Irreducible decomposition for local representations of quantum Teichmüller space

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Abstract

We give an irreducible decomposition of the so-called local representations \cite{HL07} of the quantum Teichmüller space $T_q(\Sigma)$ where $\Sigma$ is a punctured surface of genus $g > 0$ and $q$ is a $N$-th root of unity with $N$ odd.

Let $\Sigma$ be an oriented surface of genus $g > 0$ with $s$ punctures $v_1, ..., v_s$ such that $2g - 2 + s > 0$ (this condition is equivalent to the existence of an ideal triangulation of $\Sigma$, i.e. a triangulation whose vertices are exactly the $v_i$). Let $T(\Sigma)$ be the Teichmüller space of $\Sigma$, that is the moduli space of complete hyperbolic metrics on $\Sigma$. Given $\lambda$ an ideal triangulation of $\Sigma$, W.P. Thurston \cite{Thu98} constructed a parameterization of $T(\Sigma)$ by associating a strictly positive real number to each edge $\lambda_i$ of the ideal triangulation, $i \in \{1, ..., n\}$ (where $n = 6g - 6 + 3s$ is the number of edges of $\lambda$). These coordinates are called shear coordinates associated to $\lambda$. In this coordinates system, the coefficients of the Weil-Petersson form on $T(\Sigma)$ depend only on the combinatoric of $\lambda$ and are easy to compute.

For a parameter $q \in \mathbb{C}^*$, L.O. Chekhov, V.V. Fock \cite{FC99} and independently R. Kashaev \cite{Kas98} defined the so-called quantum Teichmüller space $T_q(\Sigma)$ of $\Sigma$ (the construction of R. Kashaev differs a little from the one of L.O. Chekhov and V.V. Fock), which is a deformation of the Poisson algebra of rational functions over $T(\Sigma)$. This algebraic object is obtained by gluing together a collection of non-commutative algebra $T_q(\lambda)$ (called Chekhov-Fock algebra) canonically associated to each ideal triangulation of $\Sigma$. A representation of $T_q(\Sigma)$ is then a family of representation $\{\rho_\lambda : T_q(\lambda) \to \text{End}(V)\}_{\lambda \in \Lambda(\Sigma)}$, where $\Lambda(\Sigma)$ is the space of all ideal triangulations of $\Sigma$, and $\rho_\lambda$ and $\rho_{\lambda'}$ satisfy compatibility conditions whenever $\lambda \neq \lambda'$. For $\lambda \in \Lambda(\Sigma)$, the representation $\rho_\lambda$ is an avatar of the representation of $T_q(\Sigma)$ and carries almost all the information.

When $q$ is a $N$-th root of unity, $T_q(\lambda)$ admits finite-dimensional representations. In this paper, we will consider $N$ odd. The irreducible representations of $T_q(\lambda)$ have been studied in \cite{BL07}. In particular, they show that an irreducible representation of $T_q(\lambda)$ is classified (up to isomorphism) by a weight $x_i \in \mathbb{C}^*$ assigned to each edge $\lambda_i$, a choice of $N$-th root $p_j = (x_1^{k_{j1}} ... x_n^{k_{jn}})^{1/N}$ associated to each puncture $v_j$ (where $k_{ji}$ is the number of times a small simple loop around
defines a representation of $S^N_{\text{even}}$ and find a subspace product of triangle algebras $\rho$. Here, \( T \) is a local representation of \( T \) with local representations $\rho$ and $\hat{T}_{\omega}(\lambda)$ of each edge $\lambda_i$ and a choice of $N$-th root $c = (x_1...x_n)^{1/N}$. Such a representation has dimension $N^{4g-4+2s}$.

It follows that a local representation of $T_{\omega}(\lambda)$ is not irreducible. In this paper, we address the question of the decomposition of a local representation into its irreducible components. In particular, we prove the following result:

**Theorem 1.** Let $\lambda$ be an ideal triangulation of $\Sigma$ and $\rho$ be a local representation of $T_{\omega}(\lambda)$ classified by weight $x_j \in \mathbb{C}^*$ associated to each edge $\lambda_j$ and a choice of $N$-th root $c = (x_1...x_n)^{1/N}$. We have the following decomposition:

$$\rho = \bigoplus_{i \in I} \rho^{(i)}.$$

Here, $\rho^{(i)}$ is an irreducible representation classified by the same $x_j$, a $N$-th root $p_j^{(i)} = (x_1^{k_{j,1}}...x_n^{k_{j,n}})^{1/N}$ associated to each puncture, and the same $c$. Moreover, for each choice of $N$-th root $p_j = (x_1^{k_{j,1}}...x_n^{k_{j,n}})^{1/N}$ for each puncture and $c = (p_0...p_s)^{1/2}$, there exists exactly $N^g$ elements $i \in I$ with $p_j^{(i)} = p_j$ for all $j \in \{0,...,s\}$.

