CURVATURE-DIMENSION INEQUALITIES ON SUB-RIEMANNIAN MANIFOLDS OBTAINED FROM RIEMANNIAN FOLIATIONS, PART II

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ABSTRACT. Using the curvature-dimension inequality proved in Part I, we look at consequences of this inequality in terms of the interaction between the sub-Riemannian geometry and the heat semigroup P_t corresponding to the sub-Laplacian. We give bounds for the gradient, entropy, a Poincaré inequality and a Li-Yau type inequality. These results require that the gradient of $P_t f$ remains uniformly bounded whenever the gradient of f is bounded and we give several sufficient conditions for this to hold.

1. Introduction

One of the most important relations connecting the geometric properties of a Riemannian manifold $(M, \mathbf{g})$ with the properties of its Laplace operator Δ is the curvature-dimension inequality given by

$$\frac{1}{2}\Delta\|\operatorname{grad} f\|^2_{\mathbf{g}} - \langle \operatorname{grad} f, \operatorname{grad} \Delta f\rangle_{\mathbf{g}} \geq \frac{1}{n}(\Delta f)^2 + \rho\|\operatorname{grad} f\|^2_{\mathbf{g}}.$$

In the above formula, $n = \dim M$, ρ is a lower bound for the Ricci curvature of M and f is any smooth function. In the notation of Bakry and Émery [5], this inequality is written as

$$\mathsf{\Gamma}_2(f) \geq \frac{1}{n}(Lf)^2 + \rho\mathsf{\Gamma}(f), \qquad L = \Delta,$$

where

$$\text{(1.1)} \qquad \mathsf{\Gamma}(f,g) = \frac{1}{2}\left(L(fg) - fLg - gLf\right), \qquad \mathsf{\Gamma}(f) = \mathsf{\Gamma}(f,f),$$

$$\text{(1.2)} \qquad \mathsf{\Gamma}_2(f,g) = \frac{1}{2}\left(L\mathsf{\Gamma}(f,g) - \mathsf{\Gamma}(f,Lg) - \mathsf{\Gamma}(Lf,g)\right), \qquad \mathsf{\Gamma}_2(f) = \mathsf{\Gamma}_2(f,f).$$

For a good overview of results that follow from this inequality, see [20] and references therein.

This approach has been generalized by F. Baudoin and N. Garofalo in [8] to sub-Riemannian manifolds with transverse symmetries. A sub-Riemannian manifold is a connected manifold M with a positive definite metric tensor $\mathbf{h}$ defined only on a subbundle $\mathcal{H}$ of the tangent bundle TM. As is typical, we will assume that sections of $\mathcal{H}$ and their iterated Lie brackets span the entire tangent bundle. This is a sufficient condition for the sub-Riemannian structure $(\mathcal{H}, \mathbf{h})$ to give us a metric d_{cc} on M, where the distance between two points with respect to d_{cc} is defined

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by taking the infimum of the lengths of all curves tangent to $\mathcal{H}$ that connect the mentioned points. For the definition of sub-Riemannian manifolds with transverse symmetries, see [8, Section 2.3] or Part I, Section 4.3. We extended this formalism in Part I to sub-Riemannian manifold with an integrable metric-preserving complement, consisting of all sub-Riemannian manifolds that can be obtained from Riemannian foliations.

Given such a metric-preserving complement $\mathcal{V}$ to $\mathcal{H}$, there exist a canonical corresponding choice of second order operator $\Delta'_{\mathbf{h}}$ which locally satisfies

$$\Delta'_{\mathbf{h}} = \sum_{i=1}^{n} A_i^2 + \text{lower order terms.}$$

with $A_1, \dots, A_n$ being a local orthonormal basis of $\mathcal{H}$. We proved in Part I that under mild conditions, there exist constants $n, \rho_1, \rho_{2,0}$ and $\rho_{2,1}$ such that the operator satisfies a generalized version of the curvature-dimension inequality

$$\mathsf{\Gamma}_2(f) + \ell \mathsf{\Gamma}_2^{\mathbf{v}^*}(f) \geq \frac{1}{n}(Lf) + (\rho_1 - \ell^{-1})\mathsf{\Gamma}(f) + (\rho_{2,0} + \ell\rho_{2,1})\mathsf{\Gamma}^{\mathbf{v}^*}(f),$$

for any $f \in C^\infty(M)$ and $\ell > 0$. Here, $\mathsf{\Gamma}(f)$ and $\mathsf{\Gamma}_2(f)$ is defined as in (1.1) and (1.2) with $L = \Delta'_{\mathbf{h}}$, while $\mathsf{\Gamma}^{\mathbf{v}^*}(f) = \mathbf{v}^*(df, df)$ for some $\mathbf{v}^* \in \Gamma(\mathrm{Sym}^2 TM)$ and $\mathsf{\Gamma}_2^{\mathbf{v}^*}(f)$ is defined analogously to $\mathsf{\Gamma}_2(f)$. We also gave a geometrical interpretation of these constants. A short summary of the results of Part I is given in Section 2.

In this paper, we want to explore how this inequality can be used to obtain results for the heat semigroup of $\Delta'_{\mathbf{h}}$. In Section 3, we will address the question of whether a smooth bounded function with bounded gradient under the action of the heat semigroup will continue to have a uniformly bounded gradient. This will be an important condition for the results to follow. For a complete Riemannian manifold, a sufficient condition for this to hold is that the Ricci curvature is bounded from below, see e.g. [22] and [19, Eq 1.4]. We are not able to give such a simple formulation for the sub-Laplacian, however, we are able to prove that it holds in many cases, including fiber bundles with compact fibers and totally geodesic fibers. This was only previously only known to hold for sub-Riemannian manifolds with transverse symmetries of Yang-Mills type [8, Theorem 4.3], along with some isolated examples in [21, Section 4] and [10, Appendix]. We give several results using the curvature-dimension inequality of Part I that only rely on the boundedness of the gradient under the heat flow. Our results generalize theorems found in [8, 6, 7]. In particular, if $\Delta'_{\mathbf{h}}$ is a sub-Laplacian on $(M, \mathcal{H}, \mathbf{h})$ satisfying our generalized curvature-dimension inequality, then under certain conditions (analogous to positive Ricci curvature in Riemannian geometry) we have the following version of the Poincaré inequality

$$\|f - f_M\|_{L^2(M,\mathrm{vol})} \leq \frac{1}{\sqrt{\alpha}} \|df\|_{L^2(\mathbf{h}^*)}.$$

Here, α is a positive constant, $\mathbf{h}^*$ is the co-metric of $(\mathcal{H}, \mathbf{h})$, f_M is the mean value of a compactly supported function f and for any $\eta \in \Gamma(T^*M)$ we use

$$\|\eta\|_{L^2(\mathbf{h}^*)} := \int_M \mathbf{h}^*(\eta, \eta)\, d\mathrm{vol}\,.$$

In Section 4 we look at results which require the additional assumption that $\mathsf{\Gamma}^{\mathbf{v}^*}(f, \mathsf{\Gamma}(f)) = \mathsf{\Gamma}(f, \mathsf{\Gamma}^{\mathbf{v}^*}(f))$. This is important for inequalities involving logarithms.

We give a description of what this condition means geometrically and discuss results that follow from it, such as a Li-Yau type inequality and parabolic Harnack inequality.

In Section 5 we give some concrete examples, mostly focused on case of sub-Riemannian structures appearing from totally geodesic foliations with a complete metric. Here, all previously mentioned assumptions are satisfied. In this case, we also give a comment on how the invariants in our sub-Riemannian curvature-dimension inequality compare to the Riemannian curvature of an extended metric.

In parallel with the development of our paper, part of the results of Theorem 3.4 and Lemma 4.1 was given in [9] for the case of sub-Riemannian obtained from Riemannian foliations with totally geodesic leaves that are of Yang-Mills type.

1.1. **Notations and conventions.** Unless otherwise stated, all manifolds are connected. If $\mathcal{E} \to M$ is any vector bundle over a manifold M, its space of smooth sections is written $\Gamma(\mathcal{E})$. If $s \in \Gamma(\mathcal{E})$, we generally prefer to write $s|_x$ rather than $s(x)$ for its value in $x \in M$. By a metric tensor $\mathbf{s}$ on $\mathcal{E}$, we mean smooth section of $\operatorname{Sym}^2 \mathcal{E}^*$ which is positive definite or at least positive semi-definite. For every such metric tensor, we write $\|e\|_{\mathbf{s}} = \sqrt{\mathbf{s}(e,e)}$ for any $e \in \mathcal{E}$ even if $\mathbf{s}$ is only positive semi-definite. All metric tensors are denoted by bold, lower case Latin letters (e.g. $\mathbf{h}, \mathbf{g}, \dots$). We will only use the term Riemannian metric for a positive definite metric tensor on the tangent bundle. If $\mathbf{g}$ is a Riemannian metric, we will use $\mathbf{g}^*$, $\wedge^k \mathbf{g}^*, \dots$ for the metric tensors induced on $T^*M, \bigwedge^k T^*M, \dots$.

If α is a form on a manifold M, its contraction or interior product by a vector field A will be denoted by either $\iota_A \alpha$ or $\alpha(A, \cdot)$. We use $\mathcal{L}_A$ for the Lie derivative with respect to A. If M is furnished with a Riemannian metric $\mathbf{g}$, any bilinear tensor $\mathbf{s} : TM \otimes TM \to \mathbb{R}$ can be identified with an endomorphism of TM using $\mathbf{g}$. We use the notation $\operatorname{tr} \mathbf{s}(\times, \times)$ for the trace of this corresponding endomorphism, with the metric being implicit. If $\mathcal{H}$ is a subbundle of TM, we will also use the notation $\operatorname{tr}_{\mathcal{H}} \mathbf{s}(\times,\times) := \operatorname{tr} \mathbf{s}(\operatorname{pr}_{\mathcal{H}} \times, \operatorname{pr}_{\mathcal{H}} \times)$, where $\operatorname{pr}_{\mathcal{H}}$ is the orthogonal projection to $\mathcal{H}$.

2. Summary of Part I

In this section, we briefly recall the most important definitions and results from Part I.

