Abstract. In this paper we investigate the construction of bicategories of fractions originally described by D. Pronk: given any bicategory $\mathcal{C}$ together with a suitable class of morphisms $W$, one can construct a bicategory $\mathcal{C}[W^{-1}]$, where all the morphisms of $W$ are turned into internal equivalences, and that is universal with respect to this property. Most of the descriptions leading to such a construction were long and heavily based on the axiom of choice. In this paper we simplify considerably the constructions of associators, vertical and horizontal compositions in a bicategory of fractions, thus proving that the axiom of choice is not needed under certain conditions. The simplified description of associators and 2-compositions will also play a crucial role in the next papers of this series.

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Introduction

In 1996 Dorette Pronk introduced the notion of (right) bicalculus of fractions (see [Pr]), generalizing the concept of (right) calculus of fractions (described in 1967 by Pierre Gabriel and Michel Zisman, see [GZ]) from the framework of categories to that of bicategories. To be more precise, given any bicategory $\mathcal{C}$ and any class $W$ of 1-morphisms in it, one considers the following set of axioms (where the 2-morphisms $\theta_A$’s are the associators of $\mathcal{C}$):

(BF1) for every object $A$ of $\mathcal{C}$, the 1-identity $\text{id}_A$ belongs to $W$;
(BF2) \( W \) is closed under compositions;

(BF3) for every morphism \( w : A \rightarrow B \) in \( W \) and for every morphism \( f : C \rightarrow B \), there are an object \( D \), a morphism \( w' : D \rightarrow C \) in \( W \), a morphism \( f' : D \rightarrow A \) and an invertible 2-morphism \( \alpha : f \circ w' \Rightarrow w \circ f' \);

(BF4) (a) given any morphism \( w : B \rightarrow A \) in \( W \), any pair of morphisms \( f^1, f^2 : C \rightarrow B \) and any 2-morphism \( \alpha : w \circ f^1 \Rightarrow w \circ f^2 \), there is an object \( D \), a morphism \( v : D \rightarrow C \) in \( W \) and a 2-morphism \( \beta : f^1 \circ v \Rightarrow f^2 \circ v \), such that \( \alpha \ast i_v = \theta_{w,f^2,v} \circ (i_w \ast \beta ) \circ \theta_{w,f^1,v}^{-1} \);

(b) if \( \alpha \) in (a) is invertible, then so is \( \beta \);

(c) if \( (D',v') : D' \rightarrow C, \beta' : f^1 \circ v' \Rightarrow f^2 \circ v' \) is another triple with the same properties of \((D,v,\beta)\) in (a), then there is an object \( E \), a pair of morphisms \( u : E \rightarrow D, u' : E \rightarrow D' \) and an invertible 2-morphism \( \zeta : v \circ u \Rightarrow v' \circ u' \), such that \( v \circ u \) belongs to \( W \) and

\[
\theta_{f^2,v',u'}^{-1} \circ (\beta' \ast i_{v'}) \circ \theta_{f^1,v',u'} \circ (i_{f^1} \ast \zeta) = \\
( i_{f^2} \ast \zeta ) \circ \theta_{f^2,v,u}^{-1} \circ ( \beta \ast i_v ) \circ \theta_{f^1,v,u}.
\]

(BF5) if \( w : A \rightarrow B \) is a morphism in \( W \), \( v : A \rightarrow B \) is any morphism and if there is an invertible 2-morphism \( v \Rightarrow w \), then also \( v \) belongs to \( W \).

The pair \((\mathcal{E},W)\) is said to admit a (right) bicategory of fractions if all such conditions are satisfied (actually in [Pr] § 2.1 the first condition is slightly more restrictive, but it is not necessary, for all the constructions in that paper). Pronk proved that if this is the case, then there are a bicategory \( \mathcal{E} [W^{-1}] \) (called (right) bicategory of fractions) and a pseudofunctor \( \mathcal{U}_W : \mathcal{E} \rightarrow \mathcal{E} [W^{-1}] \), satisfying a universal property (see [Pr] Theorem 21)). For the description of the morphisms and 2-morphisms of \( \mathcal{E} [W^{-1}] \) we refer directly to § 2.3 and Appendix for associators and compositions of 2-morphisms depended on the following choices (both are in general non-unique since axioms (BF3) and (BF4) do not ensure uniqueness):

C(\( W \)): for every set of data in \( \mathcal{E} \) as follows

\[
\begin{array}{ccccc}
A' & \xrightarrow{f} & B & \xrightarrow{v} & B' \\
\downarrow{\rho} & & \downarrow{\rho} & & \downarrow{\rho} \\
A'' & \xrightarrow{f'} & B' & \xrightarrow{v} & B''
\end{array}
\]  \hspace{1cm} (0.1)

D(\( W \)): given any morphism \( w : B \rightarrow A \) in \( W \), any pair of morphisms \( f^1, f^2 : C \rightarrow B \) and any 2-morphism \( \alpha : w \circ f^1 \Rightarrow w \circ f^2 \), we choose a morphism \( v : D \rightarrow C \) in \( W \) and a 2-morphism \( \beta : f^1 \circ v \Rightarrow f^2 \circ v \) as in (BF4a).

Having fixed all such choices, the description of associators and 2-compositions were very long and they did not allow not much freedom on some additional choices done
at each step of the construction (see the explicit descriptions in the next pages). Different sets of choices \((C(W), D(W))\) apparently lead to different bicategories of fractions (with the same objects, morphisms and 2-morphisms, but different compositions and associators). Such different bicategories are a priori only weakly equivalent (by the already mentioned \([P1, \text{Theorem 21}]\)).

In the present paper we will prove that actually choices \(D(W)\) are not necessary. This will be a consequence of the following 4 propositions, that are of independent interest because they allow to simplify considerably the constructions of associators and vertical and horizontal compositions in \(C(W^{-1})\). Indeed in \([P2]\) each intermediate datum \((F1) - (F10)\) mentioned in the next propositions should be obtained as a result of a (sometimes long) procedure involving the mentioned choices \((C(W), D(W))\). The next 4 propositions show that actually each such datum can vary in a much wide range, thus allowing more flexibility in the computations of associators and 2-compositions whenever it is necessary (the resulting constructions are still long, but considerably shorter than the original ones in \([P2]\)). This will serve as a key ingredient in some proofs in the next 2 papers of this series (\([T1]\) and \([T2]\)).

We remark that the choices \((F1) - (F10)\) below actually vary in a non-empty set: for \((F1) - (F3)\) we refer to Remark \(2.1\), for the remaining data this is an easy consequence either of the axioms \([BF]\) mentioned above, or of Lemmas \(1.2\) and \(1.3\) in \(\S\ 1.1\) below. In the next 4 propositions we assume all the time that \((C(W), W)\) is a pair satisfying conditions \([BF]\).

**Proposition 0.1. (associators of \(C(W^{-1})\))** Let us fix any triple of 1-morphisms in \(C(W^{-1})\) as follows:

\[
\begin{align*}
L_1 &:= \left( A \xleftarrow{u} A' \xrightarrow{f} B \right), \\
L_2 &:= \left( B \xleftarrow{v} B' \xrightarrow{g} C \right), \\
L_3 &:= \left( C \xleftarrow{w} C' \xrightarrow{h} D \right). 
\end{align*}
\]

(0.3)

Let us suppose that the fixed choices \((C(W))\) give data as in the upper parts of the following diagrams (starting from the ones on the left), with \(u^1, u^2, v^1\) and \(u^3\) in \(W\) and \(\delta, \sigma, \xi\) and \(\eta\) invertible:

\[
\begin{align*}
\xymatrix{ A^1 \ar[rr]^{f^1} \ar[rd]_{\delta} \\
A' \ar[rr]^{f} \ar[rdd]_{\delta} \ar[rd]_{\delta} \\
& B \\
& B', }
\end{align*}
\]

\[
\begin{align*}
\xymatrix{ A^2 \\
A' \ar[r]_{v \circ f^1} \\
& C \ar[l]_{w} \ar[u]_{\sigma} \ar[ru]_{\sigma} \\
& C' \ar[u]_{\sigma} \ar[l]_{w} \ar[ru]_{\sigma} }
\end{align*}
\]

(0.4)

so that by \([P1, \S\ 2.2]\) one has
\[ h \circ (g \circ f) = \left( A \xrightarrow{(u \circ u^1)u^2} A^2 \xrightarrow{h \circ l} D \right), \] (0.5)

\[ (h \circ g) \circ f = \left( A \xrightarrow{u \circ u^3} A^3 \xrightarrow{(h \circ g) \circ f^2} D \right). \] (0.6)

Then let us fix any set of choices as follows:

**(F1)** an object \( A^4 \), a morphism \( u^4 : A^4 \to A^2 \) in \( W \), a morphism \( u^5 : A^4 \to A^3 \) and an invertible 2-morphism \( \gamma : u^1 \circ (u^2 \circ u^4) \Rightarrow u^3 \circ u^5 \);

**(F2)** an invertible 2-morphism \( \omega : f^1 \circ (u^3 \circ u^4) \Rightarrow v^1 \circ (f^2 \circ u^5) \), such that \( i_v \ast \omega \) coincides with the following composition (associators of \( c \) omitted for simplicity):

\[ (F3) \text{ an invertible 2-morphism } \rho : l \circ u^4 \Rightarrow g^1 \circ (f^2 \circ u^5) \), such that \( i_w \ast \rho \) coincides with the following composition (associators of \( c \) omitted):

Then the associator \( \Theta_{\Sigma_2 \cdot f}^{W} \) from (0.5) to (0.6) is given by the class of the following diagram (associators of \( c \) omitted):

\[ (u \circ u^1)u^2 \]

\[ u^4 \]

\[ \downarrow i_u \ast \gamma \]

\[ A \]

\[ \downarrow u \circ u^2 \]

\[ \downarrow u^5 \]

\[ A^3 \]

\[ \downarrow (h \circ g) \circ f^2 \]

\[ D. \] (0.9)
Therefore, $\Theta_{C,W}$ depends only on the 4 choices of type $[C,W]$ giving the 4 diagrams in (0.4); in particular, each associator of $\mathcal{C}[W^{-1}]$ does not depend on choices $D[W]$. 

**Proposition 0.2. (vertical compositions)** Let us fix any pair of objects $A, B$, any triple of morphisms $f_m := (A_m, w_m, f_m) : A \to B$ for $m = 1, 2, 3$ and any pair of 2-morphisms in $\mathcal{C}[W^{-1}]$ 

$$\Gamma^m := \left[ A^m, u^m, z^m, \alpha^m, \beta^m \right] : f^m \Rightarrow f^{m+1} \quad \text{for} \quad m = 1, 2.$$

Then let us choose any data as follows: 

(F4) an object $A^4$, a morphism $t$ in $W$, a morphism $p$ and an invertible 2-morphism $\rho$ as below:

$$\begin{array}{ccc}
A^4 & \xrightarrow{t} & A^1 \\
| & \searrow & \searrow \\
A^1 & \xrightarrow{\rho} & p \\
\downarrow & \downarrow & \downarrow \\
A^2 & \xleftarrow{u^2} & A^2.
\end{array}$$

Then $\Gamma^2 \circ \Gamma^1$ is represented by the following diagram:

$$\begin{array}{ccc}
A & \xrightarrow{w^1} & A^1 \\
\downarrow & \searrow & \searrow \\
A^2 & \xrightarrow{\rho} & A^4 \\
\downarrow & \downarrow & \downarrow \\
A^2 & \xleftarrow{u^2} & A^2 \\
\downarrow & \downarrow & \downarrow \\
A^3 & \xleftarrow{\rho} & B.
\end{array}$$

(0.10)

In particular, vertical compositions in $\mathcal{C}[W^{-1}]$ do not depend on choices $C[W]$ and $D[W]$. 

**Proposition 0.3. (horizontal compositions with 1-morphisms on the left)** Let us fix any morphism $f := (A', w, f) : A \to B$, any pair of morphisms $g^m := (B^m, v^m, g^m) : B \to C$ for $m = 1, 2$, any 2-morphism 

$$\Delta := \left[ B^3, u^1, u^2, \alpha, \beta \right] : g^1 \Rightarrow g^2$$

in $\mathcal{C}[W^{-1}]$ and let us suppose that for each $m = 1, 2$, choices $[C,W]$ give data as in the upper part of the following diagram, with $w^m$ in $W$ and $\rho^m$ invertible

$$\begin{array}{ccc}
A' & \xrightarrow{w^m} & A^m \\
\downarrow & \searrow & \searrow \\
A' & \xleftarrow{f} & B \\
\downarrow & \downarrow & \downarrow \\
A' & \xleftarrow{v^m} & B^m
\end{array}$$

(0.11)
(so that by [21 § 2.2] we have $g^m \circ f = (A'^m, w \circ w^m, g^m \circ f^m)$ for $m = 1, 2$). Then let us fix any set of choices as follows:

(F5) for each $m = 1, 2$, we choose data as in the upper part of the following diagram, with $u^m$ in $W$ and $\sigma^m$ invertible:

(F6) we choose any set of data as in the upper part of the following diagram, with $t^1$ in $W$ and $\alpha'$ invertible:

(F7) we choose any invertible 2-morphism $\delta : f^1 \circ t^1 \Rightarrow f^2 \circ t^2$, such that $i_{v^1 \circ u^1} \ast \delta$ coincides with the following composition (associators of $C$ omitted):

Then $\Delta \ast i_L$ is represented by the following diagram (associators of $C$ omitted):

where $\beta'$ is the following composition (associators of $C$ omitted):
Therefore, each composition of the form $\Delta \ast i_f$ depends only on the 2 choices giving diagram (0.11) for $m = 1, 2$; in particular, it does not depend on choices $D(W)$.