This result may be used to define representations of the so-called **Kauffman skein algebra** $S^A(\Sigma)$ [Tur91] (where $\Sigma$ is the surface $\Sigma$ without marked points) which corresponds to a quantization by deformation of the character variety

$$\mathcal{R} := \text{Hom}(\pi_1(\Sigma), PSL(2, \mathbb{C})) \parallel PSL(2, \mathbb{C}))$$

where $PSL(2, \mathbb{C})$ acts by conjugation on the morphisms and the double slash means that we take the quotient in the sense of Geometric Invariant Theory. In [BW11, Theorem 1], the authors constructed a morphism

$$\text{Tr}_{\omega}(\lambda) : S_A(\Sigma) \rightarrow \mathcal{Z}_{\omega}(\lambda),$$

where $q = \omega^4$, $A = \omega^{-2}$ and $\mathcal{Z}_{\omega}(\lambda)$ is an algebra of non-commutative rational fractions such that $T_{\omega}(\lambda)$ consists of rational fractions in $\mathcal{Z}_{\omega}(\lambda)$ involving only even powers of the variables. This morphism, composed with a representation $\rho$ of $T_{\omega}(\lambda)$ is studied in [BW12a] and [BW12b] to define a new kind of representations of $S^A(\Sigma)$. However, if one wants to define representation of $S^A(\Sigma)$ in the same way, one has to consider the direct sum of $N^g$ irreducible components of a local representation $\rho : T_{\omega}(\lambda) \rightarrow \text{End}(V)$ arising in the decomposition of Theorem 1 and find a subspace $E \subset V$ stable by $\rho \circ \text{Tr}_{\omega}(\lambda)$ such that $(\rho \circ \text{Tr}_{\omega}(\lambda))|_E$ defines a representation of $S^A(\Sigma)$ (see [BW11] for the construction). Hopefully, this representation of $S^A(\Sigma)$ should be used to define a more intrinsic version of
the Kashaev-Baseilhac-Benedetti TQFT (see [Kas95], [Kas99], [BB04], [BB05] and [BB07]).

In the first section, we recall the definition of the Chekhov-Fock algebra, the quantum Teichmüller space, the triangle algebra and the local representations. In the second one, we prove the Theorem 1. The proof is done in two steps: we first prove the result for a special triangulation \( \lambda_0 \) and special weights; we then extend to the general case.

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1 The Chekhov-Fock algebra and the irreducible representations of \( \mathcal{T}_q(\Sigma) \)

The results of this section come from [BL07] and [HBL07]. From now, for \( n \in \mathbb{N} \), define \( \mathbb{N}_n := \mathbb{Z}/n\mathbb{Z} \) and denote by \( \mathcal{U}(N) \) the group of \( N \)-th root of unity.

1.1 The Chekhov-Fock algebra

Let \( \lambda \) be an ideal triangulation of \( \Sigma \). The **Chekhov-Fock algebra** \( \mathcal{T}_q(\lambda) \) associated to \( \lambda \) is the algebra generated by the elements \( X_i^\pm 1 \) associated to each edge \( \lambda_i \) of the triangulation \( \lambda \). These elements are subjects to the relations:

\[
X_i X_j = q^{\sigma_{ij}} X_j X_i,
\]

where the coefficients \( \sigma_{ij} \) are the coefficients of the Weil-Petersson form in the shear coordinates associated to \( \lambda \) and depend only on the combinatoric of \( \lambda \). Namely, we have \( \sigma_{ij} = a_{ij} - a_{ji} \) where \( a_{ij} \) is the number of angular sector delimited by \( \lambda_i \) and \( \lambda_j \) in the faces of \( \lambda \) with \( \lambda_i \) coming before \( \lambda_j \) counterclockwise.

In practice, elements of \( \mathcal{T}_q(\lambda) \) are just Laurent polynomials in the variables \( X_i \) satisfying non-commutativity conditions. We will sometimes denote \( \mathcal{T}_q(\lambda) \) by \( \mathbb{C}[X_1^\pm 1, ..., X_n^\pm 1]_q \) to reflect this fact.

Let \( \mathbf{k} = (k_1, ..., k_n) \in \mathbb{Z}^n \) be a multi-index; to a monomial \( X \) composed of a product of \( X_i^{k_i} \), we associate its quantum ordering:

\[
[X] := q^{-\sum_{i<j} \sigma_{ij} k_i k_j} X_1^{k_1} ... X_n^{k_n}.
\]

It allows us to associate a monomial \( X_{\mathbf{k}} \in \mathcal{T}_q(\lambda) \) to each each multi-index \( \mathbf{k} \in \mathbb{Z}^n \).

To study finite-dimensional representations of \( \mathcal{T}_q(\lambda) \), one needs to determine its center.

