $\widetilde{\alpha}'$ is the following composition:



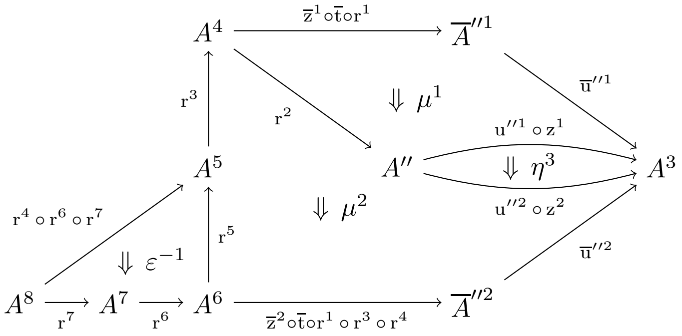
(here $\widetilde{\alpha}'$ is obtained using the identity of (5.3) and (5.4) and the identity of (5.5) and (5.6)). By construction, $\overline{\mathrm{u}}''^1$, $\overline{\mathrm{z}}^1$, $\overline{\mathrm{t}}$, r^1 and r^3 belong to $\mathbf{W}$, so we use (BF2), (BF4a) and (BF4b) in order to get an object A^8, a morphism $\mathrm{r}^7 : A^8 \to A^7$ in $\mathbf{W}$ and an invertible 2-morphism

$$\varepsilon : \mathrm{r}^5 \circ \mathrm{r}^6 \circ \mathrm{r}^7 \Longrightarrow \mathrm{r}^4 \circ \mathrm{r}^6 \circ \mathrm{r}^7,$$

such that $i_{\overline{\mathrm{u}}''^1 \circ \overline{\mathrm{z}}^1 \circ \overline{\mathrm{t}} \circ \mathrm{r}^1 \circ \mathrm{r}^3} * \varepsilon$ coincides with the following composition:

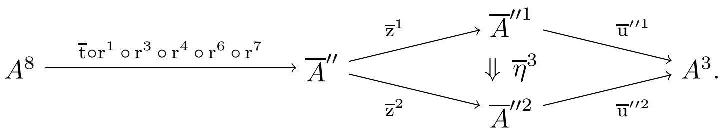


Therefore, we have that the following composition



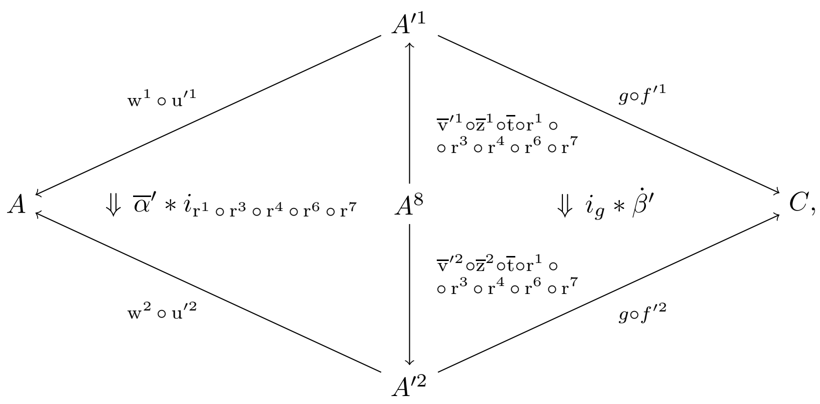
(5.8)

coincides with the following one



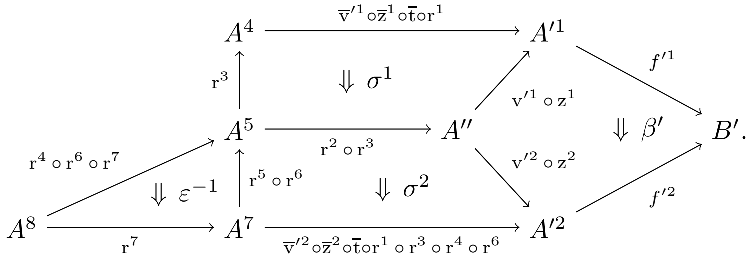
(5.9)

So we get that the class of (5.7) (hence also the class of (0.16)) coincides with the class of the following diagram:

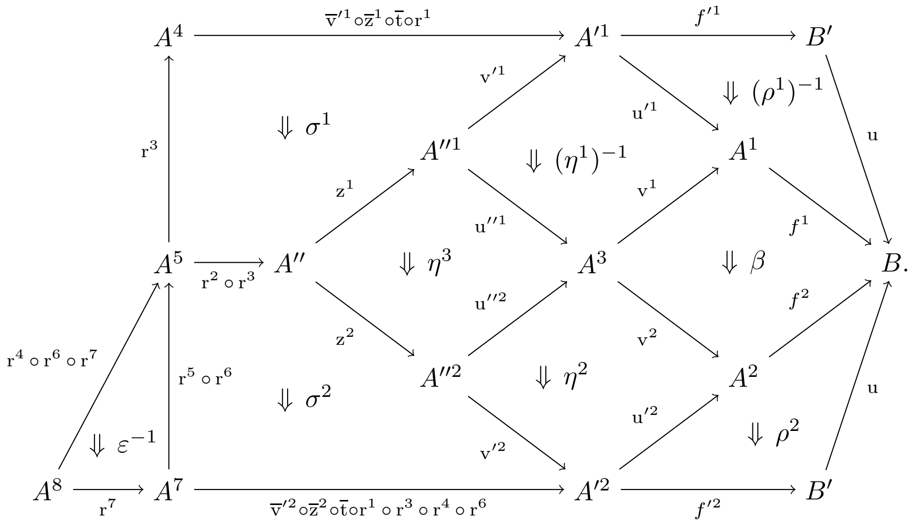


(5.10)

where $\dot{\beta}'$ is the following composition:

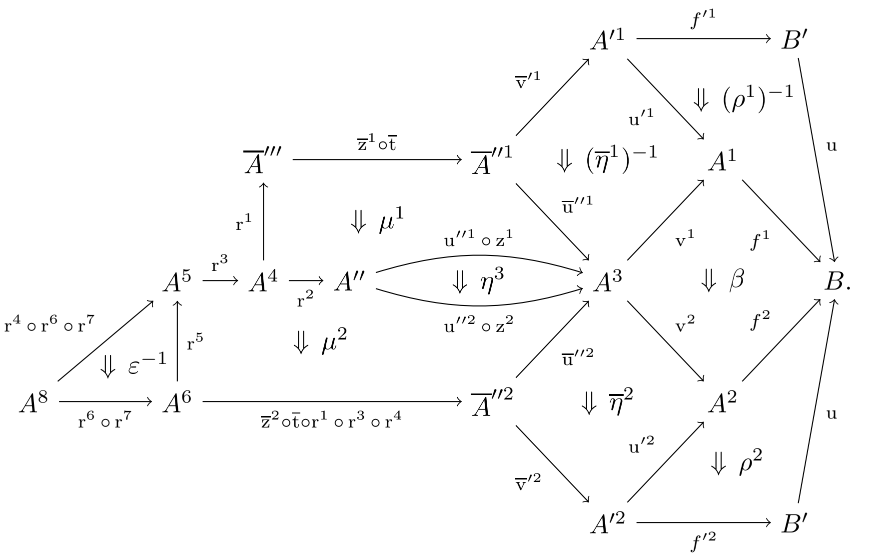


Now using (0.15) we get that $i_u*\dot{\beta}'$ coincides with the following composition:



(5.11)

Now in (5.11) we replace (5.3) with (5.4) and (5.5) with (5.6); so we get that $i_{\mathrm{u}} * \dot{\beta}'$ coincides with the following composition:



In the previous diagram we replace diagram (5.8) with (5.9). Therefore, we conclude that $i_{\mathrm{u}} * \dot{\beta}'$ coincides with $i_{\mathrm{u}} * \overline{\beta}' * i_{\mathrm{r}^1 \circ \mathrm{r}^3 \circ \mathrm{r}^4 \circ \mathrm{r}^6 \circ \mathrm{r}^7}$ (see (5.1)). So by Lemma 1.1 there are an object A^9 and a morphism $\mathrm{r}^8 : A^9 \to A^8$ in $\mathbf{W}$, such that

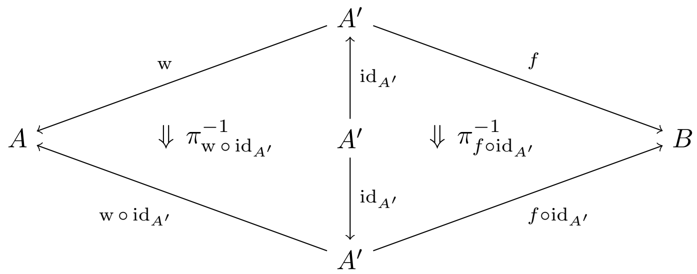
$$\dot{\beta}' * i_{\mathrm{r}^8} = \overline{\beta}' * i_{\mathrm{r}^1 \circ \mathrm{r}^3 \circ \mathrm{r}^4 \circ \mathrm{r}^6 \circ \mathrm{r}^7 \circ \mathrm{r}^8}.$$

So we conclude easily that the class of (5.10) (hence also the class of (0.16)) coincides with the class of (5.2). □

As an application of all the Propositions proved so far, we are giving the following Corollary, that will be useful in the next paper of this series ([T1]). In that paper, we will have to compare the compositions of 3 morphisms of the form (0.3) with the compositions of 3 morphisms of the following form

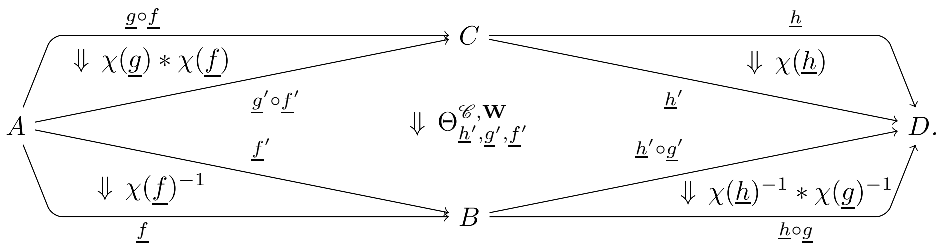
$$\underline{f}' := \Big(A \xleftarrow{\ \mathrm{u}\circ \mathrm{id}_{A'}\ } A' \xrightarrow{\ f\circ \mathrm{id}_{A'}\ } B\Big),$$
$$\underline{g}' := \Big(B \xleftarrow{\ \mathrm{v}\circ \mathrm{id}_{B'}\ } B' \xrightarrow{\ g\circ \mathrm{id}_{B'}\ } C\Big),$$
$$\underline{h}' := \Big(C \xleftarrow{\ \mathrm{w}\circ \mathrm{id}_{C'}\ } C' \xrightarrow{\ h\circ \mathrm{id}_{B'}\ } D\Big). \tag{5.12}$$