**Proposition 0.4.** (horizontal compositions with 1-morphisms on the right)

Let us fix any morphism $g = (B', u, g) : B \rightarrow C$, any pair of morphisms $f^m := (A^m, w^m, f^m) : A \rightarrow B$ for $m = 1, 2$, any 2-morphism

$$\Gamma := [A^3, v^1, v^2, \alpha, \beta] : f^1 \Rightarrow f^2$$

in $\mathcal{C}(W^{-1})$ and let us suppose that for each $m = 1, 2$, choices $W$ give data as in the upper part of the following diagram, with $u'^m$ in $W$ and $\rho^m$ invertible:

(F8) for each $m = 1, 2$, we choose data as in the upper part of the following diagram, with $u'^m$ in $W$ and $\eta^m$ invertible:

(F9) we choose data as in the upper part of the following diagram, with $z^1$ in $W$ and $\eta^3$ invertible:

(F10) we choose any 2-morphism $\beta' : f'^1 \circ (v^1 \circ z^1) \Rightarrow f'^2 \circ (v^2 \circ z^2)$, such that $i_u \ast \beta'$ coincides with the following composition (associators of $\mathcal{C}$ omitted):
Then \( i_g * \Gamma \) is represented by the following diagram (associators of \( C \) omitted):

\[
\begin{array}{c}
\xymatrix{A'' \ar[r]_{\eta^3} \ar[d]_{\eta^2} & A^3 \ar[d]_{\eta^2} \ar[r]_{\nu^2} & A^2 \ar[d]_{\beta} & B' \\
A'' \ar[r]_{\eta^3} & A^3 & \ar[r]_{\beta} & B'}
\end{array}
\]

(0.15)

So each composition of the form \( i_g * \Gamma \) depends only on the 2 choices \( C(W) \) giving diagram (0.14) for \( m = 1, 2 \); in particular, it does not depend on choices \( D(W) \).

Since each horizontal composition in any bicategory can be obtained as a suitable combination of vertical compositions and compositions of the form \( \Delta * f \) and \( i_g * \Gamma \), then Propositions 0.2, 0.3 and 0.4 prove immediately that horizontal compositions in \( C(W^{-1}) \) do not depend on choices \( D(W) \). This together with Propositions 0.1 and 0.2 implies at once that:

**Theorem 0.5.** (the structure of \( C(W^{-1}) \)) Let us fix any pair \((C, W)\) satisfying conditions \( BF \). Then the construction of \( C(W^{-1}) \) depends only on choices \( C(W) \), i.e. different sets of choices \( D(W) \) (for choices \( C(W) \) fixed) give the same bicategory of fractions, instead of only equivalent ones.

In particular, we get:

**Corollary 0.6.** Let us suppose that for each pair \((f, v)\) with \( v \) in \( W \) as in (0.1) there is a unique choice of \( (A'' , v', f', \rho) \) as in \( C(W) \). Then the construction of
does not depend on the axiom of choice. The same result holds if “unique choice” above is replaced by “canonical choice”.

As a side result of the technical lemmas used in this paper, we will also prove the following 2 useful statements.

**Proposition 0.7. (comparison of 2-morphisms in \( \mathcal{C}[W^{-1}] \))** Let us fix any pair \((\mathcal{C}, W)\) satisfying conditions \([3]\), any pair of objects \(A, B\), any pair of morphisms \(f^m := (A^m, w^m, f^m) : A \to B\) for \(m = 1, 2\) and any pair of 2-morphisms \(\Gamma^1, \Gamma^2 : f^1 \Rightarrow f^2\) in \(\mathcal{C}[W^{-1}]\). Then there are an object \(A^4\), a pair of morphisms \(v^m : A^3 \to A^m\) for \(m = 1, 2\), an invertible 2-morphism \(\alpha : w^1 \circ v^1 \Rightarrow w^2 \circ v^2\) and a pair of 2-morphisms \(\gamma^1, \gamma^2 : f^1 \circ v^1 \Rightarrow f^2 \circ v^2\), such that \(w^1 \circ v^1\) belongs to \(W\) and

\[
\Gamma^m = [A^3, v^1, v^2, \alpha, \gamma^m] \quad \text{for } m = 1, 2 \quad (0.17)
\]

(in other terms, given any pair of 2-morphisms with the same source and target in \(\mathcal{C}[W^{-1}]\), they differ at most by one term). Moreover, given any pair of 2-morphisms \(\Gamma^1, \Gamma^2 : f^1 \Rightarrow f^2\) as in \((0.17)\), the following facts are equivalent:

(i) \(\Gamma^1 = \Gamma^2\);

(ii) there are an object \(A^4\) and a morphism \(z : A^4 \to A^3\), such that \((w^1 \circ v^1) \circ z\) belongs to \(W\) and \(\gamma^1 \circ i_z = \gamma^2 \circ i_z\).

The description above simplifies considerably the comparison of 2-morphisms in \(\mathcal{C}[W^{-1}]\): just compare it with the original comparison in \([3] \S \, 2.3\).

**Proposition 0.8. (invertibility of 2-morphisms in \(\mathcal{C}[W^{-1}]\))** Let us fix any pair \((\mathcal{C}, W)\) satisfying conditions \([3]\), any pair of morphisms \((A^m, w^m, f^m) : A \to B\) in \(\mathcal{C}[W^{-1}]\) for \(m = 1, 2\) and any 2-morphism \(\Gamma : (A^1, w^1, f^1) \Rightarrow (A^2, w^2, f^2)\). Then the following facts are equivalent:

(i) \(\Gamma\) is invertible in \(\mathcal{C}[W^{-1}]\);

(ii) \(\Gamma\) has a representative

\[
\begin{array}{ccc}
A & \xleftarrow{w^1} & A^1 \\
\downarrow \alpha & & \downarrow f^1 \\
A & \xrightarrow{v^1} & B \\
\downarrow \beta & & \downarrow f^2 \\
A^2 & \xrightarrow{v^2} & B
\end{array}
\]

\[ (0.18) \]

such that \(\beta\) is invertible in \(\mathcal{C}\);

(iii) given any representative \((0.18)\) for \(\Gamma\), there are an object \(A^4\) and a morphism \(u : A^4 \to A^3\), such that

- \((w^1 \circ v^1) \circ u\) belongs to \(W\),
- \(\beta \circ i_u\) is invertible in \(\mathcal{C}\).

Note that in \((0.18)\) \(\alpha\) is always invertible and \(w^1 \circ v^1\) always belongs to \(W\) by definition of 2-morphism in \(\mathcal{C}[W^{-1}]\), see \([3] \S \, 2.3\).

We are going to apply all the results mentioned so far in the next 2 papers of this series, where we will investigate the problem of constructing pseudofunctors (and equivalences) between right bicategories of fractions.
1. Notations and basic facts

We mainly refer to \([L]\) and \([PW, \S\ 1]\) for a general overview on bicategories, pseudofunctors (i.e. homomorphisms of bicategories), Lax natural transformations and modifications. Given any bicategory \(\mathcal{C}\), we denote its objects by \(A, B, \ldots\), its morphisms by \(f, g, \ldots\) and its 2-morphisms by \(\alpha, \beta, \ldots\). Given any triple of morphisms \(f : A \to B, g : B \to C, h : C \to D\) in \(\mathcal{C}\), we denote by \(\theta_{h, g, f}\) the associator \(h \circ (g \circ f) \Rightarrow (h \circ g) \circ f\) that is part of the structure of the bicategory \(\mathcal{C}\). We denote by \(\pi_f : f \circ \text{id}_A \Rightarrow f\) and \(\upsilon_f : \text{id}_B \circ f \Rightarrow f\) the right and left unitors for \(\mathcal{C}\) relative to any morphism \(f\) as above.

1.1. Morphisms and 2-morphisms in a bicategory of fractions. We recall (see \([Pr, \S\ 2.2]\)) that the objects of \(\mathcal{C}[W^{-1}]\) are the same as those of \(\mathcal{C}\). A morphism from \(A\) to \(B\) in \(\mathcal{C}[W^{-1}]\) is any triple \((A', w, f)\), where \(A'\) is an object of \(\mathcal{C}\), \(w : A' \to A\) is an element of \(W\) and \(f : A' \to B\) is a morphism of \(\mathcal{C}\). Given any pair of morphisms from \(A\) to \(B\) and from \(B\) to \(C\) in \(\mathcal{C}[W^{-1}]\) as follows

\[ f := (A \xrightarrow{w} A' \xrightarrow{f} B) \quad \text{and} \quad g := (B \xleftarrow{v} B' \xrightarrow{g} C) \]

(with both \(w\) and \(v\) in \(W\)), following \([Pr, \S\ 2.2]\) one has to use choices \(C(W)\) for the pair \((f, v)\) in order to get data as in (0.2) and then set \(g \circ f := (A'', w \circ v', g \circ f')\).

Given any pair of objects \(A, B\) and any pair of morphisms \((A^m, w^m, f^m) : A \to B\) for \(m = 1, 2\) in \(\mathcal{C}[W^{-1}]\), a 2-morphism from \((A^1, w^1, f^1)\) to \((A^2, w^2, f^2)\) is an equivalence class of data \((A^3, v^1, v^2, \alpha, \beta)\) in \(\mathcal{C}\) as in (0.18) such that \(w^1 \circ v^1\) belongs to \(W\) and such that \(\alpha\) is invertible in \(\mathcal{C}\) (in \([Pr, \S\ 2.3]\) it is also required that \(w^2 \circ v^2\) belongs to \(W\), but this follows from (BF5)). Any other set of data

\[
\begin{array}{ccc}
A^2 & \xrightarrow{w^2} & A^1 \\
& \downarrow \alpha' & \quad \downarrow f^1 \\
A & \quad \downarrow \beta' & B \\
& \downarrow v^2 & \quad \downarrow f^2 \\
& B & \xleftarrow{v'} \quad \xrightarrow{A^3}
\end{array}
\]

(such that \(w^1 \circ v^1\) belongs to \(W\) and \(\alpha'\) is invertible) represents the same 2-morphism in \(\mathcal{C}[W^{-1}]\) if and only if there is a set of data \((A^4, z, \sigma^1, \sigma^2)\) in \(\mathcal{C}\) as in the following diagram

\[
\begin{array}{ccc}
A^2 & \xrightarrow{v^2} & A^3 \\
& \downarrow \sigma^1 & \quad \downarrow v^1 \\
A & \quad \downarrow \sigma^2 & B \\
& \downarrow z & \quad \downarrow \sigma^2 \\
& B & \xleftarrow{v'} \quad \xrightarrow{A^4}
\end{array}
\]

such that \((w^1 \circ v^1) \circ z\) belongs to \(W\), \(\sigma^1\) and \(\sigma^2\) are both invertible,
(i_{w^2} \ast \sigma^2) \circ \theta_{w^2,v^2,x}^{-1} \circ (\alpha \ast i_{w^2}) \circ \theta_{w^1,v^1,x} \circ (i_{w^1} \ast \sigma^1) = \\
= \theta_{w^2,v^2,x'}^{-1} \circ (\alpha' \ast i_{w^2}) \circ \theta_{w^1,v^1,x'}
and

(i_{f^2} \ast \sigma^2) \circ \theta_{f^2,v^2,x}^{-1} \circ (\beta \ast i_{f^2}) \circ \theta_{f^1,v^1,x} \circ (i_{f^1} \ast \sigma^1) = \\
= \theta_{f^2,v^2,x'}^{-1} \circ (\beta' \ast i_{f^2}) \circ \theta_{f^1,v^1,x'}.

For symmetric reasons, in [Pr, § 2.3] it is also required that (w^1 \circ v^1) \circ z' belongs to W, but this follows from BF3, using the invertible 2-morphism:

\[ \theta_{w^1,v^1,x} \circ (i_{w^1} \ast \sigma^1) \circ \theta_{w^1,v^1,x'} : (w^1 \circ v^1) \circ z' \Rightarrow (w^1 \circ v^1) \circ z, \]

so we will always omit this unnecessary technical condition. We denote by

\[ [A^3,v^1,v^2,\alpha,\beta] : (A^1,w^1,f^1) \Rightarrow (A^2,w^2,f^2) \]

(1.1)
the class of any data as in (0.18). We will denote morphisms of \( \mathcal{C} [W^{-1}] \) as \( f, g, \ldots \) and 2-morphisms by \( \Gamma, \Delta, \ldots \); in particular \( \Theta_{\mathcal{C}}^\circ W \) will denote any associator of \( \mathcal{C} [W^{-1}] \). Note that even if \( \mathcal{C} \) is a 2-category, in general \( \mathcal{C} [W^{-1}] \) is only a bicategory (with trivial right and left unitors, but non-trivial associators if the choices in \( \mathcal{C} [W] \) are non-unique).

In the following pages we will often use this easy lemma (see the Appendix for a proof).

**Lemma 1.1.** Let us fix any pair \((\mathcal{C}, W)\) satisfying conditions BF. Let us fix any morphism \( w : B \to A \) in \( W \), any pair of morphisms \( f^1, f^2 : C \to B \) and any pair of 2-morphisms \( \gamma, \gamma' : f^1 \Rightarrow f^2 \), such that \( i_{w^1} \circ \gamma = i_{w^2} \ast \gamma' \). Then there are an object \( D \) and a morphism \( u : D \to C \) in \( W \), such that \( \gamma \circ i_{u} = \gamma' \circ i_{u} \).

The next lemmas prove that if conditions BF hold, then conditions BF3, BF3a and BF4 hold under less restrictive conditions on the morphism \( w \) (see the Appendix for the proofs). To be more precise, instead of imposing that \( w \) belongs to \( W \), it is sufficient to impose that \( z \circ w \) belongs to \( W \) for some morphism \( z \) in \( W \) (as a special case, one gets back again BF3, BF4a and BF4b when we choose \( z \) as a 1-identity).

**Lemma 1.2.** Let us fix any pair \((\mathcal{C}, W)\) satisfying conditions BF. Let us choose any quadruple of objects \( A, B, B', C \) and any quadruple of morphisms \( w : A \to B, z : B \to B' \) and \( f : C \to B \), such that both \( z \) and \( z \circ w \) belong to \( W \). Then there are an object \( D \), a morphism \( w' : D \to C \) in \( W \), a morphism \( f' : D \to A \) and an invertible 2-morphism \( \alpha : f \circ w' \Rightarrow w \circ f' \).

**Lemma 1.3.** Let us fix any pair \((\mathcal{C}, W)\) satisfying conditions BF. Let us choose any quadruple of objects \( A, A', B, C \) and any triple of morphisms \( w : B \to A, z : A \to A', f^1, f^2 : C \to B \), such that both \( z \) and \( z \circ w \) belong to \( W \). Moreover, let us fix any 2-morphism \( \alpha : w \circ f^1 \Rightarrow w \circ f^2 \). Then there are an object \( D \), a morphism \( v : D \to C \) in \( W \) and a 2-morphism \( \beta : f^1 \circ v \Rightarrow f^2 \circ v \), such that \( \alpha \ast v = \theta_{w,f_2,v} \circ (i_{w^1} \ast \beta) \circ \theta_{w,f_1,v}^{-1} \). Moreover, if \( \alpha \) is invertible, so is \( \beta \).
2. THE ASSOCIATORS OF A BICATEGORY OF FRACTIONS

For simplicity of exposition, in this and in the next sections all the proof will be given assuming all the time that $\mathcal{C}$ is a 2-category. The general case when $\mathcal{C}$ is a bicategory follows the same ideas, adding associators and unitors whenever it is necessary and using the coherence conditions on the bicategory $\mathcal{C}$. You can have a glimpse of how to deal with the general case by looking at the proofs of Lemmas 1.1, 1.2 and 1.3 in the Appendix.