**Proposition 1** ([BL07], Proposition 15). The center of \( \mathcal{T}_q(\lambda) \) is generated by:

- \( X_i^N \) for each \( i \in \{1, ..., n\} \).

- For each puncture \( v_j \), the **puncture invariant** \( P_j \) associated to the multi-index \( \mathbf{k}_j = (k_{j_1}, ..., k_{j_n}) \) (where \( k_{ji} \) is the number of intersections of \( \lambda_i \) with a small simple loop around \( v_j \)).
The element $H$ associated to the multi index $k = (1, \ldots, 1)$.

Note that $[P_1 \ldots P_s] = H^2$.

1.2 Triangle algebra

Let $T$ be a disk with three punctures $v_1, v_2, v_3 \in \partial T$ endowed with the natural triangulation $\lambda$ composed of three counterclockwise directed edges $\lambda_1, \lambda_2$ and $\lambda_3$ (as in Figure 1).

![Figure 1: The triangle $T$](image)

Define the triangle algebra as the Chekhov-Fock algebra $T := T_q(\lambda)$. It is generated by $X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}$ with relations $X_i X_{i+1} = q^2 X_{i+1} X_i$ for all $i \in \mathbb{N}_3$. The center of $T$ is given by $X_1^N, X_2^N, X_3^N$ and $H = q^{-1} X_1 X_2 X_3$.

Irreducible finite dimensional representations of $T$ have dimension $N$ and are classified (up to isomorphism) by a choice of weight $x_i \in \mathbb{C}^*$ associated to each edge $\lambda_i$ and a central charge, that is a choice of a $N$-th root $c = (x_1 x_2 x_3)^{1/N}$ (see [HBL07, Lemma 2]).

To be more precise, for $V$ the $N$-dimensional complex vector space generated by $\{e_1, \ldots, e_N\}$ and $\rho$ an irreducible representation of $T$ classified by $x_1, x_2, x_3 \in \mathbb{C}^*$ and $c = (x_1 x_2 x_3)^{1/N}$. Up to isomorphism, the action of $T$ on $V$ defined by $\rho$ is given by:

$$
\begin{align*}
X_1 e_i &= \tilde{x}_1 q^{2i} e_i \\
X_2 e_i &= \tilde{x}_2 e_{i+1} \\
X_3 e_i &= \tilde{x}_3 q^{1-2i} e_{i-1}
\end{align*}
$$

where $\tilde{x}_i$ is an $N$-th root of $x_i$ such that $\tilde{x}_1 \tilde{x}_2 \tilde{x}_3 = c$. Note that, up to isomorphism, $\rho$ is independent of the choice of the $N$-th root $\tilde{x}_i$ with $\tilde{x}_1 \tilde{x}_2 \tilde{x}_3 = c$.

In particular, for the representation $\rho$ classified by $x_1 = x_2 = x_3 = 1$ and $c \in \mathcal{U}(N)$, as $\rho(X_i)^N = Id_V$, the spectrum of $\rho(X_i)$ is a subset of $\mathcal{U}(N)$. For $h \in \mathcal{U}(N)$, denote by $V_h(X_i)$ the eigenspace of $\rho(X_i)$ associated to the eigenvalue $h$. We have the following lemma which will be useful in the next section:

**Lemma 1.** For each $i \in \{1, 2, 3\}$ and $h \in \mathcal{U}(N)$, $\dim(V_h(X_i)) = 1$. 

Proof. We use the explicit form of the representation \( \rho \) in \( V = \text{span}\{e_1, ..., e_N\} \). Take \( \tilde{x}_1 = \tilde{x}_2 = 1 \) and \( \tilde{x}_3 = c \).

For \( i = 1 \), one sees that \( V_\lambda(X_1) = \text{span}\{e_k\} \) where \( h = q^{2k} \).

For \( i = 2 \), the vector \( \alpha_k := \sum_{i \in \mathbb{N}_k} q^{-2ki} e_i \) satisfies \( X_2 \alpha_k = q^{2k} \alpha_k \) and \( \{\alpha_1, ..., \alpha_k\} \) form a basis of \( V \). Then \( V_\lambda(X_2) \) is generated by \( \alpha_k \).

For \( i = 3 \), we use the fact that \( X_1 X_2 X_3 e_i = ce_i \), where \( c \), the central charge of \( \rho \), lies in \( \mathcal{U}(N) \).

\[ \square \]

1.3 Local representation of \( T_q(\lambda) \)

Let \( \lambda \) be an ideal triangulation of \( \Sigma \). Such a triangulation is composed of \( m \) faces \( T_1, ..., T_m \) and each face \( T_j \) determines a triangle algebra \( T_j \) whose generators are associated to the three edges of \( T_j \). It provides a canonical embedding \( i \) of \( T_q(\lambda) \) into \( T_1 \otimes ... \otimes T_m \) defined on the generators as follow:

- \( i(X_i) = X_{ji} \otimes X_{ki} \) if \( \lambda_i \) belongs to two distinct triangles \( T_j \) and \( T_k \) and \( X_{ji} \in T_j, X_{ki} \in T_k \) are the generators associated to the edge \( \lambda_i \in T_j \) and \( \lambda_i \in T_k \) respectively.