2.1. **Sub-Riemannian manifolds.** A sub-Riemannian manifold is a triple $(M, \mathcal{H}, \mathbf{h})$ where M is a connected manifold and $\mathbf{h}$ is a positive definite metric tensor defined only on the subbundle $\mathcal{H}$ of TM. Equivalently, it can be considered as a manifold with a positive semi-definite co-metric $\mathbf{h}^*$ that is degenerate along a subbundle of T^*M. This latter mentioned subbundle will be $\operatorname{Ann}(\mathcal{H})$, the annihilator of $\mathcal{H}$, that consist of all covectors vanishing on $\mathcal{H}$. Define $\sharp^{\mathbf{h}^*} : p \mapsto \mathbf{h}^*(p, \cdot) \in \mathcal{H} \subseteq TM$. We will assume that the subbundle $\mathcal{H}$ is bracket-generating, i.e. its sections and their iterated brackets span the entire tangent bundle. Then we have a well defined metric d_{cc} on M by taking the infimum over the length of curves that are tangent to $\mathcal{H}$.

2.2. **Two notions of sub-Laplacian.** Let vol be any smooth volume form on M. We then define the sub-Laplacian relative to the volume form vol as $\Delta_{\mathbf{h}} f = \operatorname{div} \sharp^{\mathbf{h}^*} df$, where the divergence is defined relative to vol. From the definition, it is clear that $\Delta_{\mathbf{h}}$ is symmetric relative the measure vol, i.e. $\int_M f \Delta_{\mathbf{h}} g \, d\mathrm{vol} = \int_M g \Delta_{\mathbf{h}} f \, d\mathrm{vol}$ for any $f, g \in C_c^\infty(M)$ of compact support.

We also introduced the concept of a sub-Laplacian defined relative to a complement $\mathcal{V}$ of $\mathcal{H}$. Let $\mathbf{g}$ be any Riemannian metric satisfying $\mathbf{g}|_{\mathcal{H}} = \mathbf{h}$ and let $\mathcal{V}$ be the orthogonal complement of $\mathcal{H}$. Consider the following connection,

$$\begin{aligned} \mathring{\nabla}_A Z = &\operatorname{pr}_{\mathcal{H}} \nabla_{\operatorname{pr}_{\mathcal{H}} A} \operatorname{pr}_{\mathcal{H}} Z + \operatorname{pr}_{\mathcal{V}} \nabla_{\operatorname{pr}_{\mathcal{V}} A} \operatorname{pr}_{\mathcal{V}} Z \\ &+ \operatorname{pr}_{\mathcal{H}} [\operatorname{pr}_{\mathcal{V}} A, \operatorname{pr}_{\mathcal{H}} Z] + \operatorname{pr}_{\mathcal{V}} [\operatorname{pr}_{\mathcal{H}} A, \operatorname{pr}_{\mathcal{V}} Z], \end{aligned} \tag{2.1}$$

where ∇ is the Levi-Civita connection of $\mathbf{g}$. We define the sub-Laplacian of $\mathcal{V}$ as

$$\Delta'_{\mathbf{h}} = \operatorname{tr}_{\mathcal{H}} \mathring{\nabla}^2_{\times,\times} f.$$

It is simple to verify that this definition is independent of $\mathbf{g}|_{\mathcal{V}}$, it only depends on $\mathbf{h}$ and the splitting $TM = \mathcal{H} \oplus \mathcal{V}$.

Remark 2.1. If $\mathcal{V}$ is the vertical bundle of a submersion $\pi : M \to B$ into a Riemannian manifold $(B, \breve{\mathbf{g}})$ and if $\mathbf{h}$ is a sub-Riemannian metric defined by pulling back $\breve{\mathbf{g}}$ to an Ehresmann connection $\mathcal{H}$ on π, then the sub-Laplacian $\Delta'_{\mathbf{h}}$ of $\mathcal{V}$ satisfies

$$\Delta'_{\mathbf{h}}(f \circ \pi) = (\breve{\Delta} f) \circ \pi,$$

where $\breve{\Delta}$ is the Laplacian of $\breve{\mathbf{g}}$ and $f \in C^\infty(B)$.

2.3. **Metric-preserving complement.** A subbundle $\mathcal{V}$ is integrable if $[\Gamma(\mathcal{V}), \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})$. By the Frobenius Theorem, such a subbundle gives us a foliation on M. We say that an integrable complement $\mathcal{V}$ of $\mathcal{H}$ is metric-preserving if

$$\mathcal{L}_V \operatorname{pr}_{\mathcal{H}}^* \mathbf{h} = 0, \quad \text{for any } V \in \Gamma(\mathcal{V}),$$

where $\operatorname{pr}_{\mathcal{H}}$ is the projection corresponding to the choice of complement $\mathcal{V}$. Let $\mathbf{g}$ be any Riemannian metric such that $\mathbf{g}|_{\mathcal{H}} = \mathbf{h}$ and $\mathcal{H}^\perp = \mathcal{V}$. If we define $\mathring{\nabla}$ as in (2.1), then $\mathcal{V}$ is metric preserving if and only if $\mathring{\nabla}\mathbf{h}^* = 0$. The foliation of $\mathcal{V}$ is then called a Riemannian foliation.

2.4. **Generalized curvature-dimension inequality.** For a given smooth second order differential operator L without constant term and for any section $\mathbf{s}^*$ of $\operatorname{Sym}^2 TM$, define

$$\begin{aligned} &\Gamma^{\mathbf{s}^*}(f,g) = \mathbf{s}^*(df, dg), && \Gamma^{\mathbf{s}^*}(f,f) = \Gamma^{\mathbf{s}^*}(f), \\ &\Gamma_2^{\mathbf{s}^*}(f,g) = \frac{1}{2}\left(L\Gamma^{\mathbf{s}^*}(f,g) - \Gamma(Lf, g) - \Gamma^{\mathbf{s}^*}(f, Lg)\right), && \Gamma_2^{\mathbf{s}^*}(f,f) = \Gamma_2^{\mathbf{s}^*}(f). \end{aligned}$$

Assume that

$$\frac{1}{2}\left(L(fg) - fLg - gLf\right) = \mathbf{h}^*(df, dg).$$

for some positive semi-definite section $\mathbf{h}^*$ of $\operatorname{Sym}^2 TM$. We say that L satisfies the generalized curvature-dimension inequality (CD*) if there is another positive semi-definite section $\mathbf{v}^*$ of $\operatorname{Sym}^2 TM$, a positive number $0 < n \leq \infty$ and real numbers $\rho_1, \rho_{2,0}$ and $\rho_{2,1}$ such that for any $\ell > 0$ and $f \in C^\infty(M)$,

$$\Gamma_2^{\mathbf{h}^* + \ell \mathbf{v}^*}(f) \geq \frac{1}{n}(Lf)^2 + (\rho_1 - \ell^{-1})\Gamma^{\mathbf{h}^*}(f) + (\rho_{2,0} + \ell\rho_{2,1})\Gamma^{\mathbf{v}^*}(f). \tag{CD*}$$

Let $(M, \mathcal{H}, \mathbf{h})$ be a sub-Riemannian manifold with $\mathcal{H}$ being a bracket-generating subbundle of TM. Assume that we have an integrable metric-preserving complement $\mathcal{V}$ and let $\mathbf{g}$ be a Riemannian metric such that $\mathcal{H}$ and $\mathcal{V}$ are orthogonal, with

$\mathbf{h} = \mathbf{g}\,|_{\mathcal{H}}$ and $\mathbf{v} := \mathbf{g}\,|_{\mathcal{V}}$. Let $\mathbf{h}^*$ and $\mathbf{v}^*$ be their respective co-metrics. Relative to these structures, we make the following assumptions.

(i) We define the curvature of $\mathcal{H}$ relative to the complement $\mathcal{V}$ as the vector valued 2-form

$$\mathcal{R}(A,Z) = \operatorname{pr}_{\mathcal{V}}[\operatorname{pr}_{\mathcal{H}} A, \operatorname{pr}_{\mathcal{H}} Z], \qquad A, Z \in \Gamma(TM).$$

We assume that there is a finite, minimal positive constant $\mathscr{M}_{\mathcal{R}} < \infty$ such that $\|\mathcal{R}(v, \centerdot)\|_{\mathbf{g}^* \otimes \mathbf{g}} \leq \mathscr{M}_{\mathcal{R}} \| \operatorname{pr}_{\mathcal{H}} v\|_{\mathbf{g}}$ for any $v \in TM$. Since $\mathscr{M}_{\mathcal{R}}$ is never zero when $\mathcal{V} \neq 0$, we can normalize $\mathbf{v}$ by requiring $\mathscr{M}_{\mathcal{R}} = 1$. Let $m_{\mathcal{R}}$ be the maximal constant satisfying $\|\alpha(\mathcal{R}(\centerdot, \centerdot))\|_{\wedge^2 \mathbf{h}^*} \geq m_{\mathcal{R}} \|\alpha\|_{\mathbf{v}^*}$ pointwise for any $\alpha \in \Gamma(T^*M)$. Note that $m_{\mathcal{R}}$ can only be non-zero if $\mathcal{H}$ is bracket-generating of step 2, i.e. if $\mathcal{H}$ and its first order brackets span the entire tangent bundle.

(ii) Define $\operatorname{Ric}_{\mathcal{H}}(Z_1, Z_2) = \operatorname{tr}\left(A \mapsto R^{\mathring{\nabla}}(\operatorname{pr}_{\mathcal{H}} A, Z_1) Z_2\right)$. This is a symmetric 2-tensor, which vanishes for vectors in $\mathcal{V}$. We assume that there is a lower bound $\rho_{\mathcal{H}}$ for $\operatorname{Ric}_{\mathcal{H}}$, i.e. for every $v \in TM$, we have

$$\operatorname{Ric}_{\mathcal{H}}(v,v) \geq \rho_{\mathcal{H}} \| \operatorname{pr}_{\mathcal{H}} v\|^2_{\mathbf{h}}.$$

(iii) Write $\mathscr{M}_{\mathring{\nabla}\mathbf{v}^*} = \sup_M \left\| \mathring{\nabla}_{\centerdot} \mathbf{v}^*(\centerdot, \centerdot) \right\|_{\mathbf{g}^* \otimes \operatorname{Sym}^2 \mathbf{g}^*}$ and assume that it is finite. Define

$$(\Delta'_{\mathbf{h}} \mathbf{v}^*)(\alpha, \alpha) = \operatorname{tr}_{\mathcal{H}} (\mathring{\nabla}^2_{\times, \times} \mathbf{v}^*)(\alpha, \alpha)$$

and assume that $(\Delta'_{\mathbf{h}} \mathbf{v}^*)(\alpha,\alpha) \geq \rho_{\Delta'_{\mathbf{h}}\mathbf{v}^*} \|\alpha\|^2_{\mathbf{v}^*}$ pointwise for any $\alpha \in \Gamma(T^*M)$.