In the initial part of this section we will give a proof of Proposition 0.1. Then we will give some interesting corollaries of this result.

Proof of Proposition 0.1. Following [Pr, Appendix], one gets immediately a set of data satisfying conditions (F1) – (F3), inducing the desired associator as in (0.9). So in order to prove the claim, it is sufficient to prove that given any 2 different sets of choices of data as in (F1) – (F3), the 2-morphism of $\mathcal{C}$ $[W^{-1}]$ induced by the first set of choices coincides with the 2-morphism induced by the second set of choices. So let us fix any other set of data satisfying the same conditions, as follows:

- an object $\tilde{A}$, a morphism $\tilde{u}^4 : \tilde{A}^4 \to A^2$ in $W$;
- an invertible 2-morphism $\tilde{\gamma} : u^1 \circ u^2 \circ \tilde{u}^4 \Rightarrow \tilde{u}^3 \circ \tilde{u}^5$;
- an invertible 2-morphism $\tilde{\omega} : f^1 \circ u^2 \circ \tilde{u}^4 \Rightarrow v^1 \circ f^2 \circ \tilde{u}^5$, such that

$$i_{\nu} \ast \tilde{\omega} = (\eta \ast i_{\tilde{u}^5}) \circ (i_f \ast \tilde{\gamma}) \circ (\delta^{-1} \ast i_{u^2 \circ \tilde{u}^4});$$

(2.1)

- an invertible 2-morphism $\tilde{\rho} : l \circ \tilde{u}^6 \Rightarrow g^1 \circ f^2 \circ \tilde{u}^5$, such that

$$i_{\omega} \ast \tilde{\rho} = (\xi \ast i_{f^2 \circ \tilde{u}^5}) \circ (i_g \ast \tilde{\omega}) \circ (\sigma^{-1} \ast i_{u^1});$$

(2.2)

Then proving the claim is equivalent to proving that the class of (0.9) coincides with the class of $\tilde{A}$.

First of all, we use axiom (BF3) in order to get data as in the upper part of the following diagram, with $z^1$ in $W$ and $\alpha$ invertible:

$$\begin{array}{ccc}
A & \xrightarrow{u \circ u^1 \circ u^2} & A^2 \\
\downarrow i_{u} \ast \tilde{\gamma} & & \downarrow h_{\alpha l} \\
\tilde{A}^4 & \xrightarrow{\tilde{u}^3} & D. \\
\downarrow i_{h} \ast \tilde{\rho} & & \\
A^3 & \xrightarrow{u^3} & \tilde{A}^4.
\end{array}$$

(2.3)

Since $u^3$ belongs to $W$ by hypothesis, then using (BF4a) and (BF4b) there are an object $\overline{A}$, a morphism $z^2 : \overline{A} \to \overline{A}$ in $W$ and an invertible 2-morphism

$$\varepsilon : u^5 \circ z^1 \circ z^2 \Rightarrow \tilde{u}^5 \circ \tilde{z}^1 \circ \tilde{z}^2,$$
such that \( i_{u^1} \ast \varepsilon \) coincides with the following composition:

\[
\begin{array}{c}
\xymatrix{ & A^4 \ar[r]^{u^5} & A^3 \\
A^2 \ar[r]^{x^2} \ar[ur]^-{\alpha} & A^1 \ar[d]^-{u^4} \ar[ur]^-{u^3} & A^2 \ar[r]_{u^1 \circ u^2} \ar[d]^-{\gamma} & A' \\
A^4 \ar[r]_{u^3} \ar[ur]^-{\gamma^{-1}} & A^3 
}\end{array}
\]

This implies that

\[
\gamma \ast i_{x^1 \circ x^2} = \left( i_{u^1} \ast \varepsilon^{-1} \right) \circ \left( \gamma \ast i_{x^1 \circ x^2} \right) \circ \left( i_{u^1} \circ u^2 \ast \alpha \ast i_{x^2} \right).
\] (2.4)

So we get:

\[
i_{v} \ast \left( \omega \ast i_{x^1 \circ x^2} \right) = \eta \ast i_{u^5 \circ x^1 \circ x^2} \circ \left( i_{f} \ast \gamma \ast i_{x^1 \circ x^2} \right) \circ \left( \delta^{-1} \ast i_{u^2 \circ u^4 \circ x^1 \circ x^2} \right)
\] (2.5)

\[
i_{v} \ast \left( i_{v_{o} \circ f_{2} \ast \varepsilon^{-1}} \circ \left( \tilde{\omega} \ast i_{x^1 \circ x^2} \right) \circ \left( i_{f_{ou} \circ x_{2} \ast \alpha \ast i_{x^2}} \right) \right) =
\]

\[
= i_{v} \ast \left( \left( i_{v_{o} \circ f_{2} \ast \varepsilon^{-1}} \circ \left( \tilde{\omega} \ast i_{x^1 \circ x^2} \right) \circ \left( i_{f_{ou} \circ x_{2} \ast \alpha \ast i_{x^2}} \right) \right) \right)
\] (2.6)

where \((\ast)\) is given by applying the interchange law twice. Using Lemma 1.11 and (2.5), there are an object \( \overline{A}^3 \) and a morphism \( x^3 : \overline{A}^3 \rightarrow \overline{A}^3 \) in \( W \), such that

\[
\omega \ast i_{x^1 \circ x^2 \circ x^3} =
\]

\[
= \left( i_{v_{o} \circ f_{2} \ast \varepsilon^{-1} \ast i_{x^2}} \circ \left( \tilde{\omega} \ast i_{x^1 \circ x^2} \right) \circ \left( i_{f_{ou} \circ x_{2} \ast \alpha \ast i_{x^2}} \right) \right)
\] (2.6)

Then we get:

\[
i_{w} \ast \rho \ast i_{x^1 \circ x^2 \circ x^3} \] (2.6)
where \((\ast)\) is given by the interchange law. So by Lemma 1.1 there are an object \(A^4\) and a morphism \(z^4 : A^4 \rightarrow A^3\) in \(W\), such that
\[
\rho \ast i_{z^1 \circ z^2 \circ z^3 \circ z^4} = (i_{f^1 \circ f^2} \ast (\varepsilon^{-1} \ast i_{z^3 \circ z^4})) \circ (\tilde{\rho} \ast i_{z^1 \circ z^2 \circ z^3 \circ z^4}) \circ \left(i_{1} \ast \left(\alpha \ast i_{z^2 \circ z^3 \circ z^4}\right)\right).
\] (2.7)
Moreover, from (2.4) we have:
\[
\gamma \ast i_{z^1 \circ z^2 \circ z^3 \circ z^4} = \left(i_{u^3} \ast (\varepsilon^{-1} \ast i_{z^3 \circ z^4})\right) \circ \left(\tilde{\gamma} \ast i_{z^1 \circ z^2 \circ z^3 \circ z^4}\right) \circ \left(i_{u^1 \circ u^2} \ast \left(\alpha \ast i_{z^2 \circ z^3 \circ z^4}\right)\right).
\] (2.8)
Then using together identities (2.7) and (2.8), the following diagram and the definition of 2-morphism in \(\mathcal{C}[W^{-1}]\) (see § 1.1), we get that the class of (0.9) coincides with the class of (2.3).

**Remark 2.1.** Apart from the procedure explained in [Pr, Appendix], when \(\mathcal{C}\) is a 2-category a simple way for finding a tuple \((A^4, u^4, u^5, \gamma, \omega, \rho)\) as in (F1) – (F3) is given as follows (in the general case of a bicategory, simply add associators wherever it is necessary). First of all, we use axiom (BF3) in order to get data as in the upper part of the following diagram, with \(\pi^4\) in \(W\) and \(\tau\) invertible:

\[
\begin{array}{ccc}
P^1 & \Rightarrow & \pi^2 \\
A^2 & \xymatrix{E \\ u^1 \circ u^2} & \xymatrix{A' \\ u^3} \\
A^3 & \xymatrix{\Rightarrow \\ \eta \\ \delta^{-1} \\ \varepsilon^{-1} \ast i_{z^3 \circ z^4}} & \pi^1
\end{array}
\]

Then we use (BF4a) and (BF4b) in order to get an object \(F\), a morphism \(z : F \rightarrow E\) in \(W\) and an invertible 2-morphism \(\pi : f^1 \circ u^2 \circ \pi^4 \circ z \Rightarrow v^1 \circ f^2 \circ \pi^3 \circ z\), such that \(i_v \ast \pi\) coincides with the following composition:
Then we use again (BF4a) and (BF4b) in order to get an object $A^4$, a morphism $r : A^4 \to F$ in $W$ and an invertible 2-morphism

$$\rho : l \circ W^2 \circ z \circ r \Rightarrow g^1 \circ f^2 \circ W^2 \circ z \circ r,$$

such that $i_w * \rho$ coincides with the following composition:

$$\begin{array}{c}
A^4 \xrightarrow{r} F \\
\xrightarrow{W^2 \circ z} A^3 \\
\xrightarrow{f^2 \circ W^2 \circ z} B^2 \\
\xrightarrow{g^2} C^2 \\
\end{array} \Rightarrow \begin{array}{c}
A^2 \\
\xrightarrow{l} C' \\
\xrightarrow{\sigma^{-1}} B' \\
\xrightarrow{g} C' \\
\end{array}$$

Then it suffices to define $u^4 := W^2 \circ z \circ r$ (this morphism belongs to $W$ by (BF2)), $u^5 := W^2 \circ z \circ r$, $\gamma := f \circ i_z \circ r$ and $\omega := \tau \circ i_r$.

**Corollary 2.2.** Let us fix any pair $(\mathcal{C}, W)$ satisfying conditions (BF), any triple of morphisms $f, g, h$ as in (0.3) and let us suppose that $B = B'$, $C = C'$, $v = id_B$ and $w = id_C$. Moreover, let us suppose that choices $\mathcal{C}(W)$ give data as in the upper part of the following diagram, with $u^3$ in $W$ and $\eta$ invertible:

$$\begin{array}{c}
A^3 \\
\xrightarrow{u^3} A' \\
\xrightarrow{f} B \\
\xrightarrow{id_B \circ id_B} B. \\
\end{array} \Rightarrow \begin{array}{c}
A^3 \\
\xrightarrow{u^3} A' \\
\xrightarrow{f^2} B \\
\end{array} \Rightarrow \begin{array}{c}
A^3 \\
\xrightarrow{\eta} f^2 \\
\xrightarrow{id_B \circ id_B} B. \\
\end{array}$$

Then

$$h \circ (g \circ f) = \left( A \xrightarrow{(u \circ id_A') \circ id_{A'}} A' \xrightarrow{ho(g \circ f)} D \right),$$

and the associator $\Theta_{\mathcal{C}, W}^{h, g, f}$ from (2.10) to (2.11) is given by the class of the following diagram:

$$\begin{array}{c}
A^3 \\
\xrightarrow{u^3} A' \\
\xrightarrow{f} B \\
\xrightarrow{id_B \circ id_B} B. \\
\end{array} \Rightarrow \begin{array}{c}
A' \\
\xrightarrow{\alpha} A^3 \\
\xrightarrow{f^2} D, \\
\end{array} \Rightarrow \begin{array}{c}
A^3 \\
\xrightarrow{\beta} D. \\
\end{array}$$

where
\[ \alpha := \pi_u^{-1} \circ (\pi_u \ast i_{u^3}) \circ \left( (\pi_u \ast i_{\text{id}_{A'}}) \ast i_{u^3} \right) \]

and

\[ \beta := \pi_u^{-1} \circ (i_{\text{hgf}} \ast (v_{f2} \circ (v_{\text{id}_{A}} \ast i_{f2}) \circ \eta)) \circ \theta_{\text{hgf,ufu}}^{-1} \circ (\theta_{\text{hgf,ufu}} \ast i_{u^3}) \]

(for the notations concerning the associators \( \theta \) and the unitors \( \pi \) and \( \upsilon \), we refer to § 1).

**Remark 2.3.** In particular, if \( \mathcal{C} \) is a 2-category, then the associators \( \theta \) and the unitors \( \pi \) and \( \upsilon \) are all trivial; moreover, \( \text{id}_B \circ \text{id}_B = \text{id}_B \). We recall that the fixed choices \( \mathbb{C}(W) \) imposed by Pronk on any pair \( (f,v) \) assume a very simple form in the case when either \( f \) or \( v \) are an identity (see [Pr, pag. 256]). So the quadruple \( (A_3, u_3, f_2, \eta) \) coincides with \( (A', \text{id}_{A'}, f, i_f) \). So when \( \mathcal{C} \) is a 2-category, (2.10) and (2.11) coincide and the associator above is the 2-identity of such morphism.

**Proof of Corollary 2.2.** As usual, for simplicity of exposition we give the proof only in the special case when \( \mathcal{C} \) is a 2-category. By the already mentioned [Pr, pag. 256], since both \( v \) and \( w \) are identities, one get that the 4 diagrams of (0.4) (chosen from left to right) assume this simple form:

\[
\begin{align*}
A^1 := A' & \quad \delta := i_f \quad f^1 := f \\
A' & \quad \Rightarrow \\
B & \quad v = \text{id}_{B} \quad B' = B,
\end{align*}
\]

\[
\begin{align*}
A^2 := A' & \quad \sigma := i_{gof} \quad t := gof \\
A' & \quad \Rightarrow \\
C & \quad w = \text{id}_{C} \quad C' = C,
\end{align*}
\]

\[
\begin{align*}
B^2 := B & \quad \xi := i_g \quad g^1 := g \\
B' & \quad \Rightarrow \\
C & \quad \quad w = \text{id}_{C} \quad C' = C,
\end{align*}
\]

\[
\begin{align*}
u^3 := \text{id}_{B} & \quad B \quad B^2 = B.
\end{align*}
\]

Then identities (2.10) and (2.11) follow at once from (0.5) and (0.6). In order to compute the associator, according to Proposition 0.1 we have to choose a set of data as in (F1) – (F3). For that, we choose:

- \( A^4 := A' \), \( u^4 := \text{id}_{A'} \), \( u^5 := \text{id}_{A'} \) and \( \gamma := i_{\text{id}_{A'}} \);
- \( \omega := i_f \);
- \( \rho := i_{gof} \).