- \( i(X_i) = [X_{ji1} X_{ji2}] \) if \( \lambda_i \) corresponds to two sides of the same face \( T_j \) and \( X_{ji1}, X_{ji2} \in T_j \) are the associated generators.

Now, a local representation of \( T_q(\lambda) \) is a representation which factorizes as \( (\rho_1 \otimes ... \otimes \rho_m) \circ i \) where \( \rho_i : T_i \to V_i \) is an irreducible representation of the triangle algebra \( T_i \). In particular, such a representation has dimension \( N^m \) where \( m = 4g - 4 + 2s \) is the number of faces of the triangulation.

1.4 Classification of these representations

Here we recall [BL07, Theorem 21] and [HBL07, Proposition 6] respectively:

**Theorem 2.** (F. Bonahon, X. Liu) An irreducible representation of \( T_q(\lambda) \) is determined by its restriction to the center of \( T_q(\lambda) \) and is classified by a non-zero complex number \( x_i \) associated to each edges \( \lambda_i \), for each puncture \( v_j \), a choice of a \( N \)-th root \( p_j = (x_1^{k_1} x_2^{k_2} ... x_n^{k_n})^{1/N} \) and a choice of a square root \( c = (p_0 p_1 ...)^{1/2} \).

Such a representation satisfies:

- \( \rho(X_i^N) = x_i \text{Id} \)
- \( \rho(P_j) = p_j \text{Id} \)
- \( \rho(H) = c \text{Id} \)

**Theorem 3.** (H. Bai, F. Bonahon, X. Liu) Up to isomorphism, a local representation of \( T_q(\lambda) \) is classified by a non-zero complex number \( x_i \) associated to the edge \( \lambda_i \) and a choice of a \( N \)-th root \( c = (x_1 ... x_n)^{1/N} \). Such a representation satisfies:

- \( \rho(X_i^N) = x_i \text{Id} \)
- \( \rho(H) = c \text{Id} \)
1.5 The quantum Teichmüller spaces and its representations

If one wants to quantize the Teichmüller space, he has to do it in a canonical way. The definition of the Chekhov-Fock algebra \( T_q(\lambda) \) involves the choice of an ideal triangulation. So we have to understand the behavior when one changes from an ideal triangulation \( \lambda \) to another one \( \lambda' \). Set \( T_q(\lambda) = \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]_q \) and \( T_q(\lambda') = \mathbb{C}[X_1'^{\pm 1}, ..., X_n'^{\pm 1}]_q \). These algebras admit a division algebras, denoted by \( \hat{T}_q(\lambda) \) and \( \hat{T}_q(\lambda') \) respectively, consisting of rational fractions in the variables \( X_i \) (respectively \( X_i' \)) satisfying some non-commutativity relations.

For each pair of ideal triangulation \( \lambda \) and \( \lambda' \), L.O. Chekhov and V.V. Fock constructed coordinates change isomorphisms

\[
\Psi_{\lambda\lambda'}^q : \hat{T}_q(\lambda') \rightarrow \hat{T}_q(\lambda),
\]

which are the unique isomorphism satisfying naturals conditions (as for example \( \Psi_{\lambda\lambda'}^q = \Psi_{\lambda'\lambda'}^q \circ \Psi_{\lambda\lambda'}^q \) for each \( \lambda, \lambda' \) and \( \lambda'' \) ideal triangulations of \( \Sigma \)). See [Liu09] for more details and explicit formulae of \( \Psi_{\lambda\lambda'}^q \).

Now, the quantum Teichmüller space \( T_q(\Sigma) \) is defined by:

\[
T_q(\Sigma) := \bigsqcup_{\lambda \in \Lambda(\Sigma)} \hat{T}_q(\lambda) / \sim,
\]

where \( \Lambda(\Sigma) \) is the set of ideal triangulation of \( \Sigma \), and the equivalence relation \( \sim \) identifies each pair of \( T_q(\lambda) \) and \( T_q(\lambda') \) by the isomorphism \( \Psi_{\lambda\lambda'}^q \). Note that, as each coordinates change \( \Psi_{\lambda\lambda'}^q \) is an algebra isomorphism, \( T_q(\Sigma) \) inherits an algebra structure, and the \( \hat{T}_q(\lambda) \) can be thought as “global coordinates” on \( T_q(\Sigma) \).