In the special case when $\mathscr{C}$ is a 2-category we have that $\underline{f} = \underline{f}'$ and so on, hence $\underline{h}\circ(\underline{g}\circ\underline{f}) = \underline{h}'\circ(\underline{g}'\circ\underline{f})$ and $(\underline{h}\circ\underline{g})\circ\underline{f} = (\underline{h}'\circ\underline{g}')\circ\underline{f}$. However, when $\mathscr{C}$ is simply a bicategory, in general the fixed choice in $\mathrm{C}(\mathbf{W})$ for the pair (f,v) (see the first diagram in (0.4)) is different from the fixed choice for the pair $(f\circ\mathrm{id}_{A'}, \mathrm{v}\circ\mathrm{id}_{B'})$, and analogously for all the remaining choices needed to compose the morphisms in (5.12). Therefore, we need a way to compare $\underline{h}\circ(\underline{g}\circ\underline{f})$ with $\underline{h}'\circ(\underline{g}'\circ\underline{f})$ and analogously for the other pair of compositions. In order to do that, first of all we compare separately $\underline{f}$ with $\underline{f}'$ by defining an invertible 2-morphism $\chi(\underline{f}) : \underline{f} \Rightarrow \underline{f}'$ as the class of the following diagram:



and analogously for $\chi(\underline{g})$ and $\chi(\underline{h})$. Then we have the following result.

Corollary 5.1. *Let us fix any pair* $(\mathscr{C}, \mathbf{W})$ *satisfying conditions* (BF) *and any set of* 6 *morphisms as in* (0.3) *and* (5.12)*. Then the associator* $\Theta^{\mathscr{C},\mathbf{W}}_{\underline{h},\underline{g},\underline{f}} : \underline{h}\circ(\underline{g}\circ\underline{f}) \Rightarrow (\underline{h}\circ\underline{g})\circ\underline{f}$ *coincides with the following composition:*



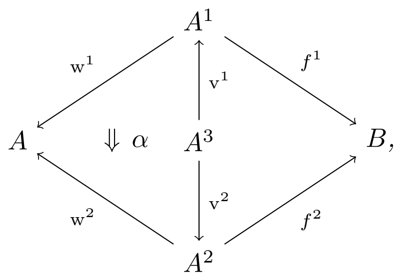
(5.13)

This result is a direct application of Propositions 0.1, 0.2 and 0.4. We refer to the Appendix for a proof.

6. Comparison of 2-morphisms and conditions for invertibility in a bicategory of fractions

In this section we are going to prove Propositions 0.7 and 0.8. As in the previous sections, *we are going to write the statements of each result in the framework of bicategories, but for simplicity of expositions we are going to give all the proofs only in the special framework of* 2*-categories.* The interested reader can easily fill the missing details when $\mathscr{C}$ is not a 2-category but simply a bicategory, by adding associators and unitors wherever it is necessary.

Lemma 6.1. *Let us fix any pair* $(\mathscr{C}, \mathbf{W})$ *satisfying conditions* (BF) *and any set of data in* $\mathscr{C}$ *as follows*



(6.1)

with $\mathrm{w}^1, \mathrm{w}^2$ *and* $\mathrm{w}^1 \circ \mathrm{v}^1$ *in* $\mathbf{W}$ *and* α *invertible. Then for each* 2*-morphism*

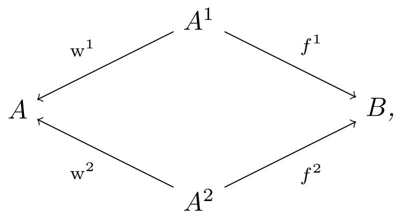
$$\Gamma : \Big(A^1, \mathrm{w}^1, f^1\Big) \Longrightarrow \Big(A^2, \mathrm{w}^2, f^2\Big)$$

in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ *there are an object* A^4*, a morphism* $\mathrm{z} : A^4 \to A^3$ *such that* $(\mathrm{w}^1 \circ \mathrm{v}^1) \circ \mathrm{z}$ *belongs to* $\mathbf{W}$*, and a* 2*-morphism* $\gamma : f^1 \circ (\mathrm{v}^1 \circ \mathrm{z}) \Rightarrow f^2 \circ (\mathrm{v}^2 \circ \mathrm{z})$ *in* $\mathscr{C}$*, such that*

$$\Gamma = \Big[A^4, \mathrm{v}^1 \circ \mathrm{z}, \mathrm{v}^2 \circ \mathrm{z}, \theta^{-1}_{\mathrm{w}^2,\mathrm{v}^2,\mathrm{z}} \odot \Big(\alpha * i_{\mathrm{z}}\Big) \odot \theta_{\mathrm{w}^1,\mathrm{v}^1,\mathrm{z}}, \gamma\Big].$$

We refer to the Appendix for the proof.

Lemma 6.2. *Let us fix any pair* $(\mathscr{C}, \mathbf{W})$ *satisfying conditions* (BF) *and any set of data in* $\mathscr{C}$ *as follows*



(6.2)

with w^1 *and* w^2 *in* $\mathbf{W}$*. Then given any pair of* 2*-morphisms*

$$\Gamma^1, \Gamma^2 : \Big(A^1, \mathrm{w}^1, f^1\Big) \Longrightarrow \Big(A^2, \mathrm{w}^2, f^2\Big)$$

in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$*, there are an object* A^3*, a pair of morphisms* $\mathrm{v}^m : A^3 \to A^m$ *for* $m = 1, 2$*, an invertible* 2*-morphism* $\alpha : \mathrm{w}^1 \circ \mathrm{v}^1 \Rightarrow \mathrm{w}^2 \circ \mathrm{v}^2$ *and a pair of* 2*-morphisms* $\gamma^1, \gamma^2 : f^1 \circ \mathrm{v}^1 \Rightarrow f^2 \circ \mathrm{v}^2$*, such that* $\mathrm{w}^1 \circ \mathrm{v}^1$ *belongs to* $\mathbf{W}$ *and*

$$\Gamma^m = \Big[A^3, \mathrm{v}^1, \mathrm{v}^2, \alpha, \gamma^m\Big] \quad \textit{for} \quad m = 1, 2. \tag{6.3}$$

In other terms, given any pair of 2-morphisms in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ with the same source and target, they differ at most by one term.

Proof. By definition of 2-morphism in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$, there are data $(A^3, \overline{\mathrm{v}}^1, \overline{\mathrm{v}}^2, \overline{\alpha}, \overline{\gamma})$ such that

$$\Gamma^1 = \Big[\overline{A}^3, \overline{\mathrm{v}}^1, \overline{\mathrm{v}}^2, \overline{\alpha}, \overline{\gamma}\Big], \tag{6.4}$$

with $\mathrm{w}^1 \circ \overline{\mathrm{v}}^1$ in $\mathbf{W}$ and $\overline{\alpha}$ invertible. Then we apply Lemma 6.1 to the set of data given by (6.2) and $(\overline{A}^3, \overline{\mathrm{v}}^1, \overline{\mathrm{v}}^2, \overline{\alpha}, \Gamma^2)$. So there are an object A^3, a morphism $\mathrm{z}: A^3 \to \overline{A}^3$ such that $\mathrm{w}^1 \circ \overline{\mathrm{v}}^1 \circ \mathrm{z}$ belongs to $\mathbf{W}$, and a 2-morphism $\gamma^2 : f^1 \circ \overline{\mathrm{v}}^1 \circ \mathrm{z} \Rightarrow f^2 \circ \overline{\mathrm{v}}^2 \circ \mathrm{z}$, such that $\Gamma^2 = [A^3, \overline{\mathrm{v}}^1 \circ \mathrm{z}, \overline{\mathrm{v}}^2 \circ \mathrm{z}, \overline{\alpha} * i_{\mathrm{z}}, \gamma^2]$. By definition of 2-morphism in a bicategory of fractions and (6.4), we have easily $\Gamma^1 = \Big[A^3, \overline{\mathrm{v}}^1 \circ \mathrm{z}, \overline{\mathrm{v}}^2 \circ \mathrm{z}, \overline{\alpha} * i_{\mathrm{z}}, \overline{\gamma} * i_{\mathrm{z}}\Big]$. Then in order to conclude, it suffices to set $\mathrm{v}^m := \overline{\mathrm{v}}^m \circ \mathrm{z}$ for $m = 1, 2$, $\alpha := \overline{\alpha} * i_{\mathrm{z}}$ and $\gamma^1 := \overline{\gamma} * i_{\mathrm{z}}$. □

By induction and using the same ideas, one can also prove easily that given finitely many 2-morphisms $\Gamma^1, \cdots, \Gamma^n$ in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$, all defined between the same pair of morphisms, there are data $A^3, \mathrm{v}^1, \mathrm{v}^2, \alpha, \gamma^1, \cdots, \gamma^n$, such that $\mathrm{w}^1 \circ \mathrm{v}^1$ belongs to $\mathbf{W}$, α is invertible and $\Gamma^m = [A^3, \mathrm{v}^1, \mathrm{v}^1, \alpha, \gamma^m]$ for each $m = 1, \cdots, n$.

Until now we have proved that any pair of 2-morphisms Γ^1, Γ^2 between the same pair of morphisms in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ can be "reduced" to a common form (as in (6.3)), where all data, except (possibly) one, coincide. The next lemma shows under which condition on the remaining datum we have $\Gamma^1 = \Gamma^2$.