Then the claim follows. In the general case when \( \mathcal{C} \) is a bicategory, the first diagram above is given by
and analogously for the second diagram and the third one. The fourth diagram must be replaced by (2.9). The data of (F1) – (F3) above have to be changed according to this. □

3. Vertical compositions

Proof of Proposition 0.2. Following [Pr, pag. 258], the composition $\Gamma^2 \circ \Gamma^1$ has to be computed as follows: firstly one has to fix data as in the upper part of the following diagram, with $t$ in $W$ and $\rho$ invertible:

these data are induced by choices $C(W)$ and $D(W)$ similarly to the construction given in the proof of Lemma 1.2 (since both $w^2$ and $w^2 \circ v^2$ belong to $W$), but we don’t need to describe how this is done explicitly. Then by [Pr], $\Gamma^2 \circ \Gamma^1$ is represented by the following diagram:

The lines below (3.1) show how we get two data as in the upper part of the following diagram, with $s$ in $W$ and $\sigma$ invertible.

We have already remarked that $w^2 \circ z^1$ belongs to $W$; moreover $w^2$ belongs to $W$ since $f^2$ is a morphism in $C(W^{-1})$. So by Lemma 1.3 for $w := z^1$, there

□
are an object $A^6$, a morphism $q : A^6 \to A^5$ in $W$ and an invertible 2-morphism $\eta : t \circ s \circ q \Rightarrow t \circ s \circ q$, such that $i_z \circ \eta$ coincides with the following composition:

$$
\begin{array}{ccc}
A^6 & \xrightarrow{q} & A^5 \\
\downarrow{s} & & \downarrow{\sigma} \\
A^4 & \xrightarrow{\tau} & A^2 \\
\downarrow{p} & & \downarrow{\rho^{-1}} \\
A^1 & \xrightarrow{\tau} & A^1 \\
\end{array}
$$

Therefore, $\rho \circ i \circ \sigma$ coincides with the following composition:

$$
\begin{array}{ccc}
A^6 & \xrightarrow{q} & A^5 \\
\downarrow{s} & & \downarrow{\sigma^{-1}} \\
A^4 & \xrightarrow{p} & A^2 \\
\downarrow{\eta} & & \downarrow{\rho} \\
A^1 & \xrightarrow{\tau} & A^2 \\
\end{array}
$$

Then the class of (3.10) coincides with the class of the following diagram

$$
\begin{array}{ccc}
A^6 & \xrightarrow{t \circ s \circ q} & A^1 \\
\downarrow{q} & & \downarrow{f^1} \\
A^5 & \xrightarrow{\tilde{\alpha}} & A^3 \\
\downarrow{s} & & \downarrow{w^3} \\
A^4 & \xrightarrow{\tilde{\beta}} & A^2 \\
\downarrow{p} & & \downarrow{f^2} \\
B, & & B \\
\end{array}
$$

(3.2)

where $\tilde{\alpha}$ is the following composition

$$
\begin{array}{ccc}
A^6 & \xrightarrow{t \circ s \circ q} & A^1 \\
\downarrow{q} & & \downarrow{u^1 \circ t \circ o \circ s \circ q} \\
A^5 & \xrightarrow{\tilde{\alpha}} & A^3 \\
\downarrow{s} & & \downarrow{w^3} \\
A^4 & \xrightarrow{\tilde{\beta}} & A^2 \\
\downarrow{p} & & \downarrow{f^2} \\
A & \xrightarrow{w^2} & A \\
\end{array}
$$

and $\tilde{\beta}$ is the following composition:
Then using the description of 2-morphisms in § [11] it is easy to see that the class of (3.2) (hence also the class of (0.10)) coincides with the class of (3.1). □

4. Horizontal compositions with 1-morphisms on the left

Proof of Proposition 0.3. By [Pr, pagg. 259–261], one has to compute $\Delta^* i_f$ as follows:

(i) for each $m = 1, 2$, we use choices $\mathcal{C}(\mathcal{W})$ in order to get data as in the upper part of the following diagram, with $\varepsilon_m$ in $\mathcal{W}$ and $\tilde{\varepsilon}$ invertible:

(ii) for each $m = 1, 2$, we use again choices $\mathcal{C}(\mathcal{W})$ in order to get data as in the upper part of the following diagram, with $\tilde{\varepsilon}_m$ in $\mathcal{W}$ and $\varepsilon_m$ invertible:

(iii) we use a long procedure (that we don’t need to explain explicitly for the purposes of this proof) in order to get for each $m = 1, 2$ an object $\breve{\mathcal{A}}_m$, a morphism $\breve{r}_m : \breve{\mathcal{A}}_m \rightarrow \mathcal{A}''_m$ in $\mathcal{W}$ and an invertible 2-morphism


such that the following composition


coincides with
(iv) we use choices $C(W)$ in order to get data as in the upper part of the following diagram, with $\pi^3$ in $W$ and $\pi$ invertible (note that here $w_2 \circ z^2 \circ \pi^2$ belongs to $W$ by (BF2) and (BF5) applied to $(\pi^3)^{-1} \circ i_{\pi^3}$):

\[
\begin{array}{ccc}
\hat{A}^m & \xrightarrow{\pi^m \circ \pi^m} & \hat{A}'^m \\
\downarrow f \circ \pi^m & & \downarrow \delta \\
\downarrow \nu^m \circ \nu^m \circ f^m & & \end{array}
\]

(v) we use choices $D(W)$ in order to get an object $\hat{A}'$, a morphism $s_3 : \hat{A}' \to \hat{A}$ in $W$ and an invertible 2-morphism $\eta : f_1 \circ w_1 \circ \pi_1 \circ \pi \circ \eta = \pi_2 \circ w_2 \circ \pi_2 \circ \pi_2$, such that $i_{\nu^1} \circ u^1 \star \eta$ coincides with the following composition:

\[
\begin{array}{ccc}
\hat{A}_1 & \xrightarrow{\pi_1} & \hat{A}'_1 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
\hat{A}_2 & \xrightarrow{\pi_2 \circ \pi\circ \eta} & \hat{A}'_2 \\
\end{array}
\]

(vi) then according to [Pr, pagg. 259–261], one defines $\Delta \star i_f$ as the class of the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{w_1 \circ w_1} & A' \\
\downarrow i_{w_1} \star \alpha \star i_{w_1} & & \downarrow \beta \\
A' & \xrightarrow{g_1 \circ f_1} & C, \\
\end{array}
\]

\[\text{(4.3)}\]
where $\delta$ is the following composition:

\[
\begin{array}{c}
\vcenter{\hbox{\xymatrix{A' \ar[r]^{\alpha} & A'' \ar[r]^{\alpha'} & A' \ar[r]^{\mu} & A'' \ar[r]^{\mu'} & A'}}}
\end{array}
\]

(4.4)

Now the claim of this proposition is equivalent to the claim that (1.16) and (3.3) are in the same equivalence class for each set of choices (P2) – (P7); we are going to prove this in the next lines.

By hypothesis, $w^1, w'^1$ and $t^1$ belong to $W$, hence also $w^1 \circ u'^1 \circ t^1$ belongs to $W$ by (BF2), so by (BF3) for $(w')^{-1}$ we conclude that also $w^2 \circ u^2 \circ t^2$ belongs to $W$. Moreover, $w^2$ belongs to $W$ by hypothesis, so by Lemma 1.2 for $w := u^2 \circ t^2$, there are data as in the upper part of the following diagram, with $q^2$ in $W$ and $\mu^2$ invertible:

\[
\begin{array}{c}
\vcenter{\hbox{\xymatrix{A^4 \ar[r]^{q^1} & A^2 \ar[r]^{u^2 \circ t^2} & A''}}}
\end{array}
\]

Then we use (BF4a) and (BF4b) in order to get an object $A^5$, a morphism $q^3 : A^5 \to A^4$ in $W$ and an invertible 2-morphism

$\mu^2 : u^1 \circ t^1 \circ q^1 \circ q^3 \Rightarrow \pi^1 \circ \pi^1 \circ \pi^2 \circ q^2 \circ q^3$,

such that $i_{w^1} \ast \mu^2$ coincides with the following composition:

\[
\begin{array}{c}
\vcenter{\hbox{\xymatrix{A^5 \ar[r]^{q^3} & A^4 \ar[r]^{u^1 \circ t^1} & A^1 \ar[r]^{w^1} & A'.}}}
\end{array}
\]

This implies that $\alpha' \ast i_{q^1} \circ q^3$ coincides with the following composition:
So using again the description of 2-morphisms in §1.1, the class of (0.13) coincides with the class of the following diagram:

\[
\begin{array}{ccc}
A^5 & \xrightarrow{u^1 \circ t^1 \circ q_1 \circ q^2} & A' \\
\downarrow \mu^2 & & \downarrow \overline{\mu} \\
A^4 & \xrightarrow{u^2 \circ t^2 \circ q_1} & A^2
\end{array}
\]

(4.5)

where $\tilde{\beta}'$ is the following composition:

\[
\begin{array}{ccc}
A^5 & \xrightarrow{g^1 \circ f^1} & C, \\
\downarrow \iota_w \ast \overline{\iota} \ast \iota \circ \sigma q^2 \circ q_3 & & \downarrow \tilde{\beta}' \\
A^2 & \xrightarrow{g^2 \circ f^2} & C
\end{array}
\]

(4.6)

By hypothesis, $\Delta$ is 2-morphism in $\mathcal{C}[W^{-1}]$, so $v^1 \circ u^1$ belongs to $W$; moreover $v^1$ belongs to $W$ because $g^1$ is a morphism in $\mathcal{C}[W^{-1}]$. So by Lemma I.3 there are an object $A^6$, a morphism $q^4 : A^6 \to A^5$ in $W$ and an invertible 2-morphism $\eta^1$ such that $i_w \ast \eta^1$ coincides with the following composition:

\[
\begin{array}{ccc}
A^5 & \xrightarrow{\overline{\sigma^1} \circ \overline{\sigma^2} \circ \overline{\sigma^3} \circ q^2 \circ q_3} & A' \\
\downarrow (\mu^2)^{-1} & & \downarrow \overline{\sigma} \\
A^4 & \xrightarrow{q^1} & A^2
\end{array}
\]

\[
\begin{array}{ccc}
A^5 & \xrightarrow{f^1} & B^1 \\
\downarrow \delta & & \downarrow \beta \\
A^2 & \xrightarrow{f_2} & B^2
\end{array}
\]

(4.7)
This implies that the following composition

\[
\begin{array}{c}
A^6 \xrightarrow{q^4} A^5 \\
\downarrow \mu_2^{-1} \quad \downarrow \gamma_1 \quad \downarrow f^1 \\
A^6 \xrightarrow{q^3} A^5 \\
\end{array}
\]

coincides with the following one:

\[
\begin{array}{c}
A^6 \xrightarrow{q^4} A^5 \\
\downarrow \eta_1 \quad \downarrow (\mu_2^{-1}) \quad \downarrow (\gamma_1) \quad \downarrow f^1 \\
A^6 \xrightarrow{q^3} A^5 \\
\end{array}
\]

Therefore, using (4.7) we get that \( \tilde{\beta} * i_{q^4} \) coincides with the following composition:

\[
\begin{array}{c}
A^6 \xrightarrow{q^4} A^5 \\
\downarrow \eta_1 \quad \downarrow \gamma_1 \quad \downarrow f^1 \\
A^6 \xrightarrow{q^3} A^5 \\
\end{array}
\]

(4.8)

We have already remarked that \( v^1 \circ u^1 \) belongs to \( W \); so by (BF5) applied to \( \alpha^{-1} \) we get that also \( v^2 \circ u^2 \) belongs to \( W \); moreover \( v^2 \) belongs to \( W \) because \( g^2 \) is a morphism in \( C \) \([W^{-1}]\). So by Lemma 1.3 there are an object \( A^7 \), a morphism \( q^5 : A^7 \to A^6 \) in \( W \) and an invertible 2-morphism

\[
\eta^2 : f'^2 \circ t^2 \circ q^1 \circ q^3 \circ q^4 \circ q^5 \Rightarrow f^2 \circ \overline{g}^2 \circ f^2 \circ \overline{g}^1 \circ q^2 \circ q^3 \circ q^4 \circ q^5,
\]

such that \( i_{q^5} * \eta^2 \) coincides with the following composition
In order to do that:

This implies that the following composition

\[ A^7 \xrightarrow{q^3 \circ q^4 \circ q^5} A^4 \xrightarrow{t^2 \circ q^1} A'' \xrightarrow{\mu^1} A^2 \xrightarrow{f^2} B^3 \xrightarrow{u^2} \]

\[ \sigma^2 \circ \sigma^1 \circ \sigma^0 \circ q^2 \]

\[ \tilde{A}^2 \xrightarrow{f^2} B^3 \xrightarrow{u^2} \]

\[ A^2 \xrightarrow{\gamma^2} B^2. \]

(4.10)

Therefore, using (4.10) we get that \( \tilde{\gamma}' \circ i_{q^4 \circ q^5} \) coincides with the following composition:

\[ A^7 \xrightarrow{f^2 \circ t^2 \circ q^1 \circ q^3 \circ q^4 \circ q^5} B^3 \xrightarrow{u^2} \]

\[ \tilde{A}^2 \xrightarrow{\gamma^2 \circ \gamma^1 \circ \gamma^0} B^2. \]

\[ A^2 \xrightarrow{\gamma^2} B^2. \]

(4.11)

Now we compute

\[ i_{q^1 \circ q^2} \circ \left( \eta^2 \circ \left( \delta \circ i_{q^3 \circ q^4 \circ q^5} \right) \circ \left( \eta^1 \circ i_{q^6} \right) \right). \]

(4.12)

In order to do that:
• we replace \( i_{u^1} \circ \eta^1 \) with diagram (4.8);
• we replace \( i_{\varphi} \circ i_{u^2} \circ \delta \) with diagram (0.12);
• we simplify \( \sigma^1 \) (from (4.8)) with its inverse (from (0.12)).