A natural definition for a finite dimensional representation of \( T_q(\Sigma) \) would be a family of finite dimensional representation \( \{ \rho_\lambda : T_q(\lambda) \rightarrow \text{End}(V_\lambda) \}_{\lambda \in \Lambda(\Sigma)} \) such that for each pair of ideal triangulation \( \lambda \) and \( \lambda' \), \( \rho_\lambda' \) is isomorphic to \( \rho_\lambda \circ \Psi_{\lambda\lambda'}^q \). Note that, as pointed out in [HBL07] Section 4.2], there exists no algebra homomorphism \( \rho_\lambda : \hat{T}_q(\lambda) \rightarrow \text{End}(V_\lambda) \) for \( V_\lambda \) finite dimensional. In fact, as \( \hat{T}_q(\lambda) \) is infinite dimensional as a vector space and \( \text{End}(V_\lambda) \) is finite dimensional, such a homomorphism \( \rho_\lambda \) would have non-zero kernel. Hence, there would exists elements \( x \in \hat{T}_q(\lambda) \) such that \( \rho_\lambda(x) = 0 \) and so, \( \rho_\lambda(x^{-1}) \) would make no sense.

So one defines a local representation (respectively irreducible representation) of \( T_q(\Sigma) \) as a family of representation \( \{ \rho_\lambda : T_q(\lambda) \rightarrow \text{End}(V_\lambda) \}_{\lambda \in \Lambda(\Sigma)} \) such that for each \( \lambda, \lambda' \in \Lambda(\Sigma) \), \( \rho_\lambda \) is a local representation (respectively irreducible representation) of \( T_q(\lambda) \), and \( \rho_\lambda' \) is isomorphic (as representation) to \( \rho_\lambda \circ \Psi_{\lambda\lambda'}^q \) whenever \( \rho_\lambda \circ \Psi_{\lambda\lambda'}^q \) makes sense. We say that \( \rho_\lambda \circ \Psi_{\lambda\lambda'}^q \) makes sense, if for each Laurent polynomial \( X' \in T_q(\lambda') \), there exists \( P, P', Q \) and \( Q' \in T_q(\lambda) \) such that:

\[
\Psi_{\lambda\lambda'}(X') = PQ^{-1} = Q'^{-1}P' \in \hat{T}_q(\lambda);
\]

now, as \( \rho_\lambda(T_q(\lambda)) \subset \text{GL}(V_\lambda) \), \( \rho_\lambda(Q) \) and \( \rho_\lambda(Q') \) are invertibles, so we can define:

\[
\rho_\lambda \circ \Psi_{\lambda\lambda'}(X') := \rho_\lambda(P) \rho_\lambda(Q)^{-1} = \rho_\lambda(Q')^{-1} \rho_\lambda(P') \cdot
\]
A fundamental result in [BL07] and [HBL07, Proposition 10] is that for each pair of ideal triangulations $\lambda$ and $\lambda'$, there exists a rational map

$$\varphi_{\lambda\lambda'} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

such that a local representation $\rho_{\lambda'}$ of $T_q(\lambda')$ classified by $x'_i \in \mathbb{C}^*$ associated to $\lambda'$ and $c' = (x'_1 \cdots x'_n)^{1/N}$ is isomorphic to $\rho_\lambda \circ \Psi_{\lambda\lambda'}$ (whenever it makes sense) for a representation $\rho_\lambda$ of $T_q(\lambda)$ classified by $x_i \in \mathbb{C}^*$ associated to $\lambda_i$ and $c = (x_1 \cdots x_n)^{1/N}$ if and only if $c = c'$ and

$$(x'_1, \ldots, x'_n) = \varphi_{\lambda\lambda'}(x_1, \ldots, x_n).$$

2 Proof of Theorem 1

2.1 Special case

Here we prove Theorem 1 for a local representation $\rho : T_q(\lambda_0) \longrightarrow \text{End}(V)$ where $\lambda_0$ is special triangulation of $\Sigma$ and $\rho$ is classified by weights $x_1 = 1$ and $c \in U(N)$. Here, $\Sigma$ is a genus $g > 0$ surface with $s + 1$ punctures $v_0, \ldots, v_s$. Recall that $m = 4g - 4 + 2(s + 1)$ and $n = 6g - 6 + 3(s + 1)$ are respectively the number of faces and edges of $\lambda_0$. Moreover, we denote by $Xu$ the action of $X \in T_q(\lambda_0)$ on $u \in V$ defined by $\rho$.

To decompose $\rho$ into irreducible factors, one has to look at the eigenspaces of $\rho(P_j)$ for each puncture invariant $P_j$ associated to the puncture $v_j$. Note that, as $\rho(P_j)^N = Id$, the spectrum of $P_j$ is contained in $U(N)$.