Lemma 6.3. *Let us fix any pair* $(\mathscr{C}, \mathbf{W})$ *satisfying conditions* (BF) *and any set of data in* $\mathscr{C}$ *as in* (6.1)*, with* $\mathrm{w}^1, \mathrm{w}^2$ *and* $\mathrm{w}^1 \circ \mathrm{v}^1$ *in* $\mathbf{W}$ *and* α *invertible. Moreover, let us fix any pair of* 2*-morphisms* $\gamma^1, \gamma^2 : f^1 \circ \mathrm{v}^1 \Rightarrow f^2 \circ \mathrm{v}^2$*; for each* $m = 1, 2$*, let us consider the* 2*-morphism in* $\mathscr{C}\left[\mathbf{W}^{-1}\right]$

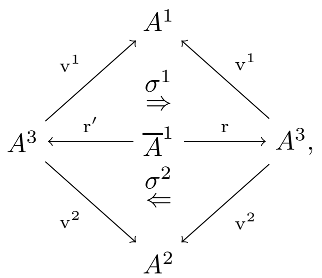
$$\Gamma^m := \Big[A^3, \mathrm{v}^1, \mathrm{v}^2, \alpha, \gamma^m\Big] : \Big(A^1, \mathrm{w}^1, f^1\Big) \Longrightarrow \Big(A^2, \mathrm{w}^2, f^2\Big).$$

Then the following facts are equivalent

(*i*) $\Gamma^1 = \Gamma^2$*;*

(*ii*) *there are an object* A^4 *and a morphism* $\mathrm{z}: A^4 \to A^3$*, such that* $(\mathrm{w}^1 \circ \mathrm{v}^1) \circ \mathrm{z}$ *belongs to* $\mathbf{W}$ *and* $\gamma^1 * i_{\mathrm{z}} = \gamma^2 * i_{\mathrm{z}}$*.*

Proof. Let us suppose that (i) holds. Then there is a set of data in $\mathscr{C}$ as in the internal part of the following diagram:



such that $\mathrm{w}^1 \circ \mathrm{v}^1 \circ \mathrm{r}$ belongs to $\mathbf{W}$, σ^1 and σ^2 are invertible,

$$\Big(i_{\mathrm{w}^2} * \sigma^2\Big) \odot \Big(\alpha * i_{\mathrm{r}}\Big) \odot \Big(i_{\mathrm{w}^1} * \sigma^1\Big) = \alpha * i_{\mathrm{r}'} \tag{6.5}$$

and

$$\Big(i_{f^2} * \sigma^2\Big) \odot \Big(\gamma^2 * i_{\mathrm{r}}\Big) \odot \Big(i_{f^1} * \sigma^1\Big) = \gamma^1 * i_{\mathrm{r}'}. \tag{6.6}$$

Since both w^1 and $\mathrm{w}^1 \circ \mathrm{v}^1$ belong to $\mathbf{W}$ by hypothesis, then by Lemma 1.3 there are an object $\overline{A}^2$, a morphism $\mathrm{p}^1 : \overline{A}^2 \to \overline{A}^1$ in $\mathbf{W}$ and an invertible 2-morphism $\widetilde{\sigma}^1 : \mathrm{r}' \circ \mathrm{p}^1 \Rightarrow \mathrm{r} \circ \mathrm{p}^1$, such that $i_{\mathrm{v}^1} * \widetilde{\sigma}^1 = \sigma^1 * i_{\mathrm{p}^1}$.

Since $\mathrm{w}^1\circ\mathrm{v}^1$ belongs to $\mathbf{W}$, then by (BF5) applied α^{-1} we get that also $\mathrm{w}^2\circ\mathrm{v}^2$ belongs to $\mathbf{W}$; since also w^2 belongs to $\mathbf{W}$ by hypothesis, then by Lemma 1.3 there are an object A^4, a morphism $\mathrm{p}^2: A^4\rightarrow\overline{A}^2$ in $\mathbf{W}$ and an invertible 2-morphism $\widetilde{\sigma}^2:\mathrm{r}\circ\mathrm{p}^1\circ\mathrm{p}^2\Rightarrow\mathrm{r}'\circ\mathrm{p}^1\circ\mathrm{p}^2$, such that $i_{\mathrm{v}^2}*\widetilde{\sigma}^2=\sigma^2*i_{\mathrm{p}^1\circ\mathrm{p}^2}$. Using (6.5), this implies that:

$$\Big(i_{\mathrm{w}^2\circ\mathrm{v}^2}*\widetilde{\sigma}^2\Big)\odot\Big(\alpha*i_{\mathrm{r}\circ\mathrm{p}^1\circ\mathrm{p}^2}\Big)\odot\Big(i_{\mathrm{w}^1\circ\mathrm{v}^1}*\widetilde{\sigma}^1*i_{\mathrm{p}^2}\Big)=\alpha*i_{\mathrm{r}'\circ\mathrm{p}^1\circ\mathrm{p}^2}.$$

Using interchange law and the fact that α is invertible by hypothesis, the previous identity implies that $\big(\widetilde{\sigma}^2\big)^{-1}=\widetilde{\sigma}^1*i_{\mathrm{p}^2}$. Then (6.6) implies that:

$$\begin{aligned}&\gamma^1*i_{\mathrm{r}'\circ\mathrm{p}^1\circ\mathrm{p}^2}=\\ &=\Big(i_{f^2}*\sigma^2*i_{\mathrm{p}^1\circ\mathrm{p}^2}\Big)\odot\Big(\gamma^2*i_{\mathrm{r}\circ\mathrm{p}^1\circ\mathrm{p}^2}\Big)\odot\Big(i_{f^1}*\sigma^1*i_{\mathrm{p}^1\circ\mathrm{p}^2}\Big)=\\ &=\Big(i_{f^2\circ\mathrm{v}^2}*\widetilde{\sigma}^2\Big)\odot\Big(\gamma^2*i_{\mathrm{r}\circ\mathrm{p}^1\circ\mathrm{p}^2}\Big)\odot\Big(i_{f^1\circ\mathrm{v}^1}*\widetilde{\sigma}^1*i_{\mathrm{p}^2}\Big)=\\ &=\Big(i_{f^2\circ\mathrm{v}^2}*\widetilde{\sigma}^2\Big)\odot\Big(\gamma^2*i_{\mathrm{r}\circ\mathrm{p}^1\circ\mathrm{p}^2}\Big)\odot\Big(i_{f^1\circ\mathrm{v}^1}*(\widetilde{\sigma}^2)^{-1}\Big)=\\ &=\gamma^2*i_{\mathrm{r}'\circ\mathrm{p}^1\circ\mathrm{p}^2}.\end{aligned}\tag{6.7}$$

Then we set $\mathrm{z}:=\mathrm{r}'\circ\mathrm{p}^1\circ\mathrm{p}^2: A^4\rightarrow A^3$; by (BF2) and construction we have that $\mathrm{w}^1\circ\mathrm{v}^1\circ\mathrm{r}\circ\mathrm{p}^1\circ\mathrm{p}^2$ belongs to $\mathbf{W}$. So using (BF5) and $i_{\mathrm{w}^1}*\sigma^1*i_{\mathrm{p}^1\circ\mathrm{p}^2}$, we conclude that also $\mathrm{w}^1\circ\mathrm{v}^1\circ\mathrm{z}$ belongs to $\mathbf{W}$. Using (6.7), we get that (ii) holds. Conversely, if (ii) holds, then (i) is obviously satisfied using the definition of 2-morphism in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$. □

Combining Lemmas 6.2 and 6.3, we get Proposition 0.7.

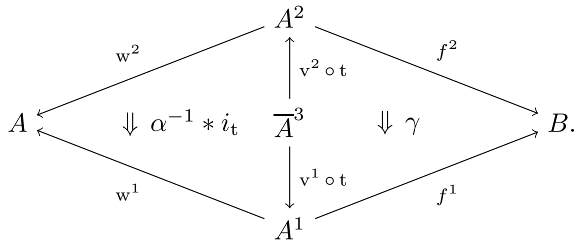
Proof of Proposition 0.8. Let us assume (iii), so let us choose any representative (0.18) for Γ and let us assume that there is a pair $(A^4,\mathrm{u}: A^4\rightarrow A^3)$ such that $\mathrm{w}^1\circ\mathrm{v}^1\circ\mathrm{u}$ belongs to $\mathbf{W}$ and $\beta*i_{\mathrm{u}}$ is invertible in $\mathscr{C}$. By the description of 2-morphisms in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$, we have

$$\Gamma=\Big[A^4,\mathrm{v}^1\circ\mathrm{u},\mathrm{v}^2\circ\mathrm{u},\alpha*i_{\mathrm{u}},\beta*i_{\mathrm{u}}\Big]$$

and the last term of such a tuple is invertible in $\mathscr{C}$ by hypothesis, so (ii) holds.

Now let us assume (ii), so let us assume that the tuple of data (0.18) is such that β is invertible in $\mathscr{C}$. Since α is invertible in $\mathscr{C}$ by definition of 2-morphism in a bicategory of fractions, then it makes sense to consider the 2-morphism $[A^3,\mathrm{v}^2,\mathrm{v}^1,\alpha^{-1},\beta^{-1}]$ in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$. Using Proposition 0.2, it is easy to see that this is an inverse for Γ, so (i) holds.

Now let us prove that (i) implies (iii), so let us assume that Γ is invertible and let us fix any representative (0.18) for it; by definition of 2-morphism in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ we have that α is invertible, so we can apply Lemma 6.1 for the 2-morphism α^{-1} of $\mathscr{C}$ and for the 2-morphism Γ^{-1} of $\mathscr{C}\left[\mathbf{W}^{-1}\right]$. So there are an object $\overline{A}^3$, a morphism $\mathrm{t}:\overline{A}^3\rightarrow A^3$, such that $\mathrm{w}^2\circ\mathrm{v}^2\circ\mathrm{t}$ belongs to $\mathbf{W}$, and a 2-morphism $\gamma: f^2\circ\mathrm{v}^2\circ\mathrm{t}\Rightarrow f^1\circ\mathrm{v}^1\circ\mathrm{t}$, such that Γ^{-1} is represented by the following diagram:



Then we have

$$\Big[\overline{A}^3, \mathrm{v}^2 \circ \mathrm{t}, \mathrm{v}^2 \circ \mathrm{t}, i_{\mathrm{w}^2 \circ \mathrm{v}^2 \circ \mathrm{t}}, i_{f^2 \circ \mathrm{v}^2 \circ \mathrm{t}}\Big] =$$

$$= i_{(A^2,\mathrm{w}^2,f^2)} = \Gamma \odot \Gamma^{-1} \stackrel{(*)}{=} \Big[\overline{A}^3, \mathrm{v}^2 \circ \mathrm{t}, \mathrm{v}^2 \circ \mathrm{t}, i_{\mathrm{w}^2 \circ \mathrm{v}^2 \circ \mathrm{t}}, \Big(\beta * i_{\mathrm{t}}\Big) \odot \gamma\Big],$$

where $(*)$ is obtained applying Proposition 0.2. So by Lemma 6.3 there are an object $\widetilde{A}^3$ and a morphism $\mathrm{r} : \widetilde{A}^3 \to \overline{A}^3$, such that $\mathrm{w}^2 \circ \mathrm{v}^2 \circ \mathrm{t} \circ \mathrm{r}$ belongs to $\mathbf{W}$ and

$$\Big(\beta * i_{\mathrm{t} \circ \mathrm{r}}\Big) \odot \Big(\gamma * i_{\mathrm{r}}\Big) = i_{f^2 \circ \mathrm{v}^2 \circ \mathrm{t} \circ \mathrm{r}}. \tag{6.8}$$