So we get that (4.12) coincides with the following composition:

Now we replace \( i_{u^2} \circ \eta^2 \) in (4.13) with diagram (4.10) and we simplify \( \sigma^2 \) (from (4.13)) with its inverse (from (4.10)). So we get that (4.12) coincides with the following composition:

Now we replace \( \alpha' \circ i_{q^2} \circ q^3 \) in (4.14) with (4.5) and we simplify \( \rho^2 \) and \( \mu^1 \). So we get that (4.14) (hence (4.12)) coincides with the composition of (4.2) with the 2-identity of \( q^2 \circ q^3 \circ q^4 \circ q^5 : A^1 \rightarrow A' \). In other terms, we have proved that
\[ i_{v^1 \cdot u^1} \ast \left( \eta^2 \odot \left( \delta \ast i_{q^1 \cdot q^3 \cdot q^5 \cdot q^6} \right) \odot \left( \eta^1 \ast i_{q^5} \right) \right) = i_{v^1 \cdot u^1} \ast \left( \eta \ast i_{q^2 \cdot q^3 \cdot q^5 \cdot q^6} \right) \]

Using Lemma 1.1 we conclude that there are an object \( A^8 \) and a morphism \( q^6 : A^8 \to A^7 \) in \( W \), such that

\[ \left( \eta^2 \odot \left( \delta \ast i_{q^1 \cdot q^3 \cdot q^5 \cdot q^6} \right) \odot \left( \eta^1 \ast i_{q^5} \right) \right) \ast i_{q^6} = \eta \ast i_{q^2 \cdot q^3 \cdot q^5 \cdot q^6} \cdot \]

By replacing this in (4.11) and comparing with (4.4), we get that

\[ \tilde{\beta} \ast i_{q^2 \cdot q^3 \cdot q^5} = \beta' \ast i_{q^2 \cdot q^3 \cdot q^5 \cdot q^6} \cdot \]

This implies easily that the class of (4.6) (hence also the class of (0.13) coincides with the class of (4.3), so we get the claim. \( \square \)

5. Horizontal compositions with 1-morphisms on the right

Proof of Proposition 0.4. According to [Pr, pag. 259], in order to compute \( i_{\gamma} \ast \Gamma \) we have to proceed as follows:

(i) for each \( m = 1, 2 \), we use choices \( \text{C}(W) \) in order to get data as in the upper part of the following diagram, with \( \pi^m \) in \( W \) and \( \eta^m \) invertible:

\[ \xymatrix{ A^m \ar[dr]_{\eta^m} \ar[rr]^{\pi^m} & & A^m \ar[dl]_{\pi^m} \ar[dr]_{\eta^m} \ar[rr] & & A^m \ar[dl]_{\eta^m} \ar[rr] & & A^m \ar[dl]_{\eta^m} \ar[rr] & & A^m \ar[dl]_{\eta^m} \ar[rr] & & A^m \ar[dl]_{\eta^m} \ar[rr] & & A^m \ar[dl]_{\eta^m} \ar[rr] & & A^m \ar[dl]_{\eta^m} \ar[rr] & & A^m \ar[dl]_{\eta^m} \ar[rr] & & A^m \ar[dl]_{\eta^m} \ar[rr] & & A^m \ar[dl]_{\eta^m} \ar[rr] & & A^m \ar[dl]_{\eta^m} \ar[rr] & & A^m \ar[dl]_{\eta^m} \ar[rr] & & A^m \ar[dl]_{\eta^m} \ar[rr] & & A^m } \]

(ii) we use again choices \( \text{C}(W) \) in order to get data as in the upper part of the following diagram, with \( \pi^1 \) in \( W \) and \( \eta^1 \) invertible:

\[ \xymatrix{ A^1 \ar[dr]_{\eta^1} \ar[rr]^{\pi^1} & & A^1 \ar[dl]_{\eta^1} \ar[dr]_{\eta^1} \ar[rr] & & A^1 \ar[dl]_{\eta^1} \ar[rr] & & A^1 \ar[dl]_{\eta^1} \ar[rr] & & A^1 \ar[dl]_{\eta^1} \ar[rr] & & A^1 \ar[dl]_{\eta^1} \ar[rr] & & A^1 \ar[dl]_{\eta^1} \ar[rr] & & A^1 \ar[dl]_{\eta^1} \ar[rr] & & A^1 } \]

(iii) we use choices \( \text{D}(W) \) in order to get an object \( \overline{A}'' \), a morphism \( \varepsilon : \overline{A}''' \to \overline{A}'' \) in \( W \) and a 2-morphism

\[ \overline{\beta} : f^1 \circ \pi^1 \circ \pi^3 \circ \eta = f^2 \circ \pi^2 \circ \pi^3 \circ \eta, \]

such that \( i_{\gamma} \ast \overline{\beta} \) coincides with the following composition:
(iv) then following [Pr], one defines \( \mathfrak{i}_g \ast \Gamma \) as the class of the following diagram

\[
\begin{array}{ccc}
\mathfrak{A}' & \xrightarrow{\pi^1} & A^1 \\
\downarrow \eta^1 & & \downarrow (\rho^1)^{-1} \\
\mathfrak{A}' & \xrightarrow{\pi^2} & \mathfrak{A}' \\
\downarrow \eta^2 & & \downarrow (\rho^2)^{-1} \\
A^2 & \xrightarrow{\pi^3} & A^3 \\
\end{array}
\]

where \( \pi' \) is the following composition:

\[
\begin{array}{ccc}
\mathfrak{A}' & \xrightarrow{\pi^1 \circ \rho^1} & A^1 \\
\downarrow \pi^2 & & \downarrow \pi^3 \\
\mathfrak{A}' & \xrightarrow{\pi^2 \circ \rho^2} & A^2 \\
\end{array}
\]

So the claim is equivalent to proving that the class of (0.16) coincides with the class of (5.2). In order to do that, we proceed as follows. By hypothesis, both \( u'^1 \) and \( z^1 \) belong to \( W \); so we use (BF4a) and (BF4b) in order to get data as in the upper part of the following diagram, with \( r^1 \) in \( W \) and \( \mu^1 \) invertible:

\[
\begin{array}{ccc}
A^4 & \xrightarrow{\mu^1} & A^3 \\
\downarrow \iota^1 & & \downarrow \iota^2 \\
\mathfrak{A}'' & \xrightarrow{\pi^1 \circ \rho^1} & A^3 \\
\end{array}
\]

By hypothesis \( u'^1 \) belongs to \( W \), so we use (BF4a) and (BF4b) in order to get an object \( A^5 \), a morphism \( r^1 : A^3 \to A^4 \) in \( W \) and an invertible 2-morphism
\[ \sigma^1 : \varpi^1 \circ \varpi^1 \circ \varpi^1 \circ r^1 \circ r^3 \Rightarrow \varpi^1 \circ \varpi^1 \circ r^2 \circ r^3, \]
such that \( i_{\nu^1} \ast \sigma^1 \) coincides with the following composition:

\[
\begin{array}{ccc}
A^5 \xrightarrow{r^5} A^4 & \xrightarrow{\varpi^1 \circ \varpi^1 \circ \varpi^1 \circ r^3} & A^1 \\
\downarrow{\sigma^1} & \Downarrow{(\eta^1)^{-1}} & \downarrow{\mu^1} \\
A'' & \xrightarrow{u''} & A^1 \\
\end{array}
\]

This implies that the following composition

\[
\begin{array}{ccc}
A^5 \xrightarrow{\varpi^1 \circ \varpi^1 \circ \varpi^1 \circ r^3} & A^1 \\
\downarrow{\sigma^1} & \Downarrow{(\eta^1)^{-1}} & \downarrow{\mu^1} \\
A'' & \xrightarrow{u''} & A^1 \\
\end{array}
\]

coincides with the following one:

\[
\begin{array}{ccc}
A^5 \xrightarrow{\varpi^1 \circ \varpi^1 \circ \varpi^1 \circ r^3} & A^1 \\
\downarrow{\sigma^1} & \Downarrow{(\eta^1)^{-1}} & \downarrow{\mu^1} \\
A'' & \xrightarrow{u''} & A^1 \\
\end{array}
\]

By construction, both \( \varpi^1 \) and \( \varpi^1 \) belong to \( W \), so using \( \text{BF2} \) and \( \text{BF5} \) (for \( (\eta^1)^{-1} \)), we get that \( \varpi^2 \circ \varpi^2 \) belongs to \( W \). Moreover, by construction, \( \tau \), \( r^2 \) and \( r^4 \) belong to \( W \). So using \( \text{BF2} \) and \( \text{BF3} \) we get data as in the upper part of the following diagram, with \( r^5 \) in \( W \) and \( \mu^2 \) invertible:

\[
\begin{array}{ccc}
A^6 \xrightarrow{r^5} A^3 \\
\downarrow{\mu^2} & \Downarrow{r^5} & \downarrow{r^5} \\
A^5 & \xrightarrow{\varpi^2 \circ \varpi^1 \circ r^3} & A^5 \\
\end{array}
\]

By hypothesis, \( u^2 \) belongs to \( W \), so using \( \text{BF4a} \) and \( \text{BF4b} \) we get an object \( A^7 \), a morphism \( r^6 : A^7 \to A^6 \) in \( W \) and an invertible 2-morphism

\[ \sigma^2 : \varpi^2 \circ \varpi^2 \circ \varpi^2 \circ r^2 \circ r^5 \circ r^6 \Rightarrow \varpi^2 \circ \varpi^2 \circ \varpi^2 \circ \varpi^1 \circ r^3 \circ r^4 \circ r^6, \]
such that \( i_{\nu^2} \ast \sigma^2 \) coincides with the following composition:
This implies that the following composition

\[ A^7 \xrightarrow{r_6} A^6 \xrightarrow{\mu_2} A^{\prime 2} \xrightarrow{(\eta^2)^{-1}} A^2 \xrightarrow{\nu^2} A^2, \]

coincides with the following one:

\[ A^6 \xrightarrow{u^\nu} A'' \xrightarrow{\eta^2} A^2 \xrightarrow{\nu^2} A^2 \]

(5.5)

Then we get easily that the class of (0.16) coincides with the class of the following diagram:

\[ A^7 \xrightarrow{r_6} A^6 \xrightarrow{u^\mu} A'' \xrightarrow{u^\nu \circ \eta^2} A^3 \xrightarrow{\nu^2} A^2 \]

(5.6)

where \( \tilde{\beta}' \) is the following composition:
and $\tilde{\alpha}'$ is the following composition:

\[ \begin{array}{ccccccccc}
A^4 & \xrightarrow{\pi^1\circ\pi^2} & A^1 \\
\downarrow\sigma^1 & & \downarrow\beta^1 \\
A^5 & \xrightarrow{\pi^2\circ\pi^3} & A^2 \\
\downarrow\sigma^2 & & \downarrow\beta^2 \\
A^7 & \xrightarrow{\pi^2\circ\pi^3\circ\pi^1\circ\pi^2\circ\pi^3} & A^2
\end{array} \]

\[ \begin{array}{ccccccccc}
A^4 & \xrightarrow{\pi^1\circ\pi^2} & A^1 \\
\downarrow\mu^1 & & \downarrow\eta^1 \\
A^5 & \xrightarrow{u^1\circ\pi^1} & A^3 \\
\downarrow\mu^2 & & \downarrow\eta^2 \\
A^7 & \xrightarrow{\pi^1\circ\pi^2\circ\pi^3\circ\pi^1\circ\pi^2\circ\pi^3} & A^3
\end{array} \]

\[ \begin{array}{ccccccccc}
A^4 & \xrightarrow{\pi^1\circ\pi^2} & A^1 \\
\downarrow\eta^1 & & \downarrow\eta^2 \\
A^5 & \xrightarrow{u^1\circ\pi^1} & A^3 \\
\downarrow\eta^2 & & \downarrow\eta^3 \\
A^7 & \xrightarrow{\pi^1\circ\pi^2\circ\pi^3\circ\pi^1\circ\pi^2\circ\pi^3} & A^3
\end{array} \]

\[ \begin{array}{ccccccccc}
A^4 & \xrightarrow{\pi^1\circ\pi^2} & A^1 \\
\downarrow\eta^1 & & \downarrow\eta^2 \\
A^5 & \xrightarrow{u^1\circ\pi^1} & A^3 \\
\downarrow\eta^2 & & \downarrow\eta^3 \\
A^7 & \xrightarrow{\pi^1\circ\pi^2\circ\pi^3\circ\pi^1\circ\pi^2\circ\pi^3} & A^3
\end{array} \]

\[ \begin{array}{ccccccccc}
A^4 & \xrightarrow{\pi^1\circ\pi^2} & A^1 \\
\downarrow\eta^1 & & \downarrow\eta^2 \\
A^5 & \xrightarrow{u^1\circ\pi^1} & A^3 \\
\downarrow\eta^2 & & \downarrow\eta^3 \\
A^7 & \xrightarrow{\pi^1\circ\pi^2\circ\pi^3\circ\pi^1\circ\pi^2\circ\pi^3} & A^3
\end{array} \]

(Here $\tilde{\alpha}'$ is obtained using the identity of (5.3) and (5.4) and the identity of (5.5) and (5.6)). By construction, $\pi^1$, $\pi^2$, $\pi^3$, $r^1$ and $r^2$ belong to $W$, so we use [BFP2], [BFP3] and [BFP4] in order to get an object $A^8$, a morphism $r^7 : A^8 \rightarrow A^7$ in $W$ and an invertible 2-morphism

\[ \varepsilon : r^5 \circ r^6 \circ r^7 \Rightarrow r^4 \circ r^6 \circ r^7, \]

such that $i_{\pi^1\circ\pi^2\circ\pi^3\circ\pi^4} \ast \varepsilon$ coincides with the following composition:

\[ \begin{array}{ccccccccc}
A^4 & \xrightarrow{\pi^1\circ\pi^2} & A^1 \\
\downarrow\mu^1 & & \downarrow\eta^1 \\
A^5 & \xrightarrow{u^1\circ\pi^1} & A^3 \\
\downarrow\mu^2 & & \downarrow\eta^2 \\
A^7 & \xrightarrow{\pi^1\circ\pi^2\circ\pi^3\circ\pi^1\circ\pi^2\circ\pi^3} & A^3
\end{array} \]

\[ \begin{array}{ccccccccc}
A^4 & \xrightarrow{\pi^1\circ\pi^2} & A^1 \\
\downarrow\eta^1 & & \downarrow\eta^2 \\
A^5 & \xrightarrow{u^1\circ\pi^1} & A^3 \\
\downarrow\eta^2 & & \downarrow\eta^3 \\
A^7 & \xrightarrow{\pi^1\circ\pi^2\circ\pi^3\circ\pi^1\circ\pi^2\circ\pi^3} & A^3
\end{array} \]

Therefore, we have that the following composition
coincides with the following one

\[
A^8 \xrightarrow{\tau_{or^7} \circ r^4 \circ r^6 \circ r^7} A^1 \xleftarrow{\pi^1} A^0 \xrightarrow{\pi^1} A^3.
\]

(5.9)

So we get that the class of (5.7) (hence also the class of 0.16) coincides with the class of the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{w^1 \circ u^1} & A^1 \\
\downarrow \tau & \downarrow g \circ f^1 & \downarrow g \circ f^2 \\
A^8 & \xrightarrow{i_g \ast \beta'} & C \\
\end{array}
\]