(iv) Finally, introduce $\operatorname{Ric}_{\mathcal{H}\mathcal{V}}$ as

$$\operatorname{Ric}_{\mathcal{H}\mathcal{V}}(A,Z) = \frac{1}{2} \operatorname{tr} \left(\mathbf{g}(A, (\mathring{\nabla}_{\times} \mathcal{R})(\times, Z)) + \mathbf{g}(Z, (\mathring{\nabla}_{\times} \mathcal{R})(\times, A)) \right).$$

Assume then that $\operatorname{Ric}_{\mathcal{H}\mathcal{V}}(Z,Z) \geq -2\mathscr{M}_{\mathcal{H}\mathcal{V}} \| \operatorname{pr}_{\mathcal{V}} Z\|_{\mathbf{v}} \| \operatorname{pr}_{\mathcal{H}} Z\|_{\mathbf{h}}$ pointwise.

These assumptions guarantee that the sub-Laplacian $\Delta'_{\mathbf{h}}$ of $\mathcal{V}$ satisfies (CD*).

Theorem 2.2. *Define $\Gamma_2^{\mathbf{s}^*}$ with respect to $L = \Delta'_{\mathbf{h}}$. Then $\Delta'_{\mathbf{h}}$ satisfies* (CD*) *with*

$$(2.2) \qquad \begin{cases} n = \operatorname{rank} \mathcal{H}, \\ \rho_1 = \rho_{\mathcal{H}} - c^{-1}, \\ \rho_{2,0} = \dfrac{1}{2} m_{\mathcal{R}}^2 - c(\mathscr{M}_{\mathcal{H}\mathcal{V}} + \mathscr{M}_{\mathring{\nabla}\mathbf{v}^*})^2, \\ \rho_{2,1} = \dfrac{1}{2} \rho_{\Delta'_{\mathbf{h}}\mathbf{v}^*} - \mathscr{M}^2_{\mathring{\nabla}\mathbf{v}^*}, \end{cases}$$

for any positive $c > 0$.

See Part I, Section 3.2 for a geometric interpretation of these constants.

2.5. **The case when $\mathring{\nabla}$ preserves the metric.** Assume that we can find a metric tensor $\mathbf{v}$ on $\mathcal{V}$ satisfying $\mathring{\nabla}\mathbf{v}^* = 0$. Then $\Delta'_{\mathbf{h}} = \Delta_{\mathbf{h}}$, where $\Delta_{\mathbf{h}}$ is defined relative to the volume form of the Riemannian metric $\mathbf{g}$ defined by $\mathbf{g}^* = \mathbf{h}^* + \mathbf{v}^*$. Hence, $L = \Delta_{\mathbf{h}}$ is symmetric with respect to this volume form and satisfies the inequality

$$(\mathsf{CD}) \qquad \Gamma_2^{\mathbf{h}^* + \ell \mathbf{v}^*}(f) \geq \frac{1}{n}(Lf)^2 + \left(\rho_1 - \ell^{-1}\right) \Gamma^{\mathbf{h}^*}(f) + \rho_2 \Gamma^{\mathbf{v}^*}(f)$$

with

$$\begin{cases} \rho_1 = \rho_{\mathcal{H}} - c^{-1}, \\ \rho_2 = \dfrac{1}{2} m_{\mathcal{R}}^2 - c \mathscr{M}_{\mathcal{H}\mathcal{V}}^2, \end{cases} \tag{2.3}$$

for any positive $c > 0$. We shall also need the following result.

Proposition 2.3. *For any $f \in C^\infty(M)$, and any $c > 0$ and $\ell > 0$,*

$$\frac{1}{4} \Gamma^{\mathbf{h}^*}(\Gamma^{\mathbf{h}^*}(f)) \leq \Gamma^{\mathbf{h}^*}(f) \left(\Gamma_2^{\mathbf{h}^* + \ell \mathbf{v}^*}(f) - (\varrho_1 - \ell^{-1}) \Gamma^{\mathbf{h}^*}(f) - \varrho_2 \Gamma^{\mathbf{v}^*}(f) \right),$$

$$\frac{1}{4} \Gamma^{\mathbf{h}^*}(\Gamma^{\mathbf{v}^*}(f)) \leq \Gamma^{\mathbf{v}^*}(f) \Gamma_2^{\mathbf{v}^*}(f),$$

where $\varrho_1 = \rho_{\mathcal{H}} - c^{-1}$ and $\varrho_2 = -c \mathscr{M}_{\mathcal{H}\mathcal{V}}^2$.

2.6. **Spectral Gap.** Let $(M, \mathcal{H}, \mathbf{h})$ be a compact sub-Riemannian manifold where $\mathcal{H}$ is bracket-generating. Let L be a smooth second order operator without constant term satisfying $\mathbf{q}_L = \mathbf{h}^*$ and assume also that L is symmetric with respect to some volume form vol on M. Assume that L satisfies (CD*) with $\rho_{2,0} > 0$. Let λ be any nonzero eigenvalue of L. Then

$$\frac{n \rho_{2,0}}{n + \rho_{2,0}(n-1)} \left(\rho_1 - \frac{k_2}{\rho_{2,0}} \right) \leq -\lambda, \qquad k_2 = \max\{0, -\rho_{2,1}\}.$$

3. Results under conditions of a uniformly bounded gradient

3.1. **Diffusions of second order operators.** Let T^2M denote the bundle of second order tangent vectors. Let L be a section of T^2M, i.e. a smooth second order differential operator on L without constant term. Consider the short exact sequence

$$0 \to TM \xrightarrow{\text{inc}} T^2M \xrightarrow{\mathbf{q}} \operatorname{Sym}^2 TM \to 0$$

where $\mathbf{q}_L = \mathbf{q}(L)$ is defined by

$$\mathbf{q}_L(df, dg) = \frac{1}{2} \left(L(fg) - fLg - gLf \right), \quad f, g \in C^\infty(M). \tag{3.1}$$

Assume that $\mathbf{q}_L$ is positive semi-definite. Then for any point $x \in M$ and relative to some filtered probability space $(\Omega, \mathscr{F}_\bullet, \mathbb{P})$, we have a $\frac{1}{2}L$-diffusion $X = X(x)$ defined up to some explosion time $\tau = \tau(x)$, see [12, Theorems 1.3.4 and 1.3.6]. In other words, there exist an $\mathscr{F}_\bullet$-adapted M-valued semimartingale $X(x)$ satisfying $X_0(x) = x$ and such that for any $f \in C^\infty(M)$,

$$d(f(X_t)) - \frac{1}{2} Lf(X_t) \, dt$$

is the differential of a local martingale up to $\tau(x)$. The diffusion $X(x)$ is defined on the stochastic interval $[0, \tau(x))$, with $\tau(x)$ being an explosion time in the sense that the event $\{\tau(x) < \infty\}$ is almost surely contained in $\{\lim_{t \uparrow \tau} X_t(x) = \infty\}$. For a construction of $X_t(x)$ in the case of $L = \Delta'_{\mathbf{h}}$, see Part I, Section 2.5.

Let P_t be the corresponding semigroup $P_t f(x) = \mathbb{E}[1_{t \leq \tau} f(X_t(x))]$ for bounded measurable functions f. Note that in general $P_t 1 \leq 1$ with equality if and only if $\tau(x) = \infty$ a.s. Also note that for any compactly supported $f \in C_c^\infty(M)$, we have $\partial_t P_t f = \frac{1}{2} L P_t f$. If $\tau = \infty$ a.s., then $u_t = P_t f$ is the unique solution to $\partial_t u_t = \frac{1}{2} L u_t$ with initial condition $u_0 = f$, where $(t, x) \mapsto u_t(x)$ is a smooth function on $\mathbb{R}_+ \times M$.

Since $\mathbf{q}_L$ is positive semi-definite, we can write L (non-uniquely) as

$$L = \sum_{i=1}^{k} Z_i^2 + Z_0,$$

where k is an integer and $Z_0, Z_1, \dots, Z_k$ are vector fields, not necessarily linearly independent at every point. If we assume that these vector fields and their brackets span the entire tangent bundle, then L is a hypoelliptic operator [11]. Hence, it has a smooth heat kernel with respect to any volume form on M. By [18], we also have $P_t f > 0$ for any nonnegative function $f \in C^\infty(M)$, not identically zero, see also [13, Introduction]. We will *only* consider such second order operators in this paper.

Write $\mathbf{h}^* = \mathbf{q}_L$. Assume that L satisfies ($\mathsf{CD*}$) for some $\mathbf{v}^*$. We want to use this inequality to obtain statements of P_t. However, we are going to need the following condition to hold to make such statements.

3.2. **Boundedness of the gradient under the action of the heat semigroup.** The most important property which we are going to need for all of our results, is the following condition. Let $C_b^\infty(M)$ be the collection of all bounded smooth functions.

(A) We have $P_t 1 = 1$ and for any $f \in C_b^\infty(M)$ with $\mathsf{\Gamma}^{\mathbf{h}^*+\mathbf{v}^*}(f) \in C_b^\infty(M)$ and any $T > 0$, it holds that $\sup_{t\in[0,T]} \|\mathsf{\Gamma}^{\mathbf{h}^*+\mathbf{v}^*}(P_t f)\|_{L^\infty} < \infty$.

To understand condition (A) better, let us first discuss the special case when $\mathbf{h} = \mathbf{g}$ is a Riemannian metric, $\mathbf{v}^* = 0$ and $L = \Delta$ is the Laplacian of $\mathbf{g}$. Then ($\mathsf{CD*}$) holds if and only if the Ricci curvature is bounded from below, see e.g. [20]. If we in addition know that $\mathbf{g}$ is complete, then (A) is satisfied. However, even if we know that $P_t 1 = 1$ and that the manifold is flat, condition (A) still may not hold if $\mathbf{g}$ is an incomplete metric. See [19] for a counter-example.

We list some cases where we are ensured that (A) is satisfied. We expect there to be more cases where this condition holds.