The idea of the proof is to look at the action of the $P_j$ on each factor of a nice decomposition of $V$ into a tensorial product of vector spaces. It is based on the following remark:

**Remark 1.** For a decomposition $V = E_1 \otimes E_2$, if $x_j \in E_j$ satisfies $Px_j = h_jx_j$ for $j = 1, 2$ where $P \in \{P_0, \ldots, P_s\}$ and $h_j \in U(N)$, then $P(x_1 \otimes x_2) = h_1h_2x_1 \otimes x_2$. That is, the eigenspace of $P$ in $V$ associated to the eigenvalue $h \in U(N)$ contains the tensorial product of eigenspaces of $P$ in $E_j$ associated to the eigenvalues $h_j$, for $j = 1, 2$, whenever $h = h_1h_2$.

For $h = (h_1, \ldots, h_s) \in U(N)^s$, set

$$V_h := \{u \in V, \ P_iu = h_iu, \ i = 1, \ldots, s\}.$$ 

**Proposition 2.** For each $h \in U(N)^s$, $\dim V_h = N^{m-s}$.

**Proof.** Take an ideal triangulation $\tilde{\lambda}$ of $\Sigma \setminus \{v_1, \ldots, v_s\}$ (which is a one punctured surface), and for a triangle $T$ of $\tilde{\lambda}$, consider the triangulation of $T \cup \{v_1, \ldots, v_s\}$ as in Picture 2.

The union of these two triangulations gives an ideal triangulation $\lambda_0$ of $\Sigma$. Denote by $\tilde{V}$ the tensorial product of all the vector spaces associated to the triangles of $\tilde{\lambda} \setminus T$. As the triangulation $\tilde{\lambda}$ contains $3g - 1$ triangles, $\dim(\tilde{V}) = N^{3g-2}$ (because we do not consider the vector space associated to $T$). Denote by $V^j$ and $V^k$ the $j$th (resp. $k$th) vector space associated to the triangle $T_j$ (resp. $T_k$) as in Figure 2 (here, $j \in \{0, \ldots, s\}$ and $k \in \{1, \ldots, s\}$).

For $h = (h_1, \ldots, h_s) \in U(N)^s$ and $j \in \{1, \ldots, s\}$, define:
We have the following lemma:

Lemma 2.

i. \( \dim \mathcal{V}_h^0 = \begin{cases} 1 & \text{if } h_k = 1 \forall k \neq 1 \\ 0 & \text{otherwise.} \end{cases} \)

ii. \( \forall j \in \{1, \ldots, s-1\} \quad \dim \mathcal{V}_h^j = \begin{cases} 1 & \text{if } h_k = 1 \forall k \notin \{j, j+1\} \\ 0 & \text{otherwise.} \end{cases} \)

iii. \( \dim \mathcal{V}_h^s = \begin{cases} N & \text{if } h_k = 1 \forall k \neq s \\ 0 & \text{otherwise.} \end{cases} \)

Proof.  

i. If \( k \neq 1, v_k \) is not a vertex of \( T_0 \). It follows that \( P_k \) acts on \( V^0 \) by the identity; so if \( h_k \neq 1, \mathcal{V}_h^0 = \{0\} \).

Now, if \( h_k = 1 \) for all \( k \neq 1 \), then \( \mathcal{V}_h^0 = V_{h_1}(P_1) \) (as defined in Lemma 1) which is one dimensional.

ii. Fix \( j \in \{1, \ldots, s-1\} \). For \( k \notin \{j, j+1\}, v_k \) is neither a vertex of \( T_j \) nor of \( T'_j \). So \( P_j \) acts on \( V^j \otimes V'^j \) as the identity. Hence, if \( h_k \neq 1 \), then \( \mathcal{V}_h^j = \{0\} \).

Take \( h_k = 1 \) for all \( k \notin \{j, j+1\} \) and denote by \( T_j = \mathbb{C}[X^\pm, Y^\pm, Z^\pm]_q, \) \( T'_j = \mathbb{C}[X'^\pm, Y'^\pm, Z'^\pm]_q \) the triangle algebras associated to the triangles \( T_j \) and \( T'_j \) respectively (as in Figure 2).
Figure 3: The generators of $T_j$ and $T'_j$

For $c_j, c'_j \in \mathcal{U}(N)$ the central charges of the restriction of the representation to $T_j$ and $T'_j$ respectively, $P_j$ acts on $V^j := \text{span}\{\epsilon_0, ..., \epsilon_{N-1}\}$ like $c_j Z^{-1}$, on $V'^j = \text{span}\{\epsilon'_0, ..., \epsilon'_{N-1}\}$ like $c'_j Z^{-1}$ and $P_{j+1}$ acts on $V_j$ like $c_j Y^{-1}$, on $V'_j$ like $c'_j Y'^{-1}$. Set $c_j = q^p$ and $c'_j = q'^{p'}$, we get the following:

$$P_j \epsilon_k = q^{2k-1+p} \epsilon_{k+1}$$
$$P_j \epsilon'_l = q^{1-2l+p'} \epsilon_{l+1}$$

It follows that the action of $P_j$ on $V^j \otimes V'^j$ is given by:

$$P_j \epsilon_{k,l} = q^{2(k-1)+p+p'} \epsilon_{k+1,l+1}$$

where $\epsilon_{k,l} := \epsilon_k \otimes \epsilon'_l$.