(5.10)

where \( \beta' \) is the following composition:

\[
\begin{array}{ccc}
A^4 & \xrightarrow{\psi^1 \circ \tau_{or^1}} & A^1 \\
\downarrow r^3 & \downarrow \sigma^1 & \downarrow \beta' \\
A^5 & \xrightarrow{\tau^2 \circ r^3} & A^0 \\
\downarrow \epsilon^{-1} & \downarrow \sigma^2 & \downarrow \psi^2 \circ \tau_{or^2} \circ r^3 \circ r^6 \\
A^8 & \xrightarrow{r^2 \circ r^7} & A^2 \\
\end{array}
\]

Now using 0.15 we get that \( i_u \ast \beta' \) coincides with the following composition:
Now in (5.11) we replace (5.3) with (5.4) and (5.5) with (5.6); so we get that $i_u \ast \hat{\beta}'$ coincides with the following composition:

$$
\begin{array}{c}
A^0 \xrightarrow{r_8 \circ r_7} A^7 \\
\scriptstyle{\mu} \downarrow \quad \scriptstyle{\mu^1} \downarrow \quad \scriptstyle{\mu^2} \downarrow \\
A^5 \xrightarrow{r^3 \circ r^2 \circ r_7} A^3 \\
\scriptstyle{\varepsilon} \downarrow \quad \scriptstyle{\varepsilon^1} \downarrow \quad \scriptstyle{\varepsilon^2} \downarrow \\
A^0 \xrightarrow{r^3 \circ r^2 \circ r_7} A^2 \\
\scriptstyle{\pi^2} \circ \pi^3 \circ r^3 \circ r^2 \circ r_7 \\
A^6 \xrightarrow{r^6 \circ r_7} A^6 \\
\end{array}
\begin{array}{c}
\xrightarrow{r^1 \circ r^2 \circ r_7} \\
\xrightarrow{f^2 \circ f^2} \\
\end{array}
\begin{array}{c}
B' \\
\end{array}
$$

In the previous diagram we replace diagram (5.5) with (5.9). Therefore, we conclude that $i_u \ast \hat{\beta}'$ coincides with $i_u \ast \hat{\beta}' \ast i_r \circ r_7 \circ r^4 \circ r^6 \circ r^7$ (see (5.1)). So by Lemma 1.1 there are an object $A^0$ and a morphism $r_8: A^0 \to A^8$ in $\mathcal{W}$, such that

$$
\hat{\beta}' \ast i_r = \hat{\beta}' \ast i_r \circ r_7 \circ r^4 \circ r^6 \circ r^7.
$$

So we conclude easily that the class of (5.10) (hence also the class of (0.16)) coincides with the class of (5.2).

As an application of all the Propositions proved so far, we are giving the following Corollary, that will be useful in the next paper of this series ([T1]). In that paper, we will have to compare the compositions of 3 morphisms of the form (0.3) with the compositions of 3 morphisms of the following form...
In the special case when \( \mathcal{C} \) is a 2-category we have that \( f = f' \) and so on, hence \( h \circ g \circ f = h' \circ (g' \circ f) \) and \( (h \circ g) \circ f = (h' \circ g') \circ f \). However, when \( \mathcal{C} \) is simply a bicategory, in general the fixed choice in \( \mathcal{C}(W) \) for the pair \((f, v)\) (see the first diagram in (0.4)) is different from the fixed choice for the pair \((f \circ \text{id}_{A'}, v \circ \text{id}_{B'})\), and analogously for all the remaining choices needed to compose the morphisms in (5.12). Therefore, we need a way to compare \( h \circ (g \circ f) \) with \( h' \circ (g' \circ f) \) and analogously for the other pair of compositions. In order to do that, first of all we compare separately \( f \) with \( f' \) by defining an invertible 2-morphism \( \chi(f) : f \Rightarrow f' \) as the class of the following diagram:

\[
\begin{array}{c}
A \\
\downarrow \pi_w \circ \text{id}_{A'} \\
A' \\
\downarrow \pi_{f \circ \text{id}_{A'}}
\end{array}
\begin{array}{c}
\text{id}_{A'} \\
\downarrow f \circ \text{id}_{A'}
\end{array}
\]

and analogously for \( \chi(g) \) and \( \chi(h) \). Then we have the following result.

**Corollary 5.1.** Let us fix any pair \( (\mathcal{C}, W) \) satisfying conditions (BF) and any set of 6 morphisms as in (0.3) and (5.12). Then the associator \( \Theta_{h, g, f, h', g', f'} : h \circ (g \circ f) \Rightarrow (h \circ g) \circ f \) coincides with the following composition:

\[
\begin{array}{c}
A \\
\downarrow \chi(g) \circ \chi(f) \\
C \\
\downarrow \chi(h)
\end{array}
\begin{array}{c}
L' \\
\downarrow \chi(h') \circ \chi(g') \\
B \\
\downarrow \chi(g)^{-1}
\end{array}
\]

This result is a direct application of Propositions 0.1, 0.2 and 0.4. We refer to the Appendix for a proof.

6. **Comparison of 2-morphisms and conditions for invertibility in a bicategory of fractions**

In this section we are going to prove Propositions 0.7 and 0.8. As in the previous sections, we are going to write the statements of each result in the framework of bicategories, but for simplicity of expositions we are going to give all the proofs only in the special framework of 2-categories. The interested reader can easily fill
the missing details when \( C \) is not a 2-category but simply a bicategory, by adding associators and unitors wherever it is necessary.

**Lemma 6.1.** Let us fix any pair \((C, W)\) satisfying conditions [BF] and any set of data in \( C \) as follows

\[
\begin{array}{ccc}
A^1 & \xrightarrow{f^1} & B,
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & A^3
\end{array}
\]

\[
\begin{array}{ccc}
A & \xleftarrow{w^2} & A^2
\end{array}
\]

with \( w^1, w^2 \) and \( w^1 \circ v^1 \) in \( W \) and \( \alpha \) invertible. Then for each 2-morphism

\[ \Gamma : (A^1, w^1, f^1) \Rightarrow (A^2, w^2, f^2) \in C[W^{-1}] \]

there are an object \( A^4 \), a morphism \( z : A^4 \to A^3 \) such that \((w^1 \circ v^1) \circ z\) belongs to \( W \), and a 2-morphism \( \gamma : f^1 \circ (v^1 \circ z) \Rightarrow f^2 \circ (v^2 \circ z) \) in \( C \), such that

\[ \Gamma = [A^4, v^1 \circ z, v^2 \circ z, \theta^{-1}_{w^1, z} \circ (\alpha \ast i_z) \circ \theta_{w^1, v^1, z}, \gamma]. \]

We refer to the Appendix for the proof.

**Lemma 6.2.** Let us fix any pair \((C, W)\) satisfying conditions [BF] and any set of data in \( C \) as follows

\[
\begin{array}{ccc}
A^1 & \xrightarrow{f^1} & B,
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{w^2} & A^2
\end{array}
\]

with \( w^1 \) and \( w^2 \) in \( W \). Then given any pair of 2-morphisms

\[ \Gamma^1, \Gamma^2 : (A^1, w^1, f^1) \Rightarrow (A^2, w^2, f^2) \]

in \( C[W^{-1}] \), there are an object \( A^3 \), a pair of morphisms \( v^m : A^3 \to A^m \) for \( m = 1, 2 \), an invertible 2-morphism \( \alpha : w^1 \circ v^1 \Rightarrow w^2 \circ v^2 \) and a pair of 2-morphisms \( \gamma^1, \gamma^2 : f^1 \circ v^1 \Rightarrow f^2 \circ v^2 \), such that \( w^1 \circ v^1 \) belongs to \( W \) and

\[ \Gamma^m = [A^3, v^1, v^2, \alpha, \gamma^m] \] for \( m = 1, 2. \]

In other terms, given any pair of 2-morphisms in \( C[W^{-1}] \) with the same source and target, they differ at most by one term.

**Proof.** By definition of 2-morphism in \( C[W^{-1}] \), there are data \((A^3, \nabla^1, \nabla^2, \pi, \tau)\) such that

\[ \Gamma^1 = [\nabla^3, \nabla^1, \nabla^2, \pi, \tau], \]
with $w^1 \circ v^1$ in $W$ and $\pi$ invertible. Then we apply Lemma 6.1 to the set of data given by $\alpha$, $\alpha$, $\alpha$, $\alpha$, $\nu$, $A$, $\gamma$, $A$, $\nu$, $\nu$, $\gamma$, $z$: 

Proof. Consider the $2$-morphism in a bicategory of fractions and $\nu$, we have easily

Then in order to conclude, it suffices to set $v^m := \nu \circ z$ for $m = 1, 2$, $\alpha := \pi \circ i_z$ and $\gamma_1 := \gamma \circ i_z$.

By induction and using the same ideas, one can also prove easily that given finitely many $2$-morphisms $\Gamma_1, \cdots, \Gamma_n$ in $\mathcal{C}[W^{-1}]$, all defined between the same pair of morphisms, there are data $A^3, v^1, v^2, \alpha, \gamma_1, \cdots, \gamma_n$, such that $w^1 \circ v^1$ belongs to $W$, $\alpha$ is invertible and $\Gamma^m = [A^3, v^1, v^1, \alpha, \gamma_m]$ for each $m = 1, \cdots, n$.

Until now we have proved that any pair of $2$-morphisms $\Gamma^1, \Gamma^2$ between the same pair of morphisms in $\mathcal{C}[W^{-1}]$ can be “reduced” to a common form (as in (6.3)), where all data, except (possibly) one, coincide. The next lemma shows under which condition on the remaining datum we have $\Gamma^1 = \Gamma^2$.

Lemma 6.3. Let us fix any pair $(\mathcal{C}, W)$ satisfying conditions (BF) and any set of data in $\mathcal{C}$ as in (6.1), with $w^1, w^2$ and $w^1 \circ v^1$ in $W$ and $\alpha$ invertible. Moreover, let us fix any pair of $2$-morphisms $\gamma_1, \gamma_2 : f^1 \circ v^1 \Rightarrow f^2 \circ v^2$; for each $m = 1, 2$, let us consider the $2$-morphism in $\mathcal{C}[W^{-1}]$

Then the following facts are equivalent

(i) $\Gamma^1 = \Gamma^2$;

(ii) there are an object $A^4$ and a morphism $z : A^4 \to A^3$, such that $(w^1 \circ v^1) \circ z$ belongs to $W$ and $\gamma_1 \circ i_z = \gamma_2 \circ i_z$.

Proof. Let us suppose that (i) holds. Then there is a set of data in $\mathcal{C}$ as in the internal part of the following diagram:

\[
\begin{array}{ccc}
A^3 & \xrightarrow{\sigma^1} & A^4 \\
\downarrow{v^1} & & \downarrow{r} \\
\bar{A} & \xleftarrow{\sigma^2} & A^3, \\
\downarrow{v^2} & & \downarrow{v^2} \\
A^2 & \xrightarrow{} & A^2,
\end{array}
\]

such that $w^1 \circ v^1 \circ r$ belongs to $W$, $\sigma^1$ and $\sigma^2$ are invertible,

\[
\left( i_{w^2} \circ \sigma^2 \right) \circ \left( \alpha \circ i_v \right) \circ \left( i_{w^1} \circ \sigma^1 \right) = \alpha \circ i_v
\]

and

\[
\left( i_{f^2} \circ \sigma^2 \right) \circ \left( \gamma^2 \circ i_v \right) \circ \left( i_{f^1} \circ \sigma^1 \right) = \gamma^1 \circ i_v.
\]

Since both $w^1$ and $w^1 \circ v^1$ belong to $W$ by hypothesis, then by Lemma 6.3 there are an object $\bar{A}$, a morphism $p^1 : \bar{A} \to \bar{A}$ in $W$ and an invertible $2$-morphism $\bar{\sigma}^1 : r^1 \circ p^1 \Rightarrow r \circ p^1$, such that $v^1 \circ \bar{\sigma}^1 = \sigma^1 \circ i_p^1$. 

\[
\begin{array}{ccc}
A^3 & \xrightarrow{\sigma^1} & A^4 \\
\downarrow{v^1} & & \downarrow{r} \\
\bar{A} & \xleftarrow{\sigma^2} & A^3, \\
\downarrow{v^2} & & \downarrow{v^2} \\
A^2 & \xrightarrow{} & A^2,
\end{array}
\]
Since \( w^1 \circ v^1 \) belongs to \( W \), then by (BF5) applied \( \alpha^{-1} \) we get that also \( w^2 \circ v^2 \) belongs to \( W \); since also \( w^2 \) belongs to \( W \) by hypothesis, then by Lemma [1.3] there are an object \( A^1 \), a morphism \( p^2 : A^1 \rightarrow A^2 \) in \( W \) and an invertible 2-morphism \( \tilde{\sigma}^2 : r \circ p^1 \circ p^2 \Rightarrow r' \circ p^1 \circ p^2 \), such that \( i_{v^2} \circ \tilde{\sigma}^2 = \sigma^2 \circ i_{p^1 \circ p^2} \). Using (6.3), this implies that:

\[
\left( i_{w^2 \circ v^2} \circ \tilde{\sigma}^2 \right) \circ \left( \alpha \circ i_{r \circ p^1 \circ p^2} \right) \circ \left( i_{w^1 \circ v^1} \circ \tilde{\sigma}^1 \circ i_{p^2} \right) = \alpha \circ i_{r' \circ p^1 \circ p^2}.
\]

Using interchange law and the fact that \( \alpha \) is invertible by hypothesis, the previous identity implies that \( (\tilde{\sigma}^2)^{-1} = \tilde{\sigma}^1 \circ i_{p^2} \). Then (6.6) implies that:

\[
\gamma^1 \circ i_{t'} \circ p^1 \circ p^2 = (i_{f^2} \circ \sigma^2 \circ i_{p^1 \circ p^2}) \circ (\gamma^2 \circ i_{r \circ p^1 \circ p^2}) \circ (i_{f^1} \circ \sigma^1 \circ i_{p^2}) = (i_{f^2 \circ v^2} \circ \tilde{\sigma}^2) \circ (\gamma^2 \circ i_{r \circ p^1 \circ p^2}) \circ (i_{f^1 \circ v^1} \circ \tilde{\sigma}^1 \circ i_{p^2}) = \gamma^2 \circ i_{r' \circ p^1 \circ p^2}.
\]

Then we set \( z := r' \circ p^1 \circ p^2 : A^2 \rightarrow A^3 \); by (BF2) and construction we have that \( w^1 \circ v^1 \circ r \circ p^1 \circ p^2 \) belongs to \( W \). So using (BF5) and \( i_{w^1} \circ \sigma^1 \circ i_{p^1 \circ p^2} \), we conclude that also \( w^1 \circ v^1 \circ z \) belongs to \( W \). Using (6.21), we get that (ii) holds. Conversely, if (ii) holds, then (i) is obviously satisfied using the definition of 2-morphism in \( \mathcal{C}[W^{-1}] \).