3.2.1. *Fiber bundles with compact fibers.* Let L be a second order operator on a manifold M with $\mathbf{q}_L = \mathbf{h}^*$. Let $\mathbf{v}^*$ be any other co-metric such that $\mathbf{h}^* + \mathbf{v}^*$ is positive definite. The following observation was given in [21, Lemma 2.1, Proof (i)].

Lemma 3.1. *Assume that there exists a function $F \in C^\infty(M)$ and a constant $C > 0$ satisfying*

- *$\{x : F(x) \leq s\}$ is compact for any $s > 0$,*
- *$LF \leq CF$,*
- *$\mathsf{\Gamma}^{\mathbf{h}^*+\mathbf{v}^*}(F) \leq CF^2$.*

Then (A) holds for the semigroup P_t of the diffusion of L.

Let $(B, \breve{\mathbf{g}})$ be a complete n-dimensional Riemannian manifold with distance $\mathsf{d}_{\breve{\mathbf{g}}}$ and Ricci bound from below by $\rho \leq 0$. For a given point $b_0 \in B$, define $r = \mathsf{d}_{\breve{\mathbf{g}}}(b_0, \,\cdot\,)$. Then the function $F = \sqrt{1 + r^2}$ (or rather an appropriately smooth approximation) satisfies the above conditions relative to $\breve{\Delta}$. This follows from the fact that (outside the cut-locus) $\mathsf{\Gamma}^{\breve{\mathbf{g}}}(r) = 1$ and from the Laplacian comparison theorem

$$\breve{\Delta} r \leq (n-1)\left(\frac{1}{r} + \sqrt{-\rho}\right).$$

Now let $\pi : M \to B$ be a fiber bundle with a compact fiber over this Riemannian manifold B. Choosing an Ehresmann connection $\mathcal{H}$ on π, we define a sub-Riemannian manifold $(M, \mathcal{H}, \mathbf{h})$ by $\mathbf{h} = \pi^* \breve{\mathbf{g}}|_{\mathcal{H}}$. Then $F \circ \pi$ clearly satisfies Lemma 3.1 with respect to $L = \Delta'_{\mathbf{h}} + Z$ where $\Delta'_{\mathbf{h}}$ is the sub-Laplacian of $\mathcal{V} = \ker \pi_*$ and Z is any vector field with values in $\mathcal{V}$. It follows that (A) holds in this case.

Remark 3.2. Let $\pi \colon M \to B$ be a surjective submersion into a Riemannian manifolds $(B, \breve{\mathbf{g}})$. Let $\mathcal{H}$ be an Ehresmann connection on π and define a sub-Riemannian structure $(\mathcal{H}, \mathbf{h})$ by $\mathbf{h} = \pi^* \breve{\mathbf{g}}|_{\mathcal{H}}$. In this case, π is a distance-decreasing map from the metric space (M, d_{cc}) to $(M, \mathsf{d}_{\breve{\mathbf{g}}})$, where the metrics d_{cc} and $\mathsf{d}_{\breve{\mathbf{g}}}$ are defined relative to $(\mathcal{H}, \mathbf{h})$ and $\breve{\mathbf{g}}$, respectively. This follows from the observation that for any horizontal curve γ in M from the point x to the point y, the curve $\pi \circ \gamma$ will be a curve of equal length in B connecting $\pi(x)$ with $\pi(y)$, hence $\mathsf{d}_{cc}(x, y) \geq \mathsf{d}_{\breve{\mathbf{g}}}(\pi(x), \pi(y))$. In particular, if d_{cc} is complete, so is $\mathsf{d}_{\breve{\mathbf{g}}}$, and the converse also hold if π is a fiber bundle with compact fibers.

Furthermore, if $\Delta_{\mathbf{h}}$ is the sub-Laplacian of $\mathcal{V} = \ker \pi_*$ satisfying (CD^*), then the Ricci curvature of B is bounded from below, since, by Remark 2.1, if we insert a function $f \circ \pi$, $f \in C^\infty(B)$ into (CD^*), we obtain the usual curvature-dimension inequality on B,

$$\Gamma_2^{\breve{\mathbf{g}}}(f) \geq \frac{1}{n} (\breve{\Delta})^2 + \rho_1 \Gamma^{\breve{\mathbf{g}}}(f).$$

A result in [4, Prop 6.2] tells us that ρ_1 must be a lower Ricci bound for B.

We summarize all the above comments in the following proposition.

Proposition 3.3. *Let $(M, \mathcal{H}, \mathbf{h})$ be a complete sub-Riemannian manifold with an integrable metric preserving complement $\mathcal{V}$. Let $\mathcal{F}$ be the foliation induced by $\mathcal{V}$ and let $\Delta'_{\mathbf{h}}$ be the sub-Laplacian of $\mathcal{V}$. Assume that the leafs of $\mathcal{F}$ are compact and that $M/\mathcal{F}$ gives us a well defined smooth manifold. Finally assume that $L = \Delta'_{\mathbf{h}} + Z$ satisfies (CD^*) with respect to some $\mathbf{v}^*$ on $\mathcal{V}$. Then (A) also hold for the corresponding semigroup P_t of L.*

Notice that in this case, unlike what we will discuss next, there is no requirement on the number of brackets needed of vector fields in $\mathcal{H}$ in order to span the entire tangent bundle.

3.2.2. *A sub-Laplacian on a totally geodesic Riemannian foliation.* Assume that $(M, \mathbf{g})$ is a complete Riemannian manifold with a foliation $\mathcal{F}$ given by an integrable subbundle $\mathcal{V}$. Let $\mathcal{H}$ be the orthogonal complement of $\mathcal{V}$ and assume that $\mathcal{H}$ is bracket-generating. Write $\mathbf{h} = \mathbf{g}\,|_{\mathcal{H}}$. Define $\mathring{\nabla}$ relative to the splitting $TM = \mathcal{H} \oplus \mathcal{V}$ as in (2.1). Assume that $\mathring{\nabla}\, \mathbf{g} = 0$, which is equivalent to stating that $\mathcal{V}$ is a metric preserving complement of $(M, \mathcal{H}, \mathbf{h})$ and that $\mathcal{F}$ is a totally geodesic foliation. Note that since $\mathbf{g}$ is complete, so is (M, d_{cc}), where d_{cc} is defined relative to the sub-Riemannian metric $\mathbf{h}$. For such sub-Riemannian manifolds, the we can deduce the following.

Theorem 3.4. *Let $\Delta_{\mathbf{h}}$ be the sub-Laplacian of the volume form of $\mathbf{g}$ or equivalently $\mathcal{V}$. Assume that $\Delta_{\mathbf{h}}$ satisfies the assumptions of Theorem 2.2 with $m_{\mathcal{R}} > 0$. Let $k = \max\{-\rho_{\mathcal{H}}, \mathscr{M}^2_{\mathcal{H}\mathcal{V}}\} \geq 0$. Then, for any compactly supported $f \in C_c^\infty(M)$, $\ell > 0$ and $t \geq 0$,*

$$\sqrt{\Gamma^{\mathbf{v}^*}(P_t f)} \leq P_t \sqrt{\Gamma^{\mathbf{v}^*}(f)}\,,$$

$$\sqrt{\Gamma^{\mathbf{h}^*}(P_t f)+\Gamma^{\mathbf{v}^*}(P_t f)} \leq e^{kt/2} P_t\left(\sqrt{\Gamma^{\mathbf{h}^*}(f)+\Gamma^{\mathbf{v}^*}(f)}\right)+\frac{2}{k}(e^{kt/2}-1),$$

where we interpret $\frac{2}{k}(e^{kt/2}-1)$ *as* t *when* $k=0$. *As a consequence* (A) *holds.*

In particular, any $\frac{1}{2}\Delta_{\mathbf{h}}$*-diffusion* $X(x)$ *with* $X_0(x)=x\in M$ *has infinite lifetime.*

We remind the reader that $m_{\mathcal{R}}>0$ can only happen if TM is spanned by $\mathcal{H}$ and first order brackets of its sections. The proof is similar to the proof given for the special case of sub-Riemannian manifolds with transverse symmetries of Yang-Mills type given in [8, Section 3 & Theorem 4.3]. In our terminology, these are sub-Riemannian manifolds with a trivial, integrable, metric-preserving complement $\mathcal{V}$ satisfying $\mathscr{M}_{\mathcal{H}\mathcal{V}}=0$. The key factors that allow us to use a similar approach are Proposition 2.3 and the relation $[\Delta_{\mathbf{h}},\Delta]f=0$, where Δ is the Laplace operator of $\mathbf{g}$ and $f\in C^\infty(M)$. The latter results follow from Lemma A.1 (c) in the Appendix. Since the proof uses spectral theory and calculus on graded forms, it is left to Appendix A.3.2. Theorem 3.4 also holds in some cases when $\mathcal{V}$ is not an integrable subbundle. See Appendix A.5 for details.

3.3. **General formulation.** Let L be an operator as in Section 3.1 with corresponding $\frac{1}{2}L$-diffusion $X(x)$ satisfying $X_0(x)=x$ and semigroup P_t. We will assume that L satisfies (CD^*) with $\mathbf{v}^*$ and the constants $n,\rho_1,\rho_{2,0}$ and $\rho_{2,1}$ being implicit. Note that if L satisfies (CD^*) for some value of the previously mentioned constants, then L also satisfies the same inequality for any larger n or smaller values of $\rho_1,\rho_{2,0}$ or $\rho_{2,1}$. For the remainder of the section, no result will depend on n, however, we will need condition (A) to hold.

Our proofs rely on the fact that, for any smooth function

$$(t,x)\mapsto u_t(x)\in C^\infty([0,\infty)\times M,\mathbb{R}),$$

we have a stochastic process $Y_t=u_t\circ X_t$ such that dY_t equals $((\partial_t+L)u_t)\circ X_t\,dt$ modulo differentials of local martingales. Hence, if $(\partial_t+L)u_t\geq 0$ and if $u_t(\cdot)$ is bounded for every fixed t, then Y_t is a (true) submartingale and $\mathbb{E}[Y_t]$ is an increasing function with respect to t.

In our presentation, we will usually state the result for a smooth, bounded function $f\in C_b^\infty(M)$ with bounded gradient $\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(f)\in C_b^\infty(M)$. Our results generalize theorems found in [8, 6, 7].

We will first construct a general type of inequality, from which many results can be obtained. See [21, Theorem 1.1 (1)] for a similar result, with somewhat different assumptions.