In the same way, one sees that the action of $P_{j+1}$ on $V^j \otimes V'^j$ is given by:

$$P_{j+1} \epsilon_{k,l} = q^{p+p'} \epsilon_{k-1,l-1}$$

Now, for $m, n \in \mathbb{N}$, set $\alpha_{m,n} := \sum_{k=0}^{N-1} q^{2km} \epsilon_{k,k+n}$, an easy calculation shows that:

$$\left\{ \begin{array}{l} P_j \alpha_{m,n} = q^{-2(m+n)+p+p'} \alpha_{m,n} \\ P_{j+1} \alpha_{m,n} = q^{2m+p+p'} \alpha_{m,n}. \end{array} \right.$$  

It follows that $\{\alpha_{m,n}, n, m \in \mathbb{N}\}$ is a base of $V^j \otimes V'^j$ and, for all $h_j, h_{j+1} \in \mathcal{U}(N)$, there exists a unique couple $(m, n) \in \mathbb{N}_N^2$ with $h_j = q^{-2(m+n)+p+p'}$ and $h_{j+1} = q^{2m+p+p'}$. So $\dim V^j_h = 1$ if and only if $h_k = 1$ for all $k \notin \{j, j+1\}$. 

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iii. If \( k \neq s \), \( v_k \) is neither a vertex of \( T_s \) nor \( T_s' \), so if \( h_k \neq 1 \), \( V^s_{h_k} = \{0\} \).

Suppose that \( h_k = 1 \) for all \( k \in \{1, \ldots, s-1\} \), then

\[
V^s_h \supset \bigoplus_{h_a h_b = h_s} V^s_{h_a}(P_s) \otimes V^r_{h_b}(P_s),
\]

(where \( V^s_{h_a}(P_s) \) and \( V^r_{h_b}(P_s) \) are defined as in Lemma I). The direct sum contains \( N \) terms of dimension one, hence \( \dim V^s_h \geq N \). But, we have

\[
\dim(V^s \otimes V^r) = N^2 = \sum_{h \in U(N)^s} \dim(V^s_h) \geq N \times N.
\]

So \( V^s_h \) is \( N \)-dimensional.

Now, the proof of Proposition [2] is straightforward: from Remark I we have

\[
\bigoplus_{h^0 h^1 \cdots h^s = h} V^0_{h^0} \otimes \cdots \otimes V^s_{h^s} \otimes \tilde{V} \subset V_h.
\]

Writing \( h^j = (h^j_1, \ldots, h^j_s) \) and \( h = (h_1, \ldots, h_s) \), one notes that the only non-zero terms in the direct sum are those who satisfy:

\[
\begin{align*}
  h_0^1 h_1^1 &= h_1 \\
  h_1^1 h_2^1 &= h_2 \\
  &\vdots \\
  h_1^{s-1} h_s^s &= h_s
\end{align*}
\]

There exists exactly \( N^s \) different choices for \( h^0, \ldots, h^s \in U(N)^s \) satisfying the above relations, and each non-zero vector space of the direct sum has dimension \( N^{m-2s} \). So \( \dim V_h \geq N^{m-s} \). Now, we have

\[
\dim V = N^m = \sum_{h \in U(N)^s} \dim V_h \geq N^s \times N^{m-s},
\]

and so \( \dim V_h = N^{m-s} \) for each \( h \in U(N) \). \( \square \)

In particular, it proves the decomposition of Theorem II for \( \rho \). In fact, let \( \rho^{(i)} : T_\rho(\mathcal{L}_0) \rightarrow \text{End}(V^{(i)}) \) be an irreducible representation in the decomposition of \( \rho \). It must satisfies \( \rho^{(i)}(X_i)^N = \text{Id}_{V^{(i)}} \) and \( \rho^{(i)}(H) = c \text{Id}_{V^{(i)}} \), in other word, \( \rho^{(i)} \) must be associated to the same weights \( x_i = 1 \) and global charge \( c \in U(N) \) than \( \rho \).