Analogously, we have:

$$\Big[\widetilde{A}^3, \mathrm{v}^1 \circ \mathrm{t} \circ \mathrm{r}, \mathrm{v}^1 \circ \mathrm{t} \circ \mathrm{r}, i_{\mathrm{w}^1 \circ \mathrm{v}^1 \circ \mathrm{t} \circ \mathrm{r}}, i_{f^1 \circ \mathrm{v}^1 \circ \mathrm{t} \circ \mathrm{r}}\Big] =$$

$$= i_{(A^1,\mathrm{w}^1,f^1)} = \Gamma^{-1} \odot \Gamma = \Big[\widetilde{A}^3, \mathrm{v}^1 \circ \mathrm{t} \circ \mathrm{r}, \mathrm{v}^1 \circ \mathrm{t} \circ \mathrm{r}, i_{\mathrm{w}^1 \circ \mathrm{v}^1 \circ \mathrm{t} \circ \mathrm{r}}, \Big(\gamma * i_{\mathrm{r}}\Big) \odot \Big(\beta * i_{\mathrm{t} \circ \mathrm{r}}\Big)\Big].$$

So again by Lemma 6.3 there are an object A^4 and a morphism $\mathrm{s} : A^4 \to \widetilde{A}^3$, such that $\mathrm{w}^1 \circ \mathrm{v}^1 \circ \mathrm{t} \circ \mathrm{r} \circ \mathrm{s}$ belongs to $\mathbf{W}$ and

$$\Big(\gamma * i_{\mathrm{r} \circ \mathrm{s}}\Big) \odot \Big(\beta * i_{\mathrm{t} \circ \mathrm{r} \circ \mathrm{s}}\Big) = i_{f^1 \circ \mathrm{v}^1 \circ \mathrm{t} \circ \mathrm{r} \circ \mathrm{s}}. \tag{6.9}$$

Then (6.8) and (6.9) prove that $\beta * i_{\mathrm{t} \circ \mathrm{r} \circ \mathrm{s}}$ has an inverse in $\mathscr{C}$ given by $\gamma * i_{\mathrm{r} \circ \mathrm{s}}$; in order to conclude that (iii) holds, it suffices to define $\mathrm{u} := \mathrm{t} \circ \mathrm{r} \circ \mathrm{s} : A^4 \to A^3$. □

Appendix

Proof of Lemma 1.1. We set

$$\alpha := i_{\mathrm{w}} * \gamma : \ \mathrm{w} \circ f^1 \Longrightarrow \mathrm{w} \circ f^2.$$

Then condition (BF4a) is obviously satisfied by the set of data:

$$C, \quad \mathrm{v} := \mathrm{id}_C, \quad \beta := \pi_{f^2}^{-1} \odot \gamma \odot \pi_{f^1} : \ f^1 \circ \mathrm{v} \Longrightarrow f^2 \circ \mathrm{v}$$

(here π_{f^1} is the unitor $f^1 \circ \mathrm{id}_C \Rightarrow f^1$, and analogously for π_{f^2}). Since we have also $\alpha = i_{\mathrm{w}} * \gamma'$, then (BF4a) is also satisfied by:

$$C, \quad \mathrm{v}' := \mathrm{id}_C, \quad \beta' := \pi_{f^2}^{-1} \odot \gamma' \odot \pi_{f^1} : \ f^1 \circ \mathrm{v} \Longrightarrow f^2 \circ \mathrm{v}\,.$$

Then by (BF4c) there are an object D, a pair of morphisms $\mathrm{u}, \mathrm{u}' : D \to C$ and an invertible 2-morphism $\zeta : \mathrm{id}_C \circ \mathrm{u} \Rightarrow \mathrm{id}_C \circ \mathrm{u}'$, such that $\mathrm{id}_C \circ \mathrm{u}$ belongs to $\mathbf{W}$ and

$$\theta^{-1}_{f^2,\mathrm{id}_C,\mathrm{u}'} \odot \Big(\Big(\pi_{f^2}^{-1} \odot \gamma' \odot \pi_{f^1}\Big) * i_{\mathrm{u}'}\Big) \odot \theta_{f^1,\mathrm{id}_C,\mathrm{u}'} \odot \Big(i_{f^1} * \zeta\Big) =$$

$$= \Big(i_{f^2} * \zeta\Big) \odot \theta^{-1}_{f^2,\mathrm{id}_C,\mathrm{u}} \odot \Big(\Big(\pi_{f^2}^{-1} \odot \gamma \odot \pi_{f^1}\Big) * i_{\mathrm{u}}\Big) \odot \theta_{f^1,\mathrm{id}_C,\mathrm{u}}.$$

Using the coherence axioms on the bicategory $\mathscr{C}$, this implies that $\gamma * i_{\mathrm{u}} = \gamma' * i_{\mathrm{u}}$. Moreover, using $\upsilon_{\mathrm{u}}^{-1} : \mathrm{u} \Rightarrow \mathrm{id}_C \circ \mathrm{u}$ and (BF5), we get that u belongs to $\mathbf{W}$. □

Proof of Lemma 1.2. We apply condition (BF3) to the pair of morphisms $(\mathrm{z} \circ f, \mathrm{z} \circ \mathrm{w})$, so we get an object E, a morphism $\mathrm{t} : E \to C$ in $\mathbf{W}$, a morphism $g : E \to A$ and an invertible 2-morphism $\beta : (\mathrm{z} \circ f) \circ \mathrm{t} \Rightarrow (\mathrm{z} \circ \mathrm{w}) \circ g$. Then we apply (BF4a) and (BF4b) to the invertible 2-morphism

$$\theta^{-1}_{\mathrm{z},\mathrm{w},g} \odot \beta \odot \theta_{\mathrm{z},f,\mathrm{t}} : \ \mathrm{z} \circ (f \circ \mathrm{t}) \Longrightarrow \mathrm{z} \circ (\mathrm{w} \circ g).$$

So there are an object D, a morphism $\mathrm{r} : D \to E$ in $\mathbf{W}$ and an invertible 2-morphism $\gamma : (f \circ \mathrm{t}) \circ \mathrm{r} \Longrightarrow (\mathrm{w} \circ g) \circ \mathrm{r}$, such that

$$\Big(\theta^{-1}_{\mathrm{z},\mathrm{w},g} \odot \beta \odot \theta_{\mathrm{z},f,\mathrm{t}}\Big) * i_{\mathrm{r}} = \theta_{\mathrm{z},\mathrm{w} \circ g,\mathrm{r}} \odot \Big(i_{\mathrm{z}} * \gamma\Big) \odot \theta^{-1}_{\mathrm{z},f \circ \mathrm{t},\mathrm{r}}. \tag{6.10}$$

Then we set $\mathrm{w}' := \mathrm{t} \circ \mathrm{r} : D \to C$; this morphism belongs to $\mathbf{W}$ by construction and (BF2). Moreover, we define $f' := g \circ \mathrm{r} : D \to A$ and

$$\alpha := \theta^{-1}_{\mathrm{w},g,\mathrm{r}} \odot \gamma \odot \theta_{f,\mathrm{t},\mathrm{r}} : \ f \circ \mathrm{w}' \Longrightarrow \mathrm{w} \circ f'.$$

□

Proof of Lemma 1.3. We use (BF4a) on the 2-morphism

$$\theta_{\mathrm{z},\mathrm{w},f^2} \odot \Big(i_{\mathrm{z}} * \alpha\Big) \odot \theta^{-1}_{\mathrm{z},\mathrm{w},f^1} : \ (\mathrm{z} \circ \mathrm{w}) \circ f^1 \Longrightarrow (\mathrm{z} \circ \mathrm{w}) \circ f^2.$$

Then there are an object E, a morphism $\mathrm{t} : E \to C$ in $\mathbf{W}$ and a 2-morphism $\gamma : f^1 \circ \mathrm{t} \Rightarrow f^2 \circ \mathrm{t}$, such that

$$\Big(\theta_{\mathrm{z},\mathrm{w},f^2} \odot \Big(i_{\mathrm{z}} * \alpha\Big) \odot \theta^{-1}_{\mathrm{z},\mathrm{w},f^1}\Big) * i_{\mathrm{t}} = \theta_{\mathrm{z} \circ \mathrm{w},f^2,\mathrm{t}} \odot \Big(i_{\mathrm{z} \circ \mathrm{w}} * \gamma\Big) \odot \theta^{-1}_{\mathrm{z} \circ \mathrm{w},f^1,\mathrm{t}}.$$

This implies that

$$i_{\mathrm{z}} * \Big(\theta_{\mathrm{w},f^2,\mathrm{t}} \odot \Big(i_{\mathrm{w}} * \gamma\Big) \odot \theta^{-1}_{\mathrm{w},f^1,\mathrm{t}}\Big) = i_{\mathrm{z}} * \Big(\alpha * i_{\mathrm{t}}\Big).$$

So by Lemma 1.1 there are an object D and a morphism $\mathrm{r} : D \to E$ in $\mathbf{W}$, such that:

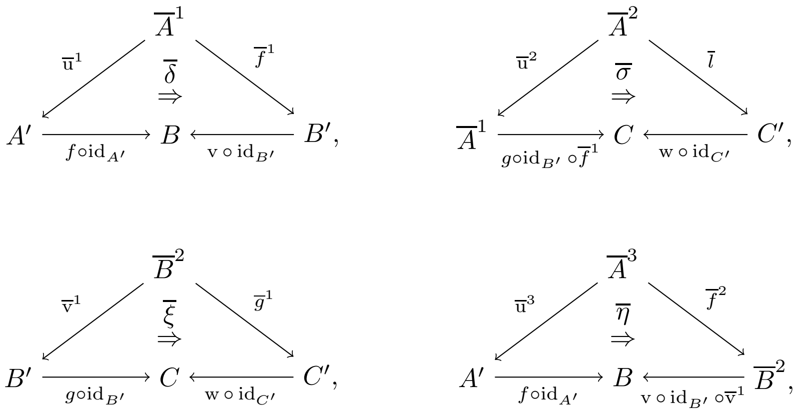
$$\Big(\theta_{\mathrm{w},f^2,\mathrm{t}} \odot \Big(i_{\mathrm{w}} * \gamma\Big) \odot \theta^{-1}_{\mathrm{w},f^1,\mathrm{t}}\Big) * i_{\mathrm{r}} = \Big(\alpha * i_{\mathrm{t}}\Big) * i_{\mathrm{r}}. \tag{6.11}$$