Lemma 3.5. *Assume that* L *satisfies the conditions* (CD^*) *and* (A). *For any* $T>0$, *let* $a,\ell\in C([0,T],\mathbb{R})$ *be two continuous functions which are smooth and positive on* $(0,T)$. *Assume that there exist a constant* C, *such that*

$$\dot{a}(t)+\left(\rho_1-\frac{1}{\ell(t)}\right)a(t)+C\geq 0,\quad \dot{\ell}(t)+\rho_{2,0}+\left(\rho_{2,1}+\frac{\dot{a}(t)}{a(t)}\right)\ell(t)\geq 0, \tag{3.2}$$

holds for every $t\in(0,T)$. *Then*

$$a(0)\Gamma^{\mathbf{h}^*+\ell(0)\mathbf{v}^*}(P_T f)\leq a(T)P_T\Gamma^{\mathbf{h}^*+\ell(T)\mathbf{v}^*}(f)+C\left(P_T f^2-(P_T f)^2\right)$$

for any $f\in C_b^\infty(M)$ *with* $\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(f)\in C_b^\infty(M)$.

Proof. Define u_t by $u_t(x) = P_{T-t}f(x)$ for any $0 \leq t \leq T$, $x \in M$. For any $x \in M$, consider the stochastic process

$$Y_t(x) := a(t)\Gamma^{\mathbf{h}^*+\ell(t)\mathbf{v}^*}(u_t) \circ X_t(x) + Cu_t^2 \circ X_t(x).$$

Write $\overset{\text{loc}}{=}$ for equivalence modulo differentials of local martingales. Then, if (3.2) holds

$$\begin{aligned} dY_t &\overset{\text{loc}}{=} \Big(\dot{a}(t)\Gamma^{\mathbf{h}^*+\ell(t)\mathbf{v}^*}(u_t) + a(t)\dot{\ell}(t)\Gamma^{\mathbf{v}^*}(u_t) + C\Gamma^{\mathbf{h}^*}(u_t)\Big) \circ X_t dt \\ &\quad + a(t)\Gamma_2^{\mathbf{h}^*+\ell(t)\mathbf{v}^*}(u_t) \circ X_t dt \\ &\geq \big(\dot{a}(t) + (\rho_1 - \ell(t)^{-1})a(t) + C\big)\,\Gamma^{\mathbf{h}^*}(u_t) \circ X_t dt \\ &\quad + a(t)\left(\dot{\ell}(t) + \frac{\dot{a}(t)}{a(t)} + \rho_{2,0} + \rho_{2,1}\ell(t)\right)\Gamma^{\mathbf{v}^*}(u_t) \circ X_t dt \geq 0. \end{aligned}$$

Since Y_t is bounded by (A), it is a true submartingale. Hence

$$\begin{aligned} \mathbb{E}[Y_T] &= a(T)P_T\Gamma^{\mathbf{h}^*+\ell(T)\mathbf{v}^*}(f) + CP_Tf^2 \\ &\geq \mathbb{E}[Y_0] = a(0)\Gamma^{\mathbf{h}^*+\ell(0)\mathbf{v}^*}(P_Tf) + C(P_Tf)^2. \end{aligned}$$

□ □

3.4. **Gradient bounds.** We give here the first results that follow from Lemma 3.5.

Proposition 3.6. *Assume that L satisfies conditions* (CD*) *and* (A). *Let $f \in C_b^\infty(M)$ be any smooth bounded function satisfying $\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(f) \in C_b^\infty(M)$.*

(a) *For any constant $\ell > 0$, if $\alpha(\ell) = \min\left\{\rho_1 - \frac{1}{\ell}, \rho_{2,1} + \frac{\rho_{2,0}}{\ell}\right\}$, then*

$$\Gamma^{\mathbf{h}^*+\ell\mathbf{v}^*}(P_tf) \leq e^{-\alpha(\ell)t}P_t\Gamma^{\mathbf{h}^*+\ell\mathbf{v}^*}(f).$$

(b) *Assume that $\rho_{2,0} > 0$ and let $k_1 = \max\{0, -\rho_1\}$ and $k_2 = \max\{0, -\rho_{2,1}\}$. Then*

$$t\Gamma^{\mathbf{h}^*}(P_tf) \leq \left(1 + \frac{2}{\rho_{2,0}} + \left(k_1 + \frac{k_2}{\rho_2}\right)t\right)(P_tf^2 - (P_tf)^2).$$

(c) *Assume that $\rho_1 \geq 0$, $\rho_{2,1} \geq 0$ and $\rho_{2,0} > 0$. Then*

$$\frac{1 - e^{-\rho_1 t}}{\rho_1}\Gamma^{\mathbf{h}^*}(P_tf) \leq \left(1 + \frac{2}{\rho_{2,0}}\right)\left(P_tf^2 - (P_tf)^2\right),$$

where we interpret $(1 - e^{-\rho_1 t})/\rho_1$ as t when $\rho_1 = 0$.

(d) *Assume that ρ_1, $\rho_{2,0}$ and $\rho_{2,1}$ are nonnegative. Then for any $\ell > 0$*

$$\frac{\ell}{\ell + t}\left(P_tf^2 - (P_tf)^2\right) \leq tP_t\Gamma^{\mathbf{h}^*+(\ell+t)\mathbf{v}^*}(f).$$

Proof. For all of our results (a)–(c), we will use Lemma 3.5.

(a) Let $\ell(t) = \ell$ be a constant, choose $C = 0$ and put $a(t) = e^{-\alpha(\ell)t}$. Then (3.2) is satisfied and we obtain

$$\Gamma^{\mathbf{h}^*+\ell\mathbf{v}^*}(P_Tf) \leq e^{-\alpha(\ell)T}P_T\Gamma^{\mathbf{h}^*+\ell\mathbf{v}^*}(P_Tf).$$

(b) For any $T \geq 0$, consider $a(t) = T - t$ and $\ell(t) = \frac{\rho_{2,0}}{Tk_2+2}(T - t)$. Then

$$\dot{\ell}(t) + \rho_{2,0} + \left(\rho_{2,1} + \frac{\dot{a}(t)}{a(t)}\right)\ell(t) \geq 0,$$

and

$$\dot{a}(t) + (\rho_1 - \ell(t)^{-1})a(t) \geq -1 - k_1 - \frac{Tk_2 + 2}{\rho_{2,0}},$$

so (3.2) is satisfied if we define $C = 1 + k_1 T + \frac{Tk_2+2}{\rho_{2,0}}$. Using Lemma 3.5, we obtain

$$T\Gamma^{\mathbf{h}^* + \frac{\rho_{2,0}}{Tk_2+2}T\mathbf{v}^*}(P_T f) \leq C(P_T f^2) - C(P_T f)^2.$$

(c) Since the case $\rho_1 = 0$ is covered in (b), we can assume $\rho_1 > 0$. Define

$$a(t) = \frac{1 - e^{-\rho_1(T-t)}}{\rho_1}$$

and let

$$\ell(t) = \rho_{2,0}\frac{\int_t^T a(s)\,ds}{a(t)} = \rho_{2,0}\frac{e^{-\rho_1(T-t)} - 1 + \rho_1(T-t)}{\rho_1(1 - e^{-\rho_1(T-t)})}.$$

Note that $\lim_{t\uparrow T}\ell(t) = 0$, while $\lim_{t\uparrow T} a(t)/\ell(t) = 2/\rho_{2,0}$. The latter number is also an upper bound for $a(t)/\ell(t)$ since

$$\frac{d}{dt}\frac{a(t)}{\ell(t)} = \frac{a(t)\left(2\dot{a}(t)\int_t^T a(s)\,ds + a(t)^2\right)}{\rho_{2,0}(\int_t^T a(s)ds)^2} > 0$$

from the fact that

$$\begin{aligned} &2\dot{a}(t)\int_t^T a(s)ds + a(t)^2 \\ &= \frac{1}{\rho_1^2}\left(-2e^{-\rho_1(T-t)}\left(e^{-\rho_1(T-T)} - 1 + \rho_1(T-t)\right) + (1 - e^{-\rho_1(T-t)})^2\right) \\ &= \frac{1}{\rho_1^2}\left(-2\rho_1(T-t)e^{-\rho_1(T-t)} + 1 - e^{-2\rho_1(T-t)}\right). \end{aligned}$$

and that $s \mapsto 1 - e^{-2s} - 2xe^{-2s}$ is an increasing function, vanishing at $s = 0$. We can then define $C = 1 + \frac{2}{\rho_{2,0}}$ such that $a(t), \ell(t)$ and C satisfies (3.2).

(d) Define $a(t) = t$, $\ell(t) = \frac{(\ell+T)t}{T}$ and $C = -\frac{\ell}{\ell+T}$, then (3.2) is satisfied. □

□

We see here that the results of (a) and (d) cannot be stated independently of a choice of co-metric $\mathbf{v}^*$. However, in the case of (a), this does help us to get global statements that are independent of $\mathbf{v}^*$.

3.5. **Bounds for the L^2-norm of the gradient and the Poincaré inequality.** We want to use an approach similar to what is used in [6, Corollary 2.4] to obtain a global inequality from the pointwise estimate in Proposition 3.6 (a) which is independent of $\mathbf{v}^*$.

Lemma 3.7. *Let $L \in \Gamma(T^2M)$ be a second order operator without constant term and with $\mathbf{q}_L = \mathbf{h}^*$ positive semi-definite. Assume also that there exists a volume form* vol*, such that*

$$\int_M fLg\,d\mathrm{vol} = \int_M gLf\,d\mathrm{vol}, \quad f, g \in C_c^\infty(M),$$

and that L is essentially self-adjoint on compactly supported functions $C_c^\infty(M)$.

Let $P_t f$ be the semigroup defined as in Section 3.1 *and let* $b : C_c^\infty(M) \times [0,\infty) \to \mathbb{R}$ *be any function such that*

$$\|\Gamma^{\mathbf{h}^*}(P_t f)\|_{L^1} \leq b(f,t), \quad \text{for any } f \in C_c^\infty(M),\ t > 0. \tag{3.3}$$

Assume that $\beta(f,t) := \lim_{T\to\infty} b(f,T)^{t/T}$ *exist for every* $t > 0$. *Then*

$$\|\Gamma^{\mathbf{h}^*}(P_t f)\|_{L^1} \leq \beta(f,t)\|\Gamma^{\mathbf{h}^*}(f)\| \quad \text{for any } f \in C_c^\infty(M).$$

Proof. Denote the unique self-adjoint extension of L an operator on $L^2(M, \mathrm{vol})$ by the same letter, and let $\mathrm{Dom}(L)$ be its domain. Then $e^{t/2L} f$ is the unique solution in $L^2(M,\mathrm{vol})$ of equation $\partial_t u_t = \frac{1}{2} L u_t$ with initial condition $u_0 = f \in C_c^\infty(M)$. Since $P_t f$ is in $L^2(M,\mathrm{vol})$ whenever f is in $L^2(M,\mathrm{vol})$, we have $P_t f = e^{t/2L} f$ (see Appendix A.3 for more details).