Set \( h_j^{(i)} \in U(N) \) the weight of \( \rho^{(i)} \) associated to the each puncture \( v_j \), that is, \( \rho^{(i)}(P_j) = h_j^{(i)} \text{Id}_{V^{(i)}} \). Note that, as \( \rho^{(i)}([P_0 \ldots P_s]) = \rho^{(i)}([H^2]) = h_0^{(i)} h_1^{(i)} \ldots h_s^{(i)} \text{Id}_{V^{(i)}} = c^2 \text{Id}_{V^{(i)}} \), a necessary condition for \( \rho^{(i)} \) to be in the decomposition of \( \rho \) is to satisfy \( h_0^{(i)} \ldots h_s^{(i)} = c^2 \). Hence, if \( \rho^{(i)} \) is in the decomposition of \( \rho \), knowing \( h_j^{(i)} \) for each \( j = 1, \ldots, s \) uniquely determine \( h_0^{(i)} \) and so fully determine \( \rho^{(i)} \).
Now, as for each \( h = (h_1, \ldots, h_s) \in \mathcal{U}(N)^s \), \( V_h \) has dimension \( N^{m-s} = N^{2g-3s+(s+1)} \) and as an irreducible representation of \( T_q(\lambda_0) \) has dimension \( N^{3g-3+s(s+1)} \), then each space \( V_h \) contains exactly \( N^g \) times the representation \( \rho^{(i)} \), classified by \( p_0 = c^2 h_1^{-1} \ldots h_s^{-1}, \ p_1 = h_1, \ldots, p_s = h_s \).

2.2 Proof in the global case

Now, to complete the proof of Theorem \( \square \) one remarks that the decomposition of \( \rho \) into irreducible factors only depends on the decomposition of \( \rho(P_j) \) into eigenspaces (for each puncture \( v_j \)), that is on the possible choices of \( N \)-th root of \( x_1^{k_1} \ldots x_n^{k_n} \) (where \( P_j \) is associated to the multi-index \( k_j = (k_{j_1}, \ldots, k_{j_n}) \)). But this choice is discrete and depends continuously on the weights \( x_i \) associated to the edge \( \lambda_i \), hence does not depend on the choice of \( x_i \in \mathbb{C}^* \). It proves Theorem \( \square \) for the triangulation \( \lambda_0 \) and every weight \( x_i \in \mathbb{C}^* \).

Note that the map \( \varphi_{\lambda_0\lambda} \) defined in Subsection \( \square \) is rational, hence defined on a Zariski dense open set of \( \mathbb{C}^n \). As we extended the decomposition for all weights \( x_i \) associated to each edge of the triangulation \( \lambda_0 \), there exists a local representation \( \{ \rho_\lambda : T_q(\lambda) \to \text{End}(V_\lambda) \}_{\lambda \in \Lambda(\Sigma)} \) of \( T_q(\Sigma) \) as defined in Subsection \( \square \). So, for each \( \lambda \in \Lambda(\Sigma) \), \( \rho_{\lambda_0} \circ \Psi_{\lambda_0\lambda}^q : T_q(\lambda) \to \text{End}(V_{\lambda_0}) \) makes sense and is isomorphic to \( \rho_\lambda : T_q(\lambda) \to \text{End}(V_\lambda) \). That is, there exists a vector space isomorphism \( L_{\lambda_0\lambda} : V_\lambda \to V_{\lambda_0} \) such that, for each \( X \in T_q(\lambda) \),

\[
\rho_{\lambda_0}(\Psi_{\lambda_0\lambda}^q(X)) = L_{\lambda_0\lambda} \circ \rho_\lambda(X) \circ L_{\lambda_0\lambda}^{-1}.
\]

However, \( \rho_{\lambda_0} \) is a local representation of \( T_q(\lambda_0) \), hence there exists an irreducible decomposition of \( \rho_{\lambda_0} \) given by the decomposition \( V_{\lambda_0} = \bigoplus_{i \in I} V_{\lambda_0}^i \) as in Theorem \( \square \). That is, for each \( i \in I \), \( V_{\lambda_0}^i \) is stable by \( \rho_{\lambda_0} \) and has dimension \( N^{3g-3+s(s+1)} \).

Using the isomorphism \( \Psi_{\lambda_0\lambda} \), one gets that for each \( X \in T_q(\lambda) \), \( \rho_{\lambda_0}(\Psi_{\lambda_0\lambda}(X))V_{\lambda_0}^i = V_{\lambda_0}^i \). Set \( V_\lambda^i := L_{\lambda_0\lambda}^{-1}(V_{\lambda_0}^i) \), we have \( \dim V_\lambda^i = \dim V_{\lambda_0}^i = 3g - 3 + s + 1 \) (because \( L_{\lambda_0\lambda} \) is an isomorphism) so for each \( X \in T_q(\lambda) \), \( \rho_\lambda(X)V_\lambda^i = V_\lambda^i \). In other words, we have a decomposition

\[
\rho_\lambda = \bigoplus_{i \in I} \rho_\lambda^{(i)},
\]

where \( \rho_\lambda^{(i)} : T_q(\lambda) \to \text{End}(V_\lambda^i) \). As each \( V_\lambda^i \) as the dimension of an irreducible representation, we get an irreducible decomposition of \( \rho_\lambda \). One easily checks that it satisfies the conditions of Theorem \( \square \). Now we extend this decomposition by continuity for all weight \( x_i \in \mathbb{C}^* \) associated to the edge \( \lambda_i \) of \( \lambda \).

References

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