Then we define $\mathrm{v} := \mathrm{t} \circ \mathrm{r} : D \to C$; this morphism belongs to $\mathbf{W}$ by construction and (BF2). Moreover, we set

$$\beta := \theta^{-1}_{f^2,\mathrm{t},\mathrm{r}} \odot \Big(\gamma * i_{\mathrm{r}}\Big) \odot \theta_{f^1,\mathrm{t},\mathrm{r}} : \ f^1 \circ \mathrm{v} \Longrightarrow f^2 \circ \mathrm{v}\,.$$

Then from (6.11) we get easily that $\alpha * i_{\mathrm{v}} = \theta_{\mathrm{w},f^2,\mathrm{v}} \odot (i_{\mathrm{w}} * \beta) \odot \theta^{-1}_{\mathrm{w},f^1,\mathrm{v}}$. Moreover, if α is invertible, then by (BF4b) so is γ, hence so is β. □

Proof of Corollary 5.1. For simplicity of exposition, let us suppose that $\mathscr{C}$ has trivial associators. If this is not the case, the proof follows the same lines, adding associators wherever it is necessary. Let us denote as in (0.4) the fixed set of choices $\mathrm{C}(\mathbf{W})$ for the triple $\underline{f}, \underline{g}, \underline{h}$, so that we have identities (0.5) and (0.6). Moreover, let us suppose that the fixed choices $\mathrm{C}(\mathbf{W})$ give data as in the upper parts of the following diagrams (starting from the ones on the left), with $\overline{\mathrm{u}}^1$, $\overline{\mathrm{u}}^2$, $\overline{\mathrm{v}}^1$ and $\overline{\mathrm{u}}^3$ in $\mathbf{W}$ and $\overline{\delta}$, $\overline{\sigma}$, $\overline{\xi}$ and $\overline{\eta}$ invertible:

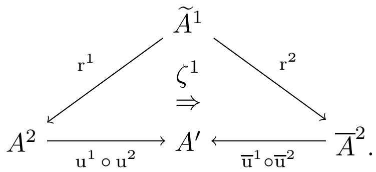


so that by [Pr, § 2.2] one has

$$\underline{h}' \circ \left(\underline{g}' \circ \underline{f}'\right) = \Big(A \xleftarrow{\mathrm{u} \circ \mathrm{id}_{A'} \circ \overline{\mathrm{u}}^1 \circ \overline{\mathrm{u}}^2} \overline{A}^2 \xrightarrow{h \circ \mathrm{id}_{C'} \circ \overline{l}} D\Big), \tag{6.12}$$

$$\left(\underline{h}' \circ \underline{g}'\right) \circ \underline{f}' = \Big(A \xleftarrow{\mathrm{u} \circ \mathrm{id}_{A'} \circ \overline{\mathrm{u}}^3} \overline{A}^3 \xrightarrow{h \circ \mathrm{id}_{C'} \circ \overline{g}^1 \circ \overline{f}^2} D\Big). \tag{6.13}$$

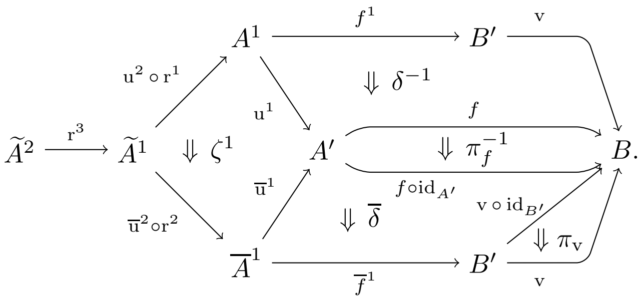
Now using (BF3) we get a set of data as in the upper part of the following diagram, with r^1 in $\mathbf{W}$ and ζ^1 invertible:



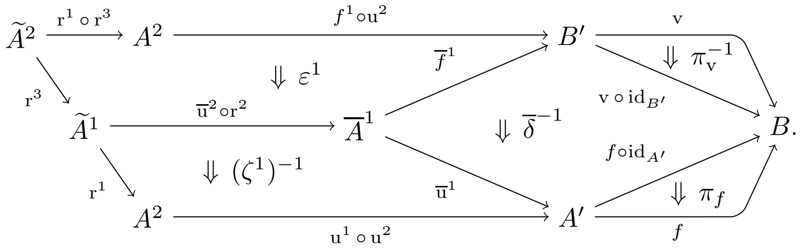
Using (BF4a) and (BF4b), there are an object $\widetilde{A}^2$, a morphism $\mathrm{r}^3 : \widetilde{A}^2 \to \widetilde{A}^1$ in $\mathbf{W}$ and an invertible 2-morphism

$$\varepsilon^1 : \, f^1 \circ \mathrm{u}^2 \circ \mathrm{r}^1 \circ \mathrm{r}^3 \Longrightarrow \overline{f}^1 \circ \overline{\mathrm{u}}^2 \circ \mathrm{r}^2 \circ \mathrm{r}^3,$$

such that $i_{\mathrm{v}} * \varepsilon^1$ coincides with the following composition:



This implies that $\delta^{-1} * i_{\mathrm{u}^2 \circ \mathrm{r}^1 \circ \mathrm{r}^3}$ coincides with the following composition:

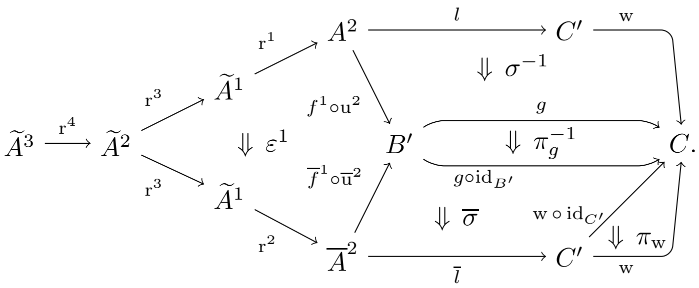


(6.14)

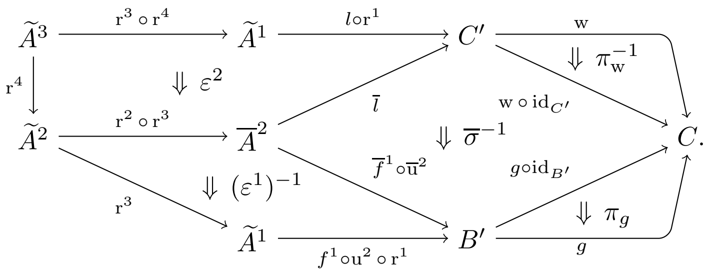
Now we use again (BF4a) and (BF4b) in order to get an object $\widetilde{A}^3$, a morphism $r^4 : \widetilde{A}^3 \to \widetilde{A}^2$ in $\mathbf{W}$ and an invertible 2-morphism

$$\varepsilon^2 : \, l \circ r^1 \circ r^3 \circ r^4 \Longrightarrow \overline{l} \circ r^2 \circ r^3 \circ r^4,$$

such that $i_w * \varepsilon^2$ coincides with the following composition:

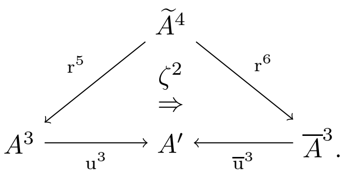


Therefore, $\sigma^{-1} * i_{r^1 \circ r^3 \circ r^4}$ coincides with the following composition:



(6.15)

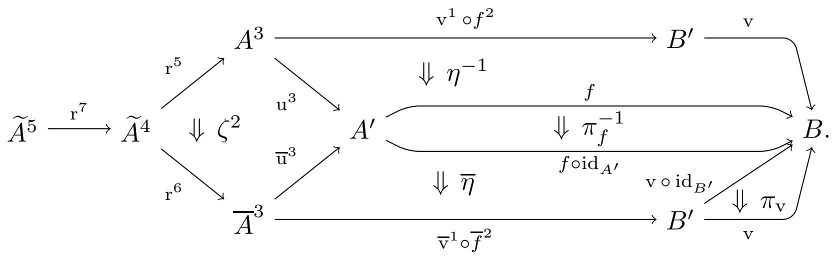
Now we use (BF3) in order to get data as in the upper part of the following diagram, with r^5 in $\mathbf{W}$ and ζ^2 invertible:



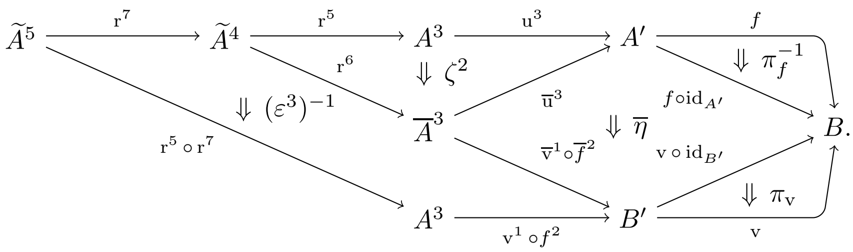
Using (BF4a) and (BF4b), there are an object $\widetilde{A}^5$, a morphism $r^7 : \widetilde{A}^5 \to \widetilde{A}^4$ in $\mathbf{W}$ and an invertible 2-morphism

$$\varepsilon^3 : \, v^1 \circ f^2 \circ r^5 \circ r^7 \Longrightarrow \overline{v}^1 \circ \overline{f}^2 \circ r^6 \circ r^7,$$

such that $i_v * \varepsilon^3$ coincides with the following composition:

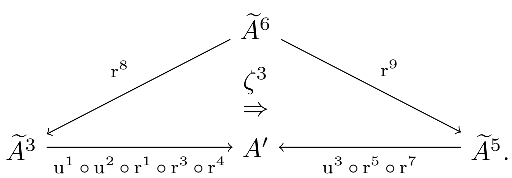


Therefore, $\eta * i_{\mathrm{r}^5 \circ \mathrm{r}^7}$ coincides with the following composition:



(6.16)