Notice that since $\mathbf{q}_L = \mathbf{h}^*$ is positive semi-definite, the self-adjoint operator L is nonpositive. Let $\langle \cdot\,, \cdot \rangle$ denote the inner product on $L^2(M,\mathrm{vol})$. Consider the spectral decomposition $L = -\int_0^\infty \lambda dE_\lambda$. Then since $\|\Gamma^{\mathbf{h}^*}(f)\|_{L^1} = -\langle f, Lf\rangle$, while

$$\|\Gamma^{\mathbf{h}^*}(P_t f)\|_{L^1} = -\langle f, LP_{2t} f\rangle,$$

the Hölder inequality tells us that for any $0 < t < T$,

$$\begin{aligned}
\|\Gamma^{\mathbf{h}^*}(P_t f)\|_{L^1} &= \int_0^\infty \lambda e^{-t\lambda} d\langle E_\lambda f, f\rangle \\
&\leq \left(\int_0^\infty \lambda e^{-\lambda T}\, d\langle E_\lambda f, f\rangle\right)^{t/T} \left(\int_0^\infty \lambda\, d\langle E_\lambda f, f\rangle\right)^{(T-t)/T} \\
&\leq b(f,T)^{t/T} \|\Gamma^{\mathbf{h}^*}(f)\|_{L^1}^{T/(T-t)}.
\end{aligned}$$

Let $T \to \infty$ for the result. □ □

We combine this result with the curvature-dimension inequality.

Proposition 3.8. *Let L be any second order operator such that the Carnot-Carathéodory metric d_{cc} defined by the sub-Riemannian co-metric $\mathbf{h}^* := \mathbf{q}_L$ is complete. Assume that L satisfies* (CD*) *and that* (A) *holds. Assume also that L is symmetric with respect to any volume form* vol, *i.e.* $\int_M fLg\, d\mathrm{vol} = \int_M gLf\, d\mathrm{vol}$ *for any* $f, g \in C_c^\infty(M)$.

(a) *For any* $f \in C_c^\infty(M)$,

$$\|\Gamma^{\mathbf{h}^*}(P_t f)\|_{L^1} \leq e^{-kt}\|\Gamma^{\mathbf{h}^*}(f)\|_{L^1},$$

where $k = \min\{\rho_1, \rho_{2,1}\}$.

(b) *Assume that* $\rho_1 \geq \rho_{2,1}$ *and* $\rho_{2,0} > -1$. *Then for any* $f \in C_c^\infty(M)$,

$$\|\Gamma^{\mathbf{h}^*}(P_t f)\|_{L^1} \leq e^{-\alpha t}\|\Gamma^{\mathbf{h}^*}(f)\|_{L^1}, \quad \alpha := \frac{\rho_{2,0}\rho_1 + \rho_{2,1}}{\rho_{2,0} + 1}.$$

Furthermore, if $\alpha > 0$ *and* $\mathbf{h}^* + \mathbf{v}^*$ *is a complete Riemannian co-metric, then* $\mathrm{vol}(M) < \infty$.

(c) *Assume that the conditions in* (b) *hold with* $\alpha > 0$ *and* $\mathrm{vol}(M) < \infty$. *Then for any* $f \in C_c^\infty(M)$,

$$\|f - f_M\|_{L^2}^2 \leq \frac{1}{\alpha}\int_M \Gamma^{\mathbf{h}^*}(f)\, d\mathrm{vol},$$

where $f_M = \mathrm{vol}(M)^{-1}\int_M f\, d\mathrm{vol}$. *As a consequence, if λ is any non-zero eigenvalue of the Friedrichs extension of L, then* $\alpha \leq -\lambda$.

Proof. (a) By Proposition 3.6 (a), we have

$$\|\Gamma^{\mathbf{h}^*}(P_t f)\|_{L^1} \le e^{-\alpha(\ell)t}\|\Gamma^{\mathbf{h}^*+\ell\mathbf{v}^*}(f)\|_{L^1}$$

with $\alpha(\ell) = \min\{\rho_1 - 1/\ell, \rho_{2,1} + \rho_{2,0}/\ell\}$ holds for any $f \in C_c^\infty(M)$. It follows that $\|\Gamma^{\mathbf{h}^*}(P_t f)\|_{L^1} \le e^{-\alpha(\ell)t}\|\Gamma^{\mathbf{h}^*}(f)\|_{L^1}$ from Lemma 3.7. For every t, we then take the infimum over ℓ to get

$$\inf_\ell e^{-\alpha t} \le e^{-kt} \quad \text{with } k = \min\{\rho_1, \rho_{2,1}\}.$$

(b) With $\alpha(\ell)$ defined as in the proof of (a), note that if $\rho_1 \ge \rho_{2,1}$ and if $\rho_2 > -1$, then

$$\inf_\ell e^{-\alpha(\ell)t} = \exp\left(-\frac{\rho_{2,0}\rho_1 + \rho_{2,1}}{\rho_{2,0}+1}t\right) = e^{-\alpha t}$$

which gives us the first part of the result.

For the second part, we assume that $\rho_1 > \rho_{2,1}$, since if $\alpha > 0$ with $\rho_1 = \rho_{2,1}$, then we can always decrease $\rho_{2,1}$ while keeping α positive. For two compactly supported functions $f, g \in C_c^\infty(M)$, note that

$$\begin{aligned}\int_M (P_t f - f) g \, d\mathrm{vol} &= \int_M \int_0^t \left(\frac{d}{ds} P_s f\right) g \, ds \, d\mathrm{vol} \\ &= \frac{1}{2}\int_0^t \int_M (\Delta_{\mathbf{h}} P_s f) g \, d\mathrm{vol} \, ds = \frac{1}{2}\int_0^t \int_M \Gamma^{\mathbf{h}^*}(P_s f, g) \, d\mathrm{vol} \, ds.\end{aligned}$$

Hence, by the Cauchy-Schwartz inequality

$$\left|\int_M (P_t f - f) g \, d\mathrm{vol}\right| \le \frac{1}{2}\int_0^t \int_M \|\Gamma^{\mathbf{h}^*}(P_s f)\|_{L^\infty}^{1/2} \Gamma^{\mathbf{h}^*}(g)^{1/2} \, d\,\mathrm{vol},$$

which has upper bound

$$\frac{1}{2}\left\|\Gamma^{\mathbf{h}^*}(f) + \frac{\rho_{2,0}+1}{\rho_1 - \rho_{2,1}}\Gamma^{\mathbf{v}^*}(f)\right\|_{L^\infty}^{1/2} \int_M \Gamma^{\mathbf{h}^*}(g)^{1/2} \, d\mathrm{vol} \int_0^t e^{-\alpha s} ds,$$

by Proposition 3.6 (a). From the spectral theorem, we know that $P_t f$ reaches an equilibrium $P_\infty f$ which is in $\mathrm{Dom}(L)$ and satisfies $LP_\infty f = 0$. Since this implies $\Gamma^{\mathbf{h}^*}(P_\infty f) = 0$, we must have that $P_\infty f$ is a constant.

Assume that $\mathrm{vol}(M) = \infty$. Then $P_\infty f = 0$ and hence, for any $f, g \in C_c^\infty(M)$, we have

$$\left|\int_M fg \, d\mathrm{vol}\right| \le \frac{1}{2\alpha}\left\|\Gamma^{\mathbf{h}^*}(f) + \frac{\rho_{2,0}+1}{\rho_1 - \rho_{2,1}}\Gamma^{\mathbf{v}^*}(f)\right\|_{L^\infty}^{1/2} \int_M \Gamma^{\mathbf{h}^*}(g)^{1/2} \, d\mathrm{vol}\,.$$

However, since $\mathbf{g}$ is complete, we can find a sequence of functions $f_n \in C_c^\infty(M)$ such that $f_n \uparrow 1$ while $\|\Gamma^{\mathbf{g}^*}(f_n)\|_{L^\infty} \to 0$. Inserting such a sequence for f in the above formula and letting $n \to \infty$, we obtain the contradiction that $\int_M g \, d\mathrm{vol} = 0$ for any $g \in C_c^\infty(M)$.

(c) Follows from the identity

$$\begin{aligned}\|f - f_M\|_{L^2}^2 &= \int_M f^2 \, d\mathrm{vol} - \frac{1}{\mathrm{vol}(M)}\left(\int_M f \, d\mathrm{vol}\right)^2 \\ &= -\int_0^\infty \frac{\partial}{\partial t}\int_M (P_t f)^2 \, d\mathrm{vol} \, dt \\ &= \int_0^\infty \int_M \Gamma^{\mathbf{h}^*}(P_t f) \, d\mathrm{vol} \, dt \le \frac{1}{\alpha}\|\Gamma^{\mathbf{h}^*}(f)\|_{L^1}.\end{aligned}$$

□
□

4. Entropy and bounds on the heat kernel

4.1. **Commutating condition on $\Gamma^{\mathbf{h}^*}$ and $\Gamma^{\mathbf{v}^*}$.** For some of our inequalities involving logarithms, we will need the following condition. Let $L \in \Gamma(T^2M)$ be a second order operator without constant term with positive semi-definite $\mathbf{q}_L = \mathbf{h}^*$ defined as in (3.1). Assume that L satisfies either (CD*) or (CD) with respect to positive semi-definite $\mathbf{v}^*$. We say that condition (B) holds if

$$\text{(B)} \qquad \Gamma^{\mathbf{h}^*}(f, \Gamma^{\mathbf{v}^*}(f)) = \Gamma^{\mathbf{v}^*}(f, \Gamma^{\mathbf{h}^*}(f)) \qquad \text{for every } f \in C^\infty(M).$$

We make the following observation.