Combining Lemmas 6.2 and 6.3 we get Proposition 0.7.

**Proof of Proposition 0.7** Let us assume (iii), so let us choose any representative (0.18) for \( \Gamma \) and let us assume that there is a pair \( (A^4, u : A^4 \rightarrow A^3) \) such that \( w^1 \circ v^1 \circ u \) belongs to \( W \) and \( \beta \circ i_u \) is invertible in \( \mathcal{C} \). By the description of 2-morphisms in \( \mathcal{C}[W^{-1}] \), we have:

\[
\Gamma = \left[ A^4, v^1 \circ u, v^2 \circ u, \alpha \circ i_u, \beta \circ i_u \right]
\]

and the last term of such a tuple is invertible in \( \mathcal{C} \) by hypothesis, so (ii) holds.

Now let us assume (ii), so let us assume that the tuple of data (0.18) is such that \( \beta \) is invertible in \( \mathcal{C} \). Since \( \alpha \) is invertible in \( \mathcal{C} \) by definition of 2-morphism in a bicategory of fractions, then it makes sense to consider the 2-morphism \( [A^3, v^2, v^1, \alpha^{-1}, \beta^{-1}] \) in \( \mathcal{C}[W^{-1}] \). Using Proposition 0.2, it is easy to see that this is an inverse for \( \Gamma \), so (i) holds.

Now let us prove that (i) implies (iii), so let us assume that \( \Gamma \) is invertible and let us fix any representative (0.18) for it; by definition of 2-morphism in \( \mathcal{C}[W^{-1}] \) we have that \( \alpha \) is invertible, so we can apply Lemma 6.1 for the 2-morphism \( \alpha^{-1} \) of \( \mathcal{C} \) and for the 2-morphism \( \Gamma^{-1} \) of \( \mathcal{C}[W^{-1}] \). So there are an object \( \overline{A}^3 \), a morphism \( t : \overline{A}^3 \rightarrow A^3 \), such that \( w^2 \circ v^2 \circ t \) belongs to \( W \), and a 2-morphism \( \gamma : f^2 \circ v^2 \circ t \Rightarrow f^1 \circ v^1 \circ t \), such that \( \Gamma^{-1} \) is represented by the following diagram:
Then we have
\[
\begin{align*}
\{ & A^1, v^2 \circ t, v^2 \circ t, i_w \circ v^2 \circ t, i_f \circ v^2 \circ t \} = \\
& = i_{(A^1, w^1, f^1)} = \Gamma \circ \Gamma^{-1} \circ (i_{A^2, w^2, f^2}) = \{ & A^1, v^2 \circ t, v^2 \circ t, i_w \circ v^2 \circ t, i_f \circ v^2 \circ t \},
\end{align*}
\]
where \((*)\) is obtained applying Proposition 1.2. So by Lemma 6.3 there are an object \(\widetilde{A}^3\) and a morphism \(r : \widetilde{A}^3 \rightarrow \overline{A}^3\), such that \(w^2 \circ v^2 \circ \text{tor} \text{ belongs to } W\) and
\[
(\beta \circ \text{tor}) \circ (\gamma \circ \text{tor}) = i_{f^2 \circ v^2 \circ \text{tor}}.
\] (6.8)

Analogously, we have:
\[
\begin{align*}
\{ & A^3, v^1 \circ \text{tor}, v^1 \circ \text{tor}, i_w \circ v^1 \circ \text{tor}, i_f \circ v^1 \circ \text{tor} \} = \\
& = i_{(A^1, w^1, f^1)} = \Gamma \circ \Gamma^{-1} \circ (i_{A^2, w^2, f^2}) = \{ & A^3, v^1 \circ \text{tor}, v^1 \circ \text{tor}, i_w \circ v^1 \circ \text{tor}, i_f \circ v^1 \circ \text{tor} \},
\end{align*}
\]
where \((*)\) is obtained applying Proposition 1.2. So by Lemma 6.3 there are an object \(A^1\) and a morphism \(s : A^1 \rightarrow \widetilde{A}^3\), such that \(w^1 \circ v^1 \circ \text{tor} \text{ belongs to } W\) and
\[
(\gamma \circ \text{tor}) \circ (\beta \circ \text{tor}) = i_{f^1 \circ v^1 \circ \text{tor}}.
\] (6.9)

Then (6.8) and (6.9) prove that \(\beta \circ \text{tor} \text{ has an inverse in } C\) given by \(\gamma \circ \text{tor}\); in order to conclude that (iii) holds, it suffices to define \(u := \text{tor} : A^3 \rightarrow A^1\). \(\Box\)

**Appendix**

**Proof of Lemma 1.4** We set
\[
\alpha := i_w \circ \gamma : w \circ f^1 \Rightarrow w \circ f^2.
\]
Then condition (BF4a) is obviously satisfied by the set of data:
\[
C, \quad \nu := \text{id}_C, \quad \beta := \pi_f^{-1} \circ \gamma \circ \pi_f, \quad f^1 \circ \nu \Rightarrow f^2 \circ \nu
\]
(here \(\pi_f\) is the unit \(f^1 \circ \text{id}_C \Rightarrow f^1\), and analogously for \(\pi_f\)). Since we have also \(\alpha = i_w \circ \gamma'\), then (BF4a) is also satisfied by:
\[
C, \quad \nu' := \text{id}_C, \quad \beta' := \pi_f^{-1} \circ \gamma' \circ \pi_f, \quad f^1 \circ \nu' \Rightarrow f^2 \circ \nu.
\]

Then by (BF4c) there are an object \(D\), a pair of morphisms \(u, u' : D \rightarrow C\) and an invertible 2-morphism \(\zeta : \text{id}_C \circ u \Rightarrow \text{id}_C \circ u'\), such that \(\text{id}_C \circ u \text{ belongs to } W\) and
\[
\theta_{f^0, \text{id}_C, u} \circ \left( (\pi_f^{-1} \circ \gamma \circ \pi_f) \circ i_u \right) \circ \theta_{f^1, \text{id}_C, u} \circ (i_f \circ \zeta) = \\
\left( i_{f^2} \circ \zeta \right) \circ \theta_{f^2, \text{id}_C, u} \circ \left( (\pi_f^{-1} \circ \gamma \circ \pi_f) \circ i_u \right) \circ \theta_{f^1, \text{id}_C, u}.
\]
Using the coherence axioms on the bicategory $C$, this implies that $\gamma \ast i_u = \gamma' \ast i_u$. Moreover, using $\nu^{-1}_u : u \Rightarrow \text{id}_C \ast u$ and $\text{BF}3$, we get that $u$ belongs to $W$.

**Proof of Lemma 5.1.** We apply condition $\text{BF}3$ to the pair of morphisms $(z \circ f, z \circ w)$, so we get an object $E$, a morphism $t : E \to C$ in $W$, a morphism $g : E \to A$ and an invertible 2-morphism $\beta : (z \circ f) \circ t \Rightarrow (z \circ w) \circ g$. Then we apply $\text{BF}1\alpha$ and $\text{BF}1\beta$ to the invertible 2-morphism

$$\theta_{z,w,g}^{-1} \circ \beta \circ \theta_{z,f,t} : z \circ (f \circ t) \Rightarrow z \circ (w \circ g).$$

So there are an object $D$, a morphism $r : D \to E$ in $W$ and an invertible 2-morphism $\gamma : (f \circ r) \circ \gamma \Rightarrow (w \circ r) \circ \gamma$, such that

$$\left( \theta_{z,w,g}^{-1} \circ \beta \circ \theta_{z,f,t} \right) \ast i_r = \theta_{z,w \circ g, r} \circ \left( i_z \ast \gamma \right) \circ \theta_{z, \circ f, t}^{-1}$$

(6.10)

Then we set $w' := t \circ r : D \to C$; this morphism belongs to $W$ by construction and $\text{BF}2$. Moreover, we define $f' := g \circ r : D \to A$ and

$$\alpha := \theta_{w,g,t}^{-1} \circ \gamma \circ \theta_{f,t,r} : f \circ w' \Rightarrow w \circ f'.$$

**Proof of Lemma 5.3.** We use $\text{BF}1\alpha$ on the 2-morphism

$$\theta_{z,w,f} \circ \left( i_z \ast \alpha \right) \circ \theta_{z,w,f}^{-1} : (z \circ w) \circ f \Rightarrow (z \circ w) \circ f^2.$$

Then there are an object $E$, a morphism $t : E \to C$ in $W$ and a 2-morphism $\gamma : f_1 \circ t \Rightarrow f_2 \circ t$, such that

$$\left( \theta_{z,w,f} \circ \left( i_z \ast \alpha \right) \circ \theta_{z,w,f}^{-1} \right) \ast i_t = \theta_{z \circ w, f_2, t} \circ \left( i_z \ast \gamma \right) \circ \theta_{z \circ w, f_1, t}^{-1}.$$ 

This implies that

$$i_z \ast \left( \theta_{w,f_2,t} \circ \left( i_w \ast \gamma \right) \circ \theta_{w,f_1,t}^{-1} \right) = i_z \ast \left( \alpha \ast i_t \right).$$

So by Lemma 1.1 there are an object $D$ and a morphism $r : D \to E$ in $W$, such that:

$$\left( \theta_{w,f_2,t} \circ \left( i_w \ast \gamma \right) \circ \theta_{w,f_1,t}^{-1} \right) \ast i_r = \left( \alpha \ast i_t \right) \ast i_r.$$ 

(6.11)

Then we define $v := t \circ r : D \to C$; this morphism belongs to $W$ by construction and $\text{BF}2$. Moreover, we set

$$\beta := \theta_{f_2,t,r} \circ \left( \gamma \ast i_t \right) \circ \theta_{f_1,t,r} : f_1 \circ v \Rightarrow f_2 \circ v.$$

Then from (6.11) we get easily that $\alpha \ast i_v = \theta_{w,f_2,v} \circ \left( i_w \ast \beta \right) \circ \theta_{w,f_1,v}^{-1}$. Moreover, if $\alpha$ is invertible, then by $\text{BF}4\beta$ so is $\gamma$, hence so is $\beta$.

**Proof of Corollary 5.4.** For simplicity of exposition, let us suppose that $C$ has trivial associators. If this is not the case, the proof follows the same lines, adding associators wherever it is necessary. Let us denote as in (6.1) the fixed set of choices $\mathcal{C}(W)$ for the triple $f, g, h$ so that we have identities (0.5) and (0.6). Moreover, let us suppose that the fixed choices $\mathcal{C}(W)$ give data as in the upper parts of the following diagrams (starting from the ones on the left), with $\nu^1, \nu^2, \nu^3$ and $\nu^4$ in $W$ and $\delta, \sigma, \xi$ and $\pi$ invertible:
so that by [Pr, § 2.2] one has

\[
\begin{align*}
    \left( h' \circ (g' \circ f') \right) &= \left( A \xrightarrow{u \circ \text{id}_{A'}} A' \xrightarrow{\text{id} \circ \text{id}_{A'}} A^2 \xrightarrow{h \circ \text{id}_{A'}} D \right), \\
    \left( (h' \circ g') \circ f' \right) &= \left( A \xrightarrow{u \circ \text{id}_{A'}} A' \xrightarrow{\text{id} \circ \text{id}_{A'}} A^2 \xrightarrow{\text{id} \circ \text{id}_{A'}} D \right).
\end{align*}
\] (6.12)

(6.13)

Now using [BF3], we get a set of data as in the upper part of the following diagram, with \( r^1 \) in \( W \) and \( \zeta^1 \) invertible:

Using [BF4a] and [BF4b], there are an object \( \tilde{A}^2 \), a morphism \( r^3 : \tilde{A}^2 \to \tilde{A}^1 \) in \( W \) and an invertible 2-morphism

\[ \varepsilon^1 : f^1 \circ u^2 \circ r^1 \circ r^3 \Rightarrow \tilde{f}^1 \circ \tilde{u}^2 \circ r^2 \circ r^3, \]

such that \( i_v \circ \varepsilon^1 \) coincides with the following composition:

This implies that \( \delta^{-1} \circ i_u \circ r^1 \circ r^3 \) coincides with the following composition:
Now we use again (BF4a) and (BF4b) in order to get an object $\tilde{A}^3$, a morphism $r^4: \tilde{A}^3 \to \tilde{A}^2$ in $W$ and an invertible 2-morphism $\varepsilon^2: l \circ r^1 \circ r^3 \circ r^4 \Rightarrow l \circ r^2 \circ r^3 \circ r^4$,

such that $i_w * \varepsilon^2$ coincides with the following composition:

$$\varepsilon^2: l \circ r^1 \circ r^3 \circ r^4 \Rightarrow l \circ r^2 \circ r^3 \circ r^4,$$

Therefore, $\sigma^{-1} * i_r \circ r^3 \circ r^4$ coincides with the following composition:

$$\sigma^{-1} * i_r \circ r^3 \circ r^4 = \sigma^{-1} * i_r \circ r^3 \circ r^4$$

Now we use (BF3) in order to get data as in the upper part of the following diagram, with $r^5$ in $W$ and $\zeta^2$ invertible:

Using (BF4a) and (BF4b), there are an object $\tilde{A}^4$, a morphism $r^7: \tilde{A}^3 \to \tilde{A}^4$ in $W$ and an invertible 2-morphism $\varepsilon^3: v^1 \circ f^2 \circ r^3 \circ r^7 \Rightarrow v^1 \circ f^2 \circ r^3 \circ r^7$,

such that $i_v * \varepsilon^3$ coincides with the following composition:
Therefore, \( \eta \ast i_{v} \circ r_{7} \) coincides with the following composition:

\[
\begin{align*}
\tilde{A}^5 & \xrightarrow{r_{7}} \tilde{A}^4 \\
\tilde{A}^4 & \xrightarrow{r_{6}} \tilde{A}^3 \\
\tilde{A}^3 & \xrightarrow{\zeta^2} \tilde{A}^2 \\
\tilde{A}^2 & \xrightarrow{r_{5}} \tilde{A}^1 \\
\tilde{A}^1 & \xrightarrow{\eta^{-1}} A' \\
A' & \xrightarrow{\eta} B' \\
B' & \xrightarrow{\pi_{v}} B.
\end{align*}
\]