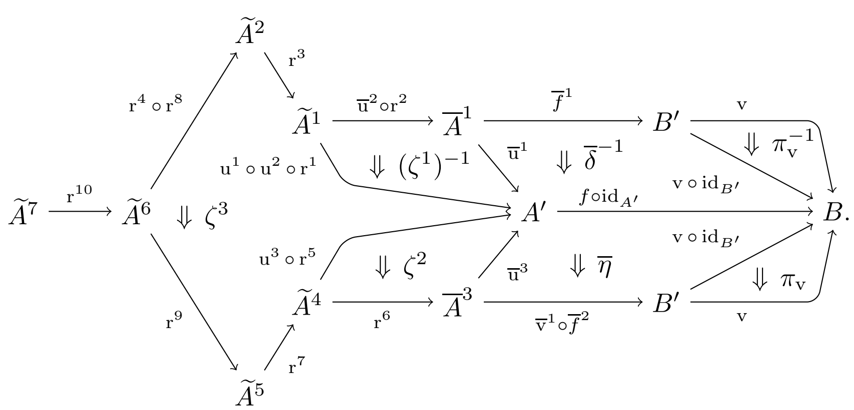
Now we use (BF3) in order to get data as in the upper part of the following diagram, with r^8 in $\mathbf{W}$ and ζ^3 invertible:



Then we use (BF4a) and (BF4b) in order to get an object $\widetilde{A}^7$, a morphism r^{10} : $\widetilde{A}^7 \to \widetilde{A}^6$ in $\mathbf{W}$ and an invertible 2-morphism

$$\varepsilon^4 : \overline{f}^1 \circ \overline{\mathrm{u}}^2 \circ \mathrm{r}^2 \circ \mathrm{r}^3 \circ \mathrm{r}^4 \circ \mathrm{r}^8 \circ \mathrm{r}^{10} \Longrightarrow \overline{\mathrm{v}}^1 \circ \overline{f}^2 \circ \mathrm{r}^6 \circ \mathrm{r}^7 \circ \mathrm{r}^9 \circ \mathrm{r}^{10}, \tag{6.17}$$

such that $i_{\mathrm{v}} * \varepsilon^4$ coincides with the following composition:

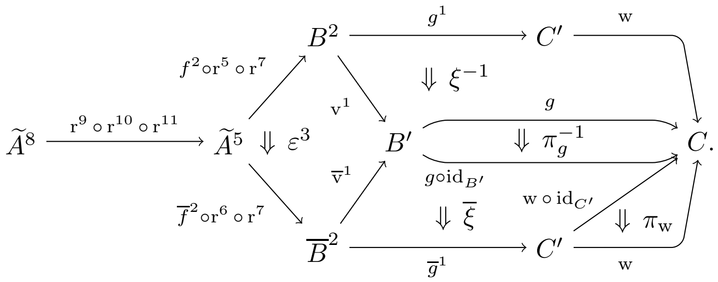


(6.18)

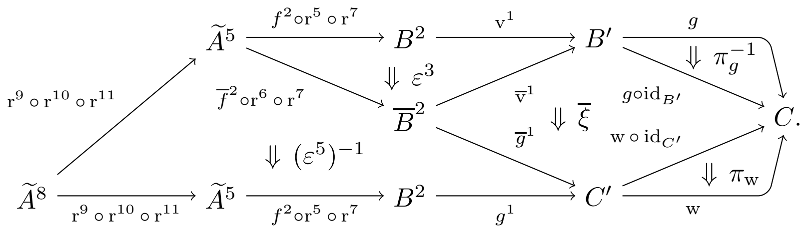
Now we use (BF4a) and (BF4b) in order to get an object $\widetilde{A}^8$, a morphism r^{11} : $\widetilde{A}^8 \to \widetilde{A}^7$ in $\mathbf{W}$ and an invertible 2-morphism

$$\varepsilon^5 :\, g^1 \circ f^2 \circ \mathrm{r}^5 \circ \mathrm{r}^7 \circ \mathrm{r}^9 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11} \Longrightarrow \overline{g}^1 \circ \overline{f}^2 \circ \mathrm{r}^6 \circ \mathrm{r}^7 \circ \mathrm{r}^9 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11},$$

such that $i_{\mathrm{w}} * \varepsilon^5$ coincides with the following composition:



Therefore, $\xi * i_{f^2 \circ \mathrm{r}^5 \circ \mathrm{r}^7 \circ \mathrm{r}^9 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11}}$ coincides with the following composition:

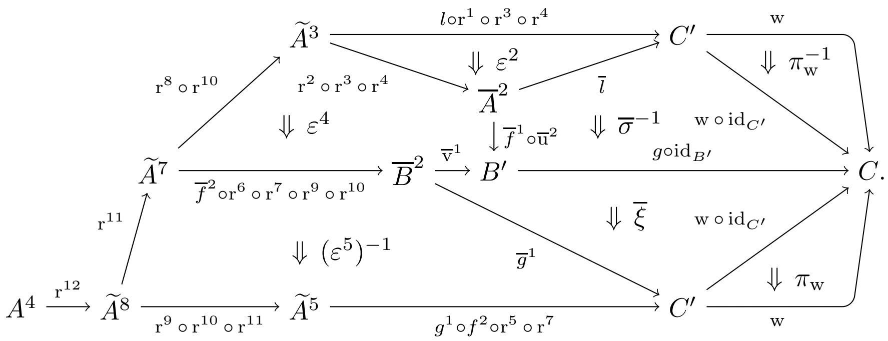


(6.19)

Lastly, we use again (BF4a) and (BF4b) in order to get an object A^4, a morphism $\mathrm{r}^{12} : A^4 \to \widetilde{A}^8$ in $\mathbf{W}$ and an invertible 2-morphism

$$\rho :\, l \circ \mathrm{r}^1 \circ \mathrm{r}^3 \circ \mathrm{r}^4 \circ \mathrm{r}^8 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11} \circ \mathrm{r}^{12} \Longrightarrow g^1 \circ f^2 \circ \mathrm{r}^5 \circ \mathrm{r}^7 \circ \mathrm{r}^9 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11} \circ \mathrm{r}^{12},$$

such that $i_{\mathrm{w}} * \rho$ coincides with the following composition:

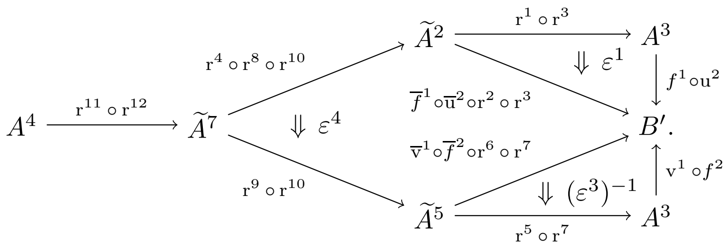


(6.20)

Then we define the following set of morphisms and 2-morphisms:

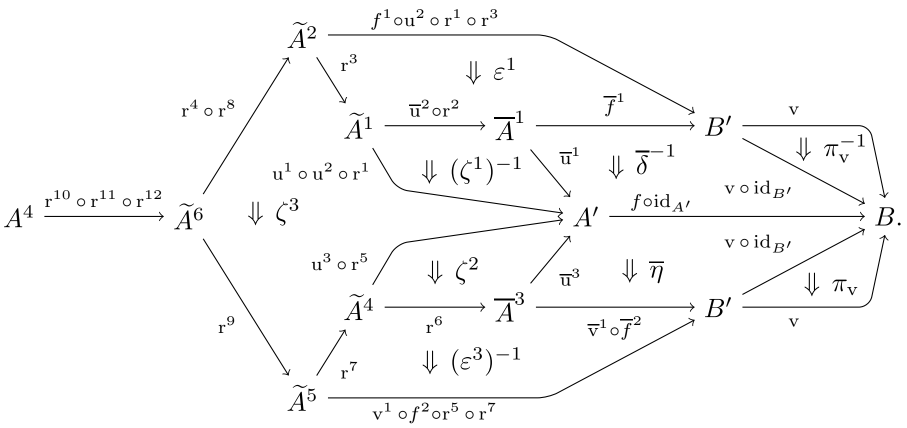
$$\mathrm{u}^4 := \mathrm{r}^1 \circ \mathrm{r}^3 \circ \mathrm{r}^4 \circ \mathrm{r}^8 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11} \circ \mathrm{r}^{12} :\, A^4 \longrightarrow A^2,$$
$$\mathrm{u}^5 := \mathrm{r}^5 \circ \mathrm{r}^7 \circ \mathrm{r}^9 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11} \circ \mathrm{r}^{12} :\, A^4 \longrightarrow A^3,$$
$$\gamma := \zeta^3 * i_{\mathrm{r}^{10} \circ \mathrm{r}^{11} \circ \mathrm{r}^{12}} :\, \mathrm{u}^1 \circ \mathrm{u}^2 \circ \mathrm{u}^4 \Longrightarrow \mathrm{u}^3 \circ \mathrm{u}^5 .$$

Moreover, we define $\omega : f^1 \circ \mathrm{u}^2 \circ \mathrm{u}^4 \Rightarrow \mathrm{v}^1 \circ f^2 \circ \mathrm{u}^5$ as the following composition:



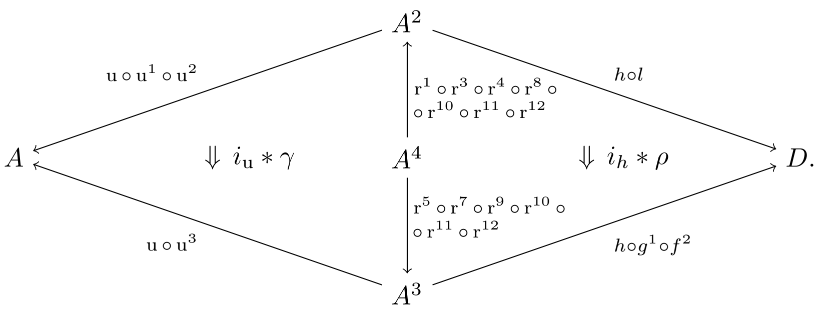
(6.21)

Clearly condition (F1) is satisfied by this set of data; we claim that also (F2) and (F3) hold. In order to prove (F2), we replace in (0.7) the definition of γ above and the expressions for $\delta^{-1}*i_{\mathrm{u}^2\circ\mathrm{r}^1\circ\mathrm{r}^3}$ and $\eta*i_{\mathrm{r}^5\circ\mathrm{r}^7}$ obtained in diagrams (6.14) and (6.16). After simplifying some terms of the form π_f, we get that the composition in (0.7) coincides with the following one:



(6.22)

Using (6.18) and (6.21), we get that (6.22) (hence (0.7)) coincides with $i_{\mathrm{v}}*\omega$. This implies that (F2) holds. Now we prove (F3), so in (0.8) we replace ω with (6.21), $\sigma^{-1}*i_{\mathrm{r}^1\circ\mathrm{r}^3\circ\mathrm{r}^4}$ with (6.15), and $\xi*i_{f^2\circ\mathrm{r}^5\circ\mathrm{r}^7\circ\mathrm{r}^9\circ\mathrm{r}^{10}\circ\mathrm{r}^{11}}$ with (6.19). After simplifying π_g,ε^1 and ε^3, we get that the composition in (0.8) coincides with (6.20), hence with $i_{\mathrm{w}}*\rho$, hence (F3) holds. So using Proposition 0.1 we conclude that the associator $\Theta^{\mathscr{C},\mathbf{W}}_{\underline{h},\underline{g},\underline{f}}$ is represented by the following diagram:



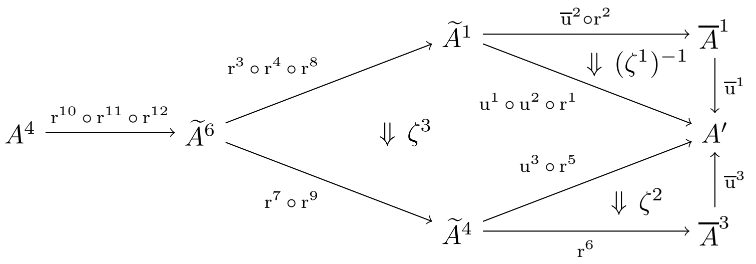
(6.23)

Now we want to compute also the associator appearing in (5.13). For that, we define the following set of data:

$$\overline{\mathrm{u}}^4:=\mathrm{r}^2\circ\mathrm{r}^3\circ\mathrm{r}^4\circ\mathrm{r}^8\circ\mathrm{r}^{10}\circ\mathrm{r}^{11}\circ\mathrm{r}^{12}:\,A^4\longrightarrow\overline{A}^2,$$

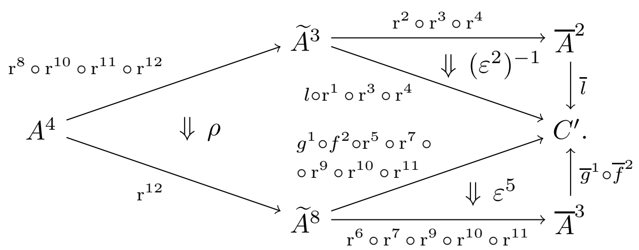
$$\overline{\mathrm{u}}^5 := \mathrm{r}^6 \circ \mathrm{r}^7 \circ \mathrm{r}^9 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11} \circ \mathrm{r}^{12} : \, A^4 \longrightarrow \overline{A}^3,$$
$$\overline{\varpi} := \varepsilon^4 * i_{\mathrm{r}^{11} \circ \mathrm{r}^{12}} : \, f^1 \circ \overline{\mathrm{u}}^2 \circ \overline{\mathrm{u}}^4 \Longrightarrow \overline{\mathrm{v}}^1 \circ f^2 \circ \overline{\mathrm{u}}^5.$$

Moreover, we define $\overline{\gamma} : \overline{\mathrm{u}}^1 \circ \overline{\mathrm{u}}^4 \Rightarrow \overline{\mathrm{u}}^3 \circ \overline{\mathrm{u}}^5$ as the following composition:



(6.24)

and $\overline{\rho} : \overline{l} \circ \overline{\mathrm{u}}^4 \Rightarrow \overline{g}^1 \circ \overline{f}^2 \circ \overline{\mathrm{u}}^5$ as the following composition:

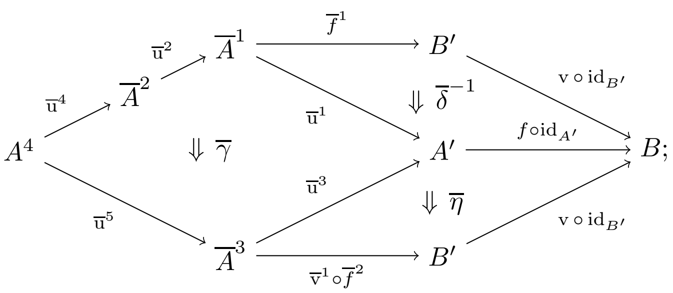


(6.25)

Using interchange law, the definition of $\overline{\varpi}$ and (6.17), we get

$$i_{\mathrm{v} \circ \mathrm{id}_{B'}} * \overline{\varpi} = \Big(\pi_{\mathrm{v}}^{-1} * i_{\overline{\mathrm{v}}^1 \circ \overline{f}^2 \circ \mathrm{r}^6 \circ \mathrm{r}^7 \circ \mathrm{r}^9 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11} \circ \mathrm{r}^{12}}\Big) \odot \Big(i_{\mathrm{v}} * \varepsilon^4 * i_{\mathrm{r}^{11} \circ \mathrm{r}^{12}}\Big) \odot$$
$$\odot \Big(\pi_{\mathrm{v}} * i_{\overline{f}^1 \circ \overline{\mathrm{u}}^2 \circ \mathrm{r}^2 \circ \mathrm{r}^3 \circ \mathrm{r}^4 \circ \mathrm{r}^8 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11} \circ \mathrm{r}^{12}}\Big). \tag{6.26}$$

Then we replace (6.18) in (6.26) and we simplify the terms of the form π_{v}. Using (6.24), we get that $i_{\mathrm{v} \circ \mathrm{id}_{B'}} * \overline{\varpi}$ coincides with the following composition:

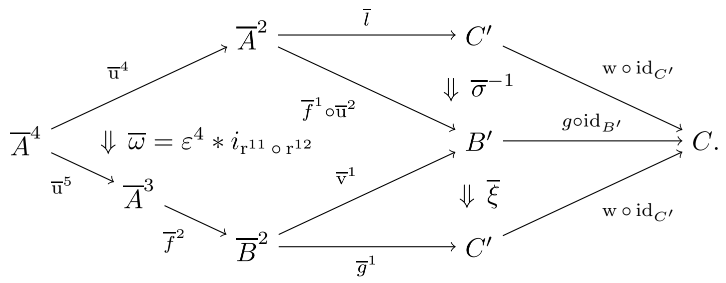


(6.27)

This proves that conditions (F1) and (F2) for the computation of the associator $\Theta^{\mathscr{C},\mathbf{W}}_{\underline{h}',\underline{g}',\underline{f}}$ are satisfied by the set of data

$$A^4, \overline{\mathrm{u}}^4, \overline{\mathrm{u}}^5, \overline{\gamma}, \overline{\varpi}, \overline{\rho}. \tag{6.28}$$

Then we need only to prove that condition (F3) for the associator mentioned above is satisfied by (6.28). In other terms, we have to prove that $i_{\mathrm{w} \circ \mathrm{id}_{C'}} * \overline{\rho}$ coincides with the following composition:

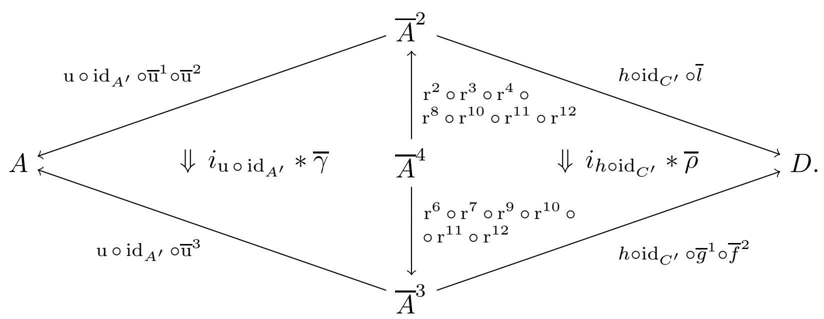


(6.29)

Now by interchange law and (6.25), we have

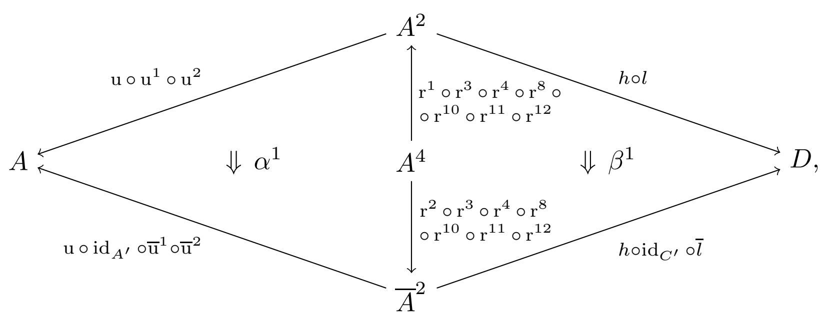
$$i_{\mathrm{w} \circ \mathrm{id}_{C'}} * \overline{\rho} = \Big(\pi_{\mathrm{w}}^{-1} * i_{\overline{g}^1 \circ \overline{f}^2 \circ \mathrm{r}^6 \circ \mathrm{r}^7 \circ \mathrm{r}^9 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11} \circ \mathrm{r}^{12}}\Big) \odot \Big(i_{\mathrm{w}} * \varepsilon^5 * i_{\mathrm{r}^{12}}\Big) \odot \Big(i_{\mathrm{w}} * \rho\Big) \odot$$
$$\odot \Big(i_{\mathrm{w}} * \big(\varepsilon^2\big)^{-1} * i_{\mathrm{r}^8 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11} \circ \mathrm{r}^{12}}\Big) \odot \Big(\pi_{\mathrm{w}} * i_{\overline{l} \circ \mathrm{r}^2 \circ \mathrm{r}^3 \circ \mathrm{r}^4 \circ \mathrm{r}^8 \circ \mathrm{r}^{10} \circ \mathrm{r}^{11} \circ \mathrm{r}^{12}}\Big).$$

Then we replace $i_{\mathrm{w}} * \rho$ above with diagram (6.20) and we simplify ε^2, ε^5 and all the terms of the form π_{w}. Then we get exactly diagram (6.29), so (F3) holds. Therefore, by Proposition 0.1, the associator $\Theta^{\mathscr{C},\mathbf{W}}_{\underline{h}',\underline{g}',\underline{f}}$ is represented by the following diagram:



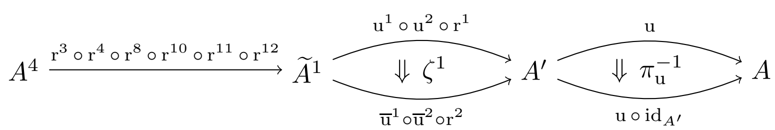
(6.30)

So until now we have computed the central term of diagram (5.13). Using Propositions 0.2 and 0.4 it is not difficult to prove that the 2-morphism $\chi(\underline{h}) * (\chi(\underline{g}) * \chi(\underline{f}))$ appearing in (5.13) is represented by the following diagram:



(6.31)

where α^1 is the following composition:



and β^1 is the following composition:

$$A^4 \xrightarrow{\mathrm{r}^8\circ \mathrm{r}^{10}\circ \mathrm{r}^{11}\circ \mathrm{r}^{12}} \widetilde{A}^3 \underset{\overline{l}\circ \mathrm{r}^2\circ \mathrm{r}^3\circ \mathrm{r}^4}{\overset{l\circ \mathrm{r}^1\circ \mathrm{r}^3\circ \mathrm{r}^4}{\Downarrow \varepsilon^2}} C' \underset{h\circ \mathrm{id}_{C'}}{\overset{h}{\Downarrow \pi_h^{-1}}} D.$$

Moreover, using again the same propositions, the composition $(\chi(\underline{h})^{-1} * \chi(\underline{g})^{-1}) * \chi(\underline{f})^{-1}$ appearing in (5.13) is represented by the following diagram

$$\begin{array}{ccccc} & & \overline{A}^3 & & \\ & \swarrow^{\mathrm{u}\circ \mathrm{id}_{A'}\circ\overline{\mathrm{u}}^3} & \uparrow{\scriptstyle \mathrm{r}^6\circ \mathrm{r}^7\circ \mathrm{r}^9\circ \mathrm{r}^{10}\circ \mathrm{r}^{11}\circ \mathrm{r}^{12}} & \searrow^{h\circ \mathrm{id}_{C'}\circ\overline{g}^1\circ\overline{f}^2} & \\ A & \Downarrow \alpha^2 & A^4 & \Downarrow \beta^2 & D, \\ & \nwarrow_{\mathrm{u}\circ \mathrm{u}^3} & \downarrow{\scriptstyle \mathrm{r}^5\circ \mathrm{r}^7\circ \mathrm{r}^9\circ \mathrm{r}^{10}\circ \mathrm{r}^{11}\circ \mathrm{r}^{12}} & \nearrow_{h\circ g^1\circ f^2} & \\ & & A^3 & & \end{array} \tag{6.32}$$

where α^2 is the following composition:

$$A^4 \xrightarrow{\mathrm{r}^7\circ \mathrm{r}^9\circ \mathrm{r}^{10}\circ \mathrm{r}^{11}\circ \mathrm{r}^{12}} \widetilde{A}^4 \underset{\mathrm{u}^3\circ \mathrm{r}^5}{\overset{\overline{\mathrm{u}}^3\circ \mathrm{r}^6}{\Downarrow (\zeta^2)^{-1}}} A' \underset{\mathrm{u}}{\overset{\mathrm{u}\circ \mathrm{id}_{A'}}{\Downarrow \pi_{\mathrm{u}}}} A$$

and β^2 is the following composition:

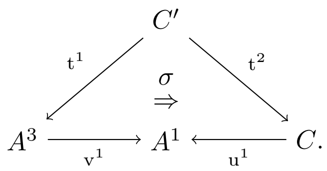
$$A^4 \xrightarrow{\mathrm{r}^{12}} \widetilde{A}^8 \underset{g^1\circ f^2\circ \mathrm{r}^5\circ \mathrm{r}^7\circ \mathrm{r}^9\circ \mathrm{r}^{10}\circ \mathrm{r}^{11}}{\overset{\overline{g}^1\circ\overline{f}^2\circ \mathrm{r}^6\circ \mathrm{r}^7\circ \mathrm{r}^9\circ \mathrm{r}^{10}\circ \mathrm{r}^{11}}{\Downarrow (\varepsilon^5)^{-1}}} C' \underset{h}{\overset{h\circ \mathrm{id}_{C'}}{\Downarrow \pi_h}} D.$$

Now we compute (5.13): for that we use Proposition 0.2 in order to compose vertically (6.30), (6.31) and (6.32). Using the definition of $\overline{\gamma}$ in (6.24) and that of $\overline{\rho}$ in (6.25), in such a composition we can simplify the terms of the form $\pi_{\mathrm{u}}, \pi_h, \zeta^1, \zeta^2, \varepsilon^2$ and ε^5, hence we get exactly diagram (6.23). So we conclude. □

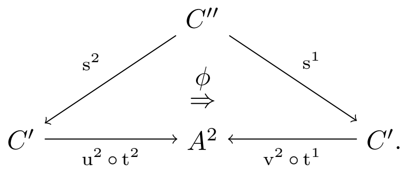
Proof of Lemma 6.1. Let us choose any representative of Γ as follows

$$\begin{array}{ccccc} & & A^1 & & \\ & \swarrow^{\mathrm{w}^1} & \uparrow{\scriptstyle \mathrm{u}^1} & \searrow^{f^1} & \\ A & \Downarrow \delta & C & \Downarrow \eta & B, \\ & \nwarrow_{\mathrm{w}^2} & \downarrow{\scriptstyle \mathrm{u}^2} & \nearrow_{f^2} & \\ & & A^2 & & \end{array}$$

with $\mathrm{w}^1\circ \mathrm{u}^1$ in $\mathbf{W}$ and δ invertible. Since w^1 belongs to $\mathbf{W}$ by hypothesis, then by Lemma 1.2 there are data as in the upper part of the following diagram, with t^1 in $\mathbf{W}$ and σ invertible:



Using (BF5) on α^{-1}, we get that $\mathrm{w}^2 \circ \mathrm{v}^2$ belongs to $\mathbf{W}$. By (BF2), this implies that also $\mathrm{w}^2 \circ \mathrm{v}^2 \circ \mathrm{t}^1$ belongs to $\mathbf{W}$. Moreover, w^2 belongs to $\mathbf{W}$ by hypothesis. So again by Lemma 1.2 there are data as in the upper part of the following diagram, with s^2 in $\mathbf{W}$ and ϕ invertible:



Now we consider the following invertible 2-morphism:

$$\mu := \Big(i_{\mathrm{w}^2} * \phi\Big) \odot \Big(\delta * i_{\mathrm{t}^2 \circ \mathrm{s}^2}\Big) \odot \Big(i_{\mathrm{w}^1} * \sigma * i_{\mathrm{s}^2}\Big) \odot \Big(\alpha^{-1} * i_{\mathrm{t}^1 \circ \mathrm{s}^2}\Big) : \\ \mathrm{w}^2 \circ \mathrm{v}^2 \circ \mathrm{t}^1 \circ \mathrm{s}^2 \Longrightarrow \mathrm{w}^2 \circ \mathrm{v}^2 \circ \mathrm{t}^1 \circ \mathrm{s}^1 .$$

Since $\mathrm{w}^2 \circ \mathrm{v}^2 \circ \mathrm{t}^1$ belongs to $\mathbf{W}$, then by (BF4a) and (BF4b) there are an object A^4, a morphism $\mathrm{r} : A^4 \to C''$ in $\mathbf{W}$ and an invertible 2-morphism

$$\nu :\ \mathrm{s}^2 \circ \mathrm{r} \Longrightarrow \mathrm{s}^1 \circ \mathrm{r},$$

such that $\mu * i_{\mathrm{r}} = i_{\mathrm{w}^2 \circ \mathrm{v}^2 \circ \mathrm{t}^1} * \nu$. We set $\mathrm{z} := \mathrm{t}^1 \circ \mathrm{s}^2 \circ \mathrm{r} : A^4 \to A^3$; this morphism belongs to $\mathbf{W}$ by (BF2). By definition of μ, this implies that

$$\Big(i_{\mathrm{w}^2} * \Big(\Big(\phi^{-1} * i_{\mathrm{r}}\Big) \odot \Big(i_{\mathrm{v}^2 \circ \mathrm{t}^1} * \nu\Big)\Big)\Big) \odot \Big(\alpha * i_{\mathrm{z}}\Big) \odot \\ \odot \Big(i_{\mathrm{w}^1} * \Big(\sigma^{-1} * i_{\mathrm{s}^2 \circ \mathrm{r}}\Big)\Big) = \delta * i_{\mathrm{t}^2 \circ \mathrm{s}^2 \circ \mathrm{r}}. \tag{6.33}$$

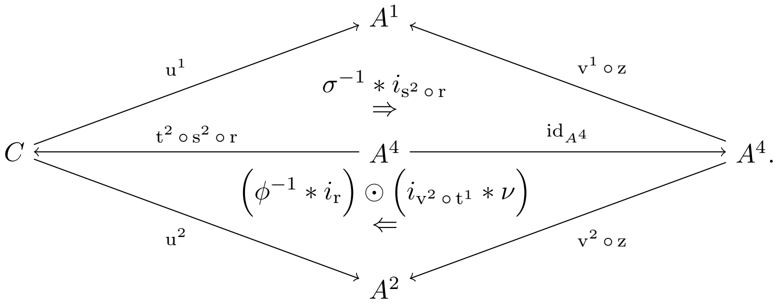
Then we define:

$$\gamma := \Big(i_{f^2 \circ \mathrm{v}^2 \circ \mathrm{t}^1} * \nu^{-1}\Big) \odot \Big(i_{f^2} * \phi * i_{\mathrm{r}}\Big) \odot \Big(\eta * i_{\mathrm{t}^2 \circ \mathrm{s}^2 \circ \mathrm{r}}\Big) \odot \\ \odot \Big(i_{f^1} * \sigma * i_{\mathrm{s}^2 \circ \mathrm{r}}\Big) :\ f^1 \circ \mathrm{v}^1 \circ \mathrm{z} \Longrightarrow f^2 \circ \mathrm{v}^2 \circ \mathrm{z}\,.$$

This implies that:

$$\Big(i_{f^2} * \Big(\Big(\phi^{-1} * i_{\mathrm{r}}\Big) \odot \Big(i_{\mathrm{v}^2 \circ \mathrm{t}^1} * \nu\Big)\Big)\Big) \odot \gamma \odot \\ \odot \Big(i_{f^1} * \Big(\sigma^{-1} * i_{\mathrm{s}^2 \circ \mathrm{r}}\Big)\Big) = \eta * i_{\mathrm{t}^2 \circ \mathrm{s}^2 \circ \mathrm{r}}. \tag{6.34}$$

Lastly, we consider the following diagram



Then using (6.33), (6.34) and the previous diagram and comparing with § 1.1, we conclude that

$$\Gamma = \Big[C, \mathrm{u}^1, \mathrm{u}^2, \delta, \eta\Big] = \Big[A^4, \mathrm{v}^1 \circ \mathrm{z}, \mathrm{v}^2 \circ \mathrm{z}, \alpha * i_{\mathrm{z}}, \gamma\Big].$$

□

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Mathematics Research Unit
University of Luxembourg
6, rue Richard Coudenhove-Kalergi
L-1359 Luxembourg

website: http://matteotommasini.altervista.org/

email: matteo.tommasini2@gmail.com