Lemma 4.1. *Let $\mathbf{g}$ be a Riemannian metric on a manifold M, with an orthogonal splitting $TM = \mathcal{H} \oplus_\perp \mathcal{V}$ and use this decomposition to define the connection $\mathring{\nabla}$ as in* (2.1)*. Write $\mathbf{g}\,|_{\mathcal{H}} = \mathbf{h}$ and $\mathbf{g}\,|_{\mathcal{V}} = \mathbf{v}$ and let $\mathbf{h}^*$ and $\mathbf{v}^*$ be their respective corresponding co-metrics. Then*

$$\Gamma^{\mathbf{h}^*}(f, \Gamma^{\mathbf{v}^*}(f)) = \Gamma^{\mathbf{v}^*}(f, \Gamma^{\mathbf{h}^*}(f))$$

holds for every $f \in C^\infty(M)$ if and only if $\mathring{\nabla}\mathbf{v}^* = \mathring{\nabla}\mathbf{h}^* = 0$.

Proof. It is simple to verify that for any $A \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$, we have

$$\mathring{\nabla}_A \mathbf{h}^* = 0, \qquad \mathring{\nabla}_V \mathbf{v}^* = 0, \qquad T^{\mathring{\nabla}}(A, V) = 0,$$

where $T^{\mathring{\nabla}}$ is the torsion of $\mathring{\nabla}$. Define $\sharp^{\mathbf{h}^*}$ as in Section 2 and let $\sharp^{\mathbf{v}^*}$ be defined analogously. Using the properties of $\mathring{\nabla}$, we get

$$\begin{aligned} \Gamma^{\mathbf{h}^*}(f, \Gamma^{\mathbf{v}^*}(f)) - \Gamma^{\mathbf{v}^*}(f, \Gamma^{\mathbf{h}^*}(f)) &= (\sharp^{\mathbf{h}^*} df)\|df\|^2_{\mathbf{v}^*} - (\sharp^{\mathbf{v}^*} df)\|df\|^2_{\mathbf{h}^*} \\ &= 2\mathring{\nabla}_{\sharp^{\mathbf{h}^*} df} df(\sharp^{\mathbf{v}^*} df) - 2\mathring{\nabla}_{\sharp^{\mathbf{v}^*} df} df(\sharp^{\mathbf{h}^*} df) \\ &\quad + (\mathring{\nabla}_{\sharp^{\mathbf{h}^*} df} \mathbf{v}^*)(df, df) - (\mathring{\nabla}_{\sharp^{\mathbf{v}^*} df} \mathbf{h}^*)(df, df) \\ &= (\mathring{\nabla}_{\sharp^{\mathbf{h}^*} df} \mathbf{v}^*)(df, df) - (\mathring{\nabla}_{\sharp^{\mathbf{v}^*} df} \mathbf{h}^*)(df, df). \end{aligned}$$

Since $T^*M = \ker \mathbf{h}^* \oplus \ker \mathbf{v}^*$ and since $\mathring{\nabla}$ preserves these kernels, the above expression can only vanish for all $f \in C^\infty(M)$ if $\mathring{\nabla}\mathbf{h}^* = 0$ and $\mathring{\nabla}\mathbf{v}^* = 0$. □ □

Let L, P_t and $X(x)$ be as in Section 3.1. In this section, we explore the results we obtain when both conditions (A) and (B) hold. We will also assume that L satisfies (CD) rather than (CD*). The reason for this is that in the concrete case when L is the sub-Laplacian of a sub-Riemannian manifold with an integrable metric-preserving complement, the condition (B) along with the assumptions of Theorem 2.2 imply (CD), see Section 2. For most of the results, we also need the requirement that $\rho_2 > 0$. This means that we can use the results of [8, 6, 7].

Let us first establish some necessary identities. Let P_t be the minimal semigroup of $\frac{1}{2}L$ where $\mathbf{q}_L = \mathbf{h}^*$. For a given $T > 0$, let $u_t := P_{T-t}f$ with $f \in C^\infty(M) \cap L^\infty(M)$. It is clear that $(\frac{1}{2}L + \frac{\partial}{\partial t})\Gamma^{\mathbf{s}^*}(u_t) = \Gamma_2^{\mathbf{s}^*}(u_t)$ for any $\mathbf{s}^* \in \Gamma(\mathrm{Sym}^2 TM)$. Also note that if $F \colon U \subseteq \mathbb{R} \to \mathbb{R}$ be a smooth function, then for any $f \in C^\infty(M)$ with values in U, we obtain

$$LF(f) = F'(f)Lf + F''(f)\Gamma^{\mathbf{h}^*}(f).$$

Straight-forward calculations lead to the following identities.

Lemma 4.2.

(a) *If $u_t = P_{T-t}f$ has values in the domain of F, then*

$$\left(\frac{1}{2}L + \frac{\partial}{\partial t}\right) F(u_t) = \frac{1}{2}F''(u_t)\Gamma^{\mathbf{h}^*}(u_t).$$

In particular, if u_t is positive then

$$\left(\frac{1}{2}L + \frac{\partial}{\partial t}\right) \log u_t = -\frac{\Gamma^{\mathbf{h}^*}(u_t)}{2u_t^2},$$

$$\left(\frac{1}{2}L + \frac{\partial}{\partial t}\right) u_t \log u_t = \frac{\Gamma^{\mathbf{h}^*}(u_t)}{2u_t} = \frac{1}{2}u_t\Gamma^{\mathbf{h}^*}(\log u_t).$$

(b) *For any $\mathbf{s}^* \in \Gamma(\mathrm{Sym}^2 TM)$, we have*

$$\left(\frac{1}{2}L + \frac{\partial}{\partial t}\right) u_t\Gamma^{\mathbf{s}^*}(\log u_t)$$
$$= u_t\,\Gamma_2^{\mathbf{s}^*}(\log u_t) + u_t\left(\Gamma^{\mathbf{h}^*}(\log u_t, \Gamma^{\mathbf{s}^*}(\log u_t)) - \Gamma^{\mathbf{s}^*}(\log u_t, \Gamma^{\mathbf{h}^*}(\log u_t))\right).$$

In particular, $\left(\frac{1}{2}L + \frac{\partial}{\partial t}\right) u_t\Gamma^{\mathbf{h}^*}(\log u_t) = u_t\Gamma_2^{\mathbf{h}^*}(\log u_t)$. If $\mathbf{v}^*$ is any co-metric such that $\Gamma^{\mathbf{h}^*}(f, \Gamma^{\mathbf{v}^*}(f)) = \Gamma^{\mathbf{v}^*}(f, \Gamma^{\mathbf{h}^*}(f))$, then $\left(\frac{1}{2}L + \frac{\partial}{\partial t}\right) u_t\Gamma^{\mathbf{v}^*}(\log u_t) = u_t\Gamma_2^{\mathbf{v}^*}(\log u_t)$ as well.

4.2. **Entropy bounds and Li-Yau type inequality.** We follow the approach of [3], [8, Theorem 5.2] and [21, Theorem 1.1].

Lemma 4.3. *Assume that L satisfies* (CD). *Also assume that* (A) *and* (B) *hold. Consider three continuous functions $a, b, \ell : [0,T] \to \mathbb{R}$ with $a(t)$ and $\ell(t)$ being non-negative. Let C be a constant. Assume that $a(t), b(t)$ and $\ell(t)$ are smooth for $t \in (0,T)$ and on the same domain satisfy*

$$\begin{cases} 0 \leq \dot{a}(t) + \left(\rho_1 - \frac{1}{\ell(t)} - 2b(t)\right) a(t) + C \\ 0 \leq \dot{\ell}(t) + \rho_2 + \frac{\dot{a}(t)}{a(t)}\ell(t). \end{cases} \tag{4.1}$$

Consider a positive function $f \in C_b^\infty(M)$, $f > 0$ with bounded gradient $\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(f) \in C_b^\infty(M)$. Then we have

$$a(0)\,P_Tf\,\Gamma^{\mathbf{h}^*+\ell(0)\mathbf{v}^*}(\log P_Tf) - a(T)P_T\left(f\Gamma^{\mathbf{h}^*+\ell(T)\mathbf{v}^*}(\log f)\right)$$
$$\leq 2C\left(P_T(f\log f) - (P_Tf)\log P_Tf\right)$$
$$+ n\left(\int_0^T a(t)b(t)^2dt\right)P_Tf - 2\left(\int_0^T a(t)b(t)dt\right)P_TLf.$$

Proof. We have $P_tf > 0$ from our assumptions on L and f. For any $T > 0$, define $u_t = P_{T-t}f$ for $0 \leq t \leq T$ and

$$Y_t = a(t)\left(u_t\,\Gamma^{\mathbf{h}^*}(\log u_t) + \ell(t)u_t\,\Gamma^{\mathbf{v}^*}(\log u_t)\right)\circ X_t$$
$$+ 2C\left(u_t\log u_t\right)\circ X_t + \int_0^t a(s)\left(nb(s)^2u_s - 2b(s)Lu_s\right)\circ X_s\,ds.$$

Let us write $\overset{\text{loc}}{=}$ for equivalence modulo differentials of local martingales. We use that

$$Lu_t = u_t L \log u_t + \frac{\Gamma^{\mathbf{h}^*}(u_t)}{u_t} = u_t L \log u_t + u_t \Gamma^{\mathbf{h}^*}(\log u_t)$$

and (CD) to obtain

$$\begin{aligned}
dY_t &\overset{\text{loc}}{=} (\dot{a}(t) - 2a(t)b(t) + C)\, u_t \Gamma^{\mathbf{h}^*}(\log u_t) \circ X_t \, dt \\
&\quad + \left(\dot{a}(t)\ell(t) + a(t)\dot{\ell}(t)\right) \Gamma^{\mathbf{v}^*}(\log u_t) \circ X_t \, dt \\
&\quad + a(t) u_t \Gamma_2^{\mathbf{h}^*+\ell(t)\mathbf{v}^*}(\log u_t) \circ X_t \, dt \\
&\quad + a(t) u_t \left(nb(t)^2 - 2b(t) L \log u_t\right) \circ X_t \, dt \\
&\geq \left(\dot{a}(t) + \left(\rho_1 - \frac{1}{\ell(t)} - 2b(t)\right) a(t) + C\right) u_t \Gamma^{\mathbf{h}^*}(\log u_t) \circ X_t \, dt \\
&\quad + a(t) \left(\dot{\ell}(t) + \rho_2 + \frac{\dot{a}(t)}{a(t)} \ell(t)\right) \Gamma^{\mathbf{v}^*}(\log u_t) \circ X_t \, dt \\
&\quad + n a(t) u_t \left(b(t) - L \log u_t\right)^2 \circ X_t \, dt.
\end{aligned}$$

Y is then a submartingale from (4.1). The result follows from $\mathbb{E}[Y_T] \geq \mathbb{E}[Y_0]$. □
□

We look at some of the consequences of Lemma 4.3.