Now we use (BF3) in order to get data as in the upper part of the following diagram, with \( r_{8} \) in \( W \) and \( \zeta^3 \) invertible:

\[
\begin{align*}
\tilde{A}^3 & \xrightarrow{r_{8}} \tilde{A}^4 \\
\tilde{A}^4 & \xrightarrow{r_{7}} \tilde{A}^5 \\
\tilde{A}^5 & \xrightarrow{\zeta^3} \tilde{A}^6 \\
\tilde{A}^6 & \xrightarrow{r_{9}} \tilde{A}^7 \\
\tilde{A}^7 & \xrightarrow{\eta_{v} \ast \epsilon^4} \tilde{A}^8 \\
\tilde{A}^8 & \xrightarrow{r_{10}} \tilde{A}^9 \\
\tilde{A}^9 & \xrightarrow{r_{11}} \tilde{A}^{10} \\
\tilde{A}^{10} & \xrightarrow{\epsilon^4} \tilde{A}^{11} \\
\tilde{A}^{11} & \xrightarrow{r_{12}} \tilde{A}^{12}.
\end{align*}
\]

Then we use (BF3a) and (BF3b) in order to get an object \( \tilde{A}^{7} \), a morphism \( r_{10} : \tilde{A}^{7} \to \tilde{A}^{6} \) in \( W \) and an invertible 2-morphism

\[
\epsilon^4 : \tilde{f} \circ \eta^2 \circ r_{2} \circ r_{3} \circ r_{4} \circ r_{10} \Rightarrow \eta_{v} \circ \eta_{v} \circ r_{7} \circ r_{8} \circ r_{10}^{11},
\]

such that \( i_{v} \ast \epsilon^4 \) coincides with the following composition:

\[
\begin{align*}
\tilde{A}^{2} & \xrightarrow{r_{3}} \tilde{A}^{3} \\
\tilde{A}^{3} & \xrightarrow{\eta_{v} \ast \epsilon^2} \tilde{A}^{4} \\
\tilde{A}^{4} & \xrightarrow{r_{6}} \tilde{A}^{5} \\
\tilde{A}^{5} & \xrightarrow{r_{7}} \tilde{A}^{6} \\
\tilde{A}^{6} & \xrightarrow{r_{8}} \tilde{A}^{7} \\
\tilde{A}^{7} & \xrightarrow{r_{9}} \tilde{A}^{8} \\
\tilde{A}^{8} & \xrightarrow{r_{10}} \tilde{A}^{9} \\
\tilde{A}^{9} & \xrightarrow{r_{11}} \tilde{A}^{10} \\
\tilde{A}^{10} & \xrightarrow{\epsilon^4} \tilde{A}^{11} \\
\tilde{A}^{11} & \xrightarrow{r_{12}} \tilde{A}^{12}.
\end{align*}
\]

Now we use (BF3a) and (BF3b) in order to get an object \( \tilde{A}^{8} \), a morphism \( r_{11} : \tilde{A}^{8} \to \tilde{A}^{7} \) in \( W \) and an invertible 2-morphism.
Moreover, we define 
\varepsilon^5 : g^1 \circ f^2 \circ r^5 \circ r^7 \circ r^9 \circ r^{10} \circ r^{11} \Rightarrow f^1 \circ f^2 \circ r^6 \circ r^7 \circ r^9 \circ r^{10} \circ r^{11},
such that \(i_w \ast \varepsilon^5\) coincides with the following composition:

Therefore, \(\xi \ast i f^2 \circ r^5 \circ r^7 \circ r^9 \circ r^{10} \circ r^{11}\) coincides with the following composition:

Lastly, we use again \(\textbf{B} \Rightarrow \textbf{a}\) and \(\textbf{B} \Rightarrow \textbf{b}\) in order to get an object \(A^4\), a morphism 
\(r^{12} : A^4 \to \tilde{A}^8\) in \(\mathbf{W}\) and an invertible 2-morphism
\(\rho : l \circ r^1 \circ r^3 \circ r^4 \circ r^8 \circ r^{10} \circ r^{11} \circ r^{12} \Rightarrow g^1 \circ f^2 \circ r^5 \circ r^7 \circ r^9 \circ r^{10} \circ r^{11} \circ r^{12},\)
such that \(i_w \ast \rho\) coincides with the following composition:

Then we define the following set of morphisms and 2-morphisms:

\(u^4 := r^1 \circ r^3 \circ r^4 \circ r^8 \circ r^{10} \circ r^{11} \circ r^{12} : A^4 \to A^2,\)
\(u^5 := r^5 \circ r^7 \circ r^9 \circ r^{10} \circ r^{11} \circ r^{12} : A^4 \to A^3,\)
\(\gamma := \xi^3 \ast i_{f^2 \circ r^5 \circ r^7 \circ r^9 \circ r^{10} \circ r^{11} \circ r^{12}} : u^1 \circ u^2 \circ u^4 \Rightarrow u^3 \circ u^5.\)
Moreover, we define \(\omega : f^1 \circ u^2 \circ u^4 \Rightarrow v^1 \circ f^2 \circ u^5\) as the following composition:
Clearly condition (F1) is satisfied by this set of data; we claim that also (F2) and (F3) hold. In order to prove (F2), we replace in (0.7) the definition of $\gamma$ above and the expressions for $\delta^{-1} \ast i_{r^2 \circ r^3}$ and $\eta \ast i_{r^2 \circ r^7}$ obtained in diagrams (6.14) and (6.18). After simplifying some terms of the form $\pi_f$, we get that the composition in (0.7) coincides with the following one:

Using (6.13) and (6.21), we get that (6.22) (hence (0.7)) coincides with $i_v \ast \omega$. This implies that (F2) holds. Now we prove (F3), so in (0.8) we replace $\omega$ with (0.21), $\sigma^{-1} \ast i_{r^2 \circ r^3}$ with (6.15), and $\xi \ast i_{f^2 \circ r^5 \circ r^9 \circ r^{10} \circ r^{12}}$ with (6.19). After simplifying $\pi_g$, $\varepsilon^1$ and $\varepsilon^3$, we get that the composition in (0.8) coincides with (6.20), hence with $i_w \ast \rho$, hence (F3) holds. So using Proposition 0.1 we conclude that the associator $\Theta^{C_{W\mu}}_{\rho_{\rho_{\rho}}}$ is represented by the following diagram:

Now we want to compute also the associator appearing in (5.13). For that, we define the following set of data:

$$\pi^2 := r^2 \circ r^3 \circ r^4 \circ r^8 \circ r^{10} \circ r^{11} \circ r^{12} : A^4 \rightarrow A^2,$$
Moreover, we define \( \pi : \pi^3 \circ \pi^4 \Rightarrow \pi^3 \circ \pi^4 \) as the following composition:

\[
\begin{array}{c}
\pi^5 := r^6 \circ r^7 \circ r^9 \circ r^{10} \circ r^{11} \circ r^{12} : A^4 \rightarrow A^4,
\end{array}
\]

and \( \pi : I \circ \pi^4 \Rightarrow \pi^3 \circ \pi^4 \) as the following composition:

\[
(6.24)
\]

Using interchange law, the definition of \( \pi \) and (6.17), we get

\[
(6.25)
\]

Then we replace (6.13) in (6.26) and we simplify the terms of the form \( \pi \). Using (6.24), we get that \( \pi \circ \text{id}_{A^4} \ast \pi \) coincides with the following composition:

\[
(6.27)
\]

This proves that conditions (F1) and (F2) for the computation of the associator \( \Theta \) are satisfied by the set of data

\[
(6.28)
\]

Then we need only to prove that condition (F3) for the associator mentioned above is satisfied by (6.25). In other terms, we have to prove that \( \pi \circ \text{id}_{A^4} \ast \pi \) coincides with the following composition:
Now by interchange law and (6.25), we have

\[
\pi_1 \circ \rho = (\pi_{i_2} - 1 \ast i_1) \circ (i_{w} \ast \rho) \circ (\pi_{i_2} - 1 \ast i_1) \circ (\pi_{w} \ast \rho) \circ (\pi_{i_2} - 1 \ast i_1).
\]

Then we replace \(i_{w} \ast \rho\) above with diagram (6.20) and we simplify \(\pi_{i_2} - 1\) and all the terms of the form \(\pi_{w}\). Then we get exactly diagram (6.29), so (F3) holds. Therefore, by Proposition 0.1, the associator \(\Theta_{C, W, h'}\) is represented by the following diagram:

\[
\begin{array}{cccc}
A & \xleftarrow{\pi_1} & A & \\
\downarrow \pi_1 & & \downarrow \pi_1 & \\
A & \xleftarrow{\pi_1} & B
\end{array}
\]

(6.30)

So until now we have computed the central term of diagram (5.13). Using Propositions 0.2 and 0.4 it is not difficult to prove that the 2-morphism \(\chi(h) \ast (\chi(g) \ast \chi(f))\) appearing in (5.13) is represented by the following diagram:

\[
\begin{array}{cccc}
A & \xleftarrow{\pi_1} & A & \\
\downarrow \pi_1 & & \downarrow \pi_1 & \\
A & \xleftarrow{\pi_1} & D
\end{array}
\]

\[
\begin{array}{cccc}
A & \xleftarrow{\pi_1} & A & \\
\downarrow \pi_1 & & \downarrow \pi_1 & \\
A & \xleftarrow{\pi_1} & D
\end{array}
\]

(6.31)

where \(\alpha^1\) is the following composition:

\[
A \xleftarrow{\pi_1} \xrightarrow{\rho} A \xrightarrow{\pi_1} \xrightarrow{\rho} A
\]

\[
\begin{array}{cccc}
A & \xleftarrow{\pi_1} & A & \\
\downarrow \pi_1 & & \downarrow \pi_1 & \\
A & \xleftarrow{\pi_1} & D
\end{array}
\]

(6.31)
and $\beta^1$ is the following composition:

$$A^4 \xrightarrow{r^8 \circ r^{10} \circ r^{11} \circ r^{12}} A^3 \xrightarrow{\tau_{r^4} \circ r^3 \circ r^4} C' \xrightarrow{\pi_{h^{-1}}} D.$$  

Moreover, using again the same propositions, the composition $(\chi(h)^{-1} \ast \chi(g)^{-1}) \ast \chi(f)^{-1}$ appearing in (5.13) is represented by the following diagram

where $\alpha^2$ is the following composition:

$$A^4 \xrightarrow{r^7 \circ r^5 \circ r^{10} \circ r^{12}} A^1 \xrightarrow{(\zeta^2)^{-1}} A' \xrightarrow{\pi u} A$$

and $\beta^2$ is the following composition:

$$A^1 \xrightarrow{r^{12}} A^0 \xrightarrow{g^1 \circ f^2 \circ a^5 \circ r^7 \circ r^9 \circ r^{10} \circ r^{11}} C' \xrightarrow{\pi h} D.$$  

Now we compute (5.13): for that we use Proposition 0.2 in order to compose vertically (6.30), (6.31) and (6.32). Using the definition of $\gamma$ in (6.24) and that of $\rho$ in (6.25), in such a composition we can simplify the terms of the form $\pi u, \pi h, \zeta^1, \zeta^2, \varepsilon^2$ and $\varepsilon^5$, hence we get exactly diagram (6.23). So we conclude. □

**Proof of Lemma 6.1.** Let us choose any representative of $\Gamma$ as follows

$$w^1 \circ u^1 \text{ in } W \text{ and } \delta \text{ invertible. Since } w^1 \text{ belongs to } W \text{ by hypothesis, then by Lemma 1.2 there are data as in the upper part of the following diagram, with } t^1 \text{ in } W \text{ and } \sigma \text{ invertible:}$$

with $w^1 \circ u^1$ in $W$ and $\delta$ invertible. Since $w^1$ belongs to $W$ by hypothesis, then by Lemma 1.2 there are data as in the upper part of the following diagram, with $t^1$ in $W$ and $\sigma$ invertible:
Using (BF5) on $\alpha^{-1}$, we get that $w^2 \circ v^2$ belongs to $W$. By (BF2), this implies that also $w^2 \circ v^2 \circ t^1$ belongs to $W$. Moreover, $w^2$ belongs to $W$ by hypothesis. So again by Lemma 1.2 there are data as in the upper part of the following diagram, with $s^2$ in $W$ and $\phi$ invertible:

Now we consider the following invertible 2-morphism:

$$
\mu := \left( i_{w^2} \ast \phi \right) \ast \left( \delta \ast i_{t^2 \circ s^2} \right) \ast \left( i_{w^1} \ast \sigma \ast i_{s^2} \right) \ast \left( \alpha^{-1} \ast i_{t^1 \circ s^2} \right) : w^2 \circ v^2 \circ t^1 \circ s^2 \Rightarrow w^2 \circ v^2 \circ t^1 \circ s^1.
$$

Since $w^2 \circ v^2 \circ t^1$ belongs to $W$, then by (BF4a) and (BF4b) there are an object $A^4$, a morphism $r : A_4 \to C'$ in $W$ and an invertible 2-morphism $\nu : s^2 \circ r \Rightarrow s^1 \circ r$, such that $\mu \ast i_r = i_{w^2 \circ v^2 \circ t^1} \ast \nu$. We set $z := t^1 \circ s^2 \circ r : A^4 \to A^3$; this morphism belongs to $W$ by (BF2). By definition of $\mu$, this implies that

$$
\left( i_{w^2} \ast \left( \phi^{-1} \ast i_r \right) \ast \left( i_{v^2 \circ t^1} \ast \nu \right) \right) \ast \left( \alpha \ast i_s \right) \ast \left( \sigma^{-1} \ast i_{s^2 \circ r} \right) = \delta \ast i_{t^2 \circ s^2 \circ r}.
$$

Then we define:

$$
\gamma := \left( i_{f^2 \circ v^2 \circ t^1} \ast \nu^{-1} \right) \ast \left( i_{f^2} \ast \phi \ast i_r \right) \ast \left( \eta \ast i_{t^2 \circ s^2 \circ r} \right) \ast \left( i_{f^1} \ast \sigma \ast i_{s^2 \circ r} \right) : f^1 \circ v^1 \circ z \Rightarrow f^2 \circ v^2 \circ z.
$$

This implies that

$$
\left( i_{f^2} \ast \left( \phi^{-1} \ast i_r \right) \ast \left( i_{v^2 \circ t^1} \ast \nu \right) \right) \ast \gamma \ast \left( i_{f^1} \ast \left( \sigma^{-1} \ast i_{s^2 \circ r} \right) \right) = \eta \ast i_{t^2 \circ s^2 \circ r}.
$$

Lastly, we consider the following diagram
Then using (6.33), (6.34) and the previous diagram and comparing with § 1.1, we conclude that

\[ \Gamma = \left[ C, u^1, u^2, \delta, \eta \right] = \left[ A^4, v^1 \circ z, v^2 \circ z, \alpha \star iz, \gamma \right]. \]

References


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