Corollary 4.4. *Assume that L satisfies* (CD) *with $\rho_2 > 0$, and that* (A) *and* (B) *also hold. Let $f \in C_b^\infty(M)$ be any bounded smooth function with $\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(f) \in C_b^\infty(M)$.*

(a) (Entropy bound) *Assume that $\rho_1 \geq 0$ and that $f > 0$. Then for any $x \in M$,*

$$\frac{1 - e^{-\rho_1 t}}{2\rho_1} \Gamma^{\mathbf{h}^*}(\log P_t f)(x) \leq \left(1 + \frac{2}{\rho_2}\right) P_t \left(\frac{f}{P_t f(x)} \log \frac{f}{P_t f(x)}\right)(x).$$

(b) (Li-Yau inequality) *Assume that $n < \infty$ in* (CD) *and that $f \geq 0$, not identically zero. Then for any $1 < \beta < 2$ and for any $t \geq 0$,*

$$\frac{\Gamma^{\mathbf{h}^*}(P_t f)}{(P_t f)^2} - (a_\beta - b_\beta \rho_1 t) \frac{P_t L f}{P_t f} \leq \frac{n}{4t} \left(\frac{a_\beta^2}{(2-\beta)(\beta-1)} - \rho_1 t (2a_\beta - b_\beta \rho_1 t)\right) \tag{4.2}$$

where $a_\beta = \dfrac{\rho_2 + \beta}{\rho_2}$ and $b_\beta = \dfrac{\beta - 1}{\beta}$.

The special case of $\beta = 2/3$ in (4.2) is described with consequences in [8, Theorem 6.1]. If $\rho_1 \geq 0$, then for many application $\beta = \sqrt{(2+\rho_2)(1+\rho_2)} - \rho_2$ is a better choice, as this minimizes the ratio of $a_\beta^2/(2-\beta)(\beta-1)$ over a_β. With this choice, we obtain relation

$$\frac{1}{D} \frac{\Gamma^{\mathbf{h}^*}(P_t f)}{(P_t f)^2} - \frac{P_t L f}{P_t f} \leq \frac{N}{t}, \tag{4.3}$$

where

$$N := \frac{n}{4} \frac{(\sqrt{2+\rho_2} + \sqrt{1+\rho_2})^2}{\rho_2}, \quad D = \frac{\sqrt{(2+\rho_2)(1+\rho_2)}}{\rho_2}. \tag{4.4}$$

Proof. Recall that if $f \in C_b^\infty(M)$ is non-negative and non-zero, then $P_t f$ is strictly positive.

(a) We will use Lemma 4.3. As in Proposition 3.6 (c), for any $T \geq 0$, define

$$a(t) = \frac{1 - e^{-\rho_1(T-t)}}{\rho_1}, \quad \ell(t) = \rho_{2,0}\frac{\int_t^T a(s)\,ds}{a(t)} = \rho_{2,0}\frac{e^{-\rho_1(T-t)} - 1 + \rho_1(T-t)}{\rho_1(1 - e^{-\rho_1(T-t)})}$$

and $C = 1 + 2/\rho_2$. If we define $b(t) \equiv 0$, condition (4.1) is satisfied. Hence,

$$\frac{1 - e^{-\rho_1 T}}{\rho_1}\frac{\Gamma^{\mathbf{h}^* + \frac{\rho_2 T}{2}\mathbf{v}^*}(P_T f)}{P_T f} \leq \left(1 + \frac{2}{\rho_2}\right)\big(P_T(f \log f) - (P_T f)\log P_T f\big).$$

Divide by $P_T f$ and evaluate at x for the result.

(b) For any $\varepsilon > 0$, define $f_\varepsilon = f + \varepsilon > 0$. For any $\alpha > 0$ and $T > 0$, define $\ell(t) = \frac{\rho_2}{\alpha+2}(T-t)$, $a(t) = (T-t)^{\alpha+1}$ and

$$b(t) = \frac{1}{2}\left(\rho_1 + \frac{\dot{a}}{a} - \frac{1}{\ell}\right) = \frac{1}{2}\left(\rho_1 - \left(\alpha + 1 + \frac{\alpha+2}{\rho_2}\right)\frac{1}{T-t}\right).$$

Note that

$$\int_0^T a(t)b(t)\,dt = \frac{1}{2}\left(\frac{\rho_1}{\alpha+2}T^{\alpha+2} - \left(1 + \frac{\alpha+2}{\rho_2(\alpha+1)}\right)T^{\alpha+1}\right),$$

$$\int_0^T a(t)b(t)^2\,dt = \frac{1}{4}\left(\frac{\rho_1^2}{\alpha+2}T^{\alpha+2} - 2\rho_1\left(1 + \frac{\alpha+2}{\rho_2(\alpha+1)}\right)T^{\alpha+1} + \frac{(\alpha+1)^2}{\alpha}\left(1 + \frac{\alpha+2}{\rho_2(\alpha+1)}\right)^2 T^\alpha\right).$$

If we put $C = 0$, then (4.1) is satisfied and so if we use f_ε in Lemma 4.3 and let $\varepsilon \downarrow 0$, we get

$$\frac{\Gamma^{\mathbf{h}^* + \frac{\rho_2 T}{\alpha+2}\mathbf{v}^*}(P_T f)}{P_T f} + \left(\frac{\rho_1}{\alpha+2}T - \left(1 + \frac{\alpha+2}{\rho_2(\alpha+1)}\right)\right)P_T L f$$
$$\leq \frac{n}{4}\left(\frac{\rho_1^2}{\alpha+2}T - 2\rho_1\left(1 + \frac{\alpha+2}{\rho_2(\alpha+1)}\right) + \frac{(\alpha+1)^2}{\alpha}\left(1 + \frac{\alpha+2}{\rho_2(\alpha+1)}\right)^2\frac{1}{T}\right)P_T f.$$

Define $\beta := (\alpha+2)/(\alpha+1)$ to obtain (4.2). □

□

Using (4.3) and the approach found in [8, Remark 6.2 and Section 7] and [7], we obtain the following results.

Corollary 4.5. *Assume that L satisfies* (CD) *relative to $\mathbf{v}^*$ with $\rho_1 \geq 0, \rho_2 > 0$ and $n < \infty$. Write $\mathbf{g}^* = \mathbf{h}^* + \mathbf{v}^*$. Also assume that* (A) *and* (B) *hold and that L is symmetric with respect to the volume form* vol. *Let $p_t(x,y)$ be the heat kernel of $\frac{1}{2}L$ with respect to* vol. *Finally, let N and D be as in (4.4). Then the following holds.*

(a) *$p_t(x,x) \leq t^{-N/2}p_1(x,x)$ for any $x \in M$.*

(b) *For any $0 < t_0 < t_1$ and any $f \in C_b^\infty(M)$ non-negative, not identically zero,*

$$P_{t_0} f(x) \leq (P_{t_1} f)(y)\left(\frac{t_1}{t_0}\right)^{N/2}\exp\left(D\,\frac{\mathsf{d}_{cc}(x,y)^2}{2(t_1 - t_0)}\right) \tag{4.5}$$

where d_{cc} *is the Carnot-Carathéodory distance. If* $\mathbf{g}^*$ *is the co-metric of a complete Riemannian metric, then*

$$p_{t_0}(x,y) \leq p_{t_1}(x,z) \left(\frac{t_1}{t_0}\right)^{N/2} \exp\left(D \,\frac{\mathsf{d}_{cc}(y,z)^2}{2(t_1-t_0)}\right).$$

There are several more results which we can obtain when (A) and (B) hold, along with the fact that L satisfies (CD) with $\rho_2 > 0$, which can be found in [8, 6, 7]. We list some of the most important results here, found in [8, Theorem 10.1] and [7, Theorem 1.5].

Theorem 4.6. *Let* L *be a second order operator satisfying* (CD) *with respect to* $\mathbf{v}^*$ *and with* $\rho_2 > 0$*. Assume that it is symmetric with respect to some volume form* vol*. Define* $\mathbf{g}^* = \mathbf{h}^* + \mathbf{v}^*$ *and assume that this is a complete Riemannian metric. Finally, assume that conditions* (A) *and* (B) *hold. Let* $B_r(x)$ *be the ball of radius* r *centered at* $x \in M$ *with respect to the metric* d_{cc}*.*

(a) (Sub-Riemannian Bonnet-Myers Theorem) *If* $\rho_1 > 0$*, then* M *is compact.*
(b) (Volume doubling property) *If* $\rho_1 \geq 0$*, there exist a constant* C *such that*

$$\mathrm{vol}(B_{2r}(x)) \leq C\,\mathrm{vol}(B_r(x)), \quad \textit{for any } r \geq 0.$$

(c) (Poincaré inequality on metric-balls) *If* $\rho_1 \geq 0$*, there exist a constant* C *such that*

$$\int_{B_r(x)} \|f - f_{B_r}\|^2 \,d\mathrm{vol} \leq Cr^2 \int_{B_r(x)} \Gamma^{\mathbf{h}^*}(f) d\,\mathrm{vol},$$

for any $r \geq 0$ *and* $f \in C^1\left(\bar{B}_r(x)\right)$ *where* $f_{B_r} = \mathrm{vol}(B_r(x))^{-1} \int_{B_r(x)} f\,d\mathrm{vol}.$

5. Examples and comments

5.1. **Results in the case of totally geodesic Riemannian foliations.** Let us consider the following case. Let $(M, \mathbf{g})$ be a Riemannian manifold, and let $\mathcal{H}$ be a subbundle that is bracket generating of step 2, i.e. the tangent bundle is spanned by the sections of $\mathcal{H}$ and their first order brackets. Let $\mathcal{V}$ be the orthogonal complement of $\mathcal{H}$ with respect to $\mathbf{g}$. Define $\mathring{\nabla}$ with respect to the decomposition $TM = \mathcal{H} \oplus \mathcal{V}$ and let $\mathbf{h}$ and $\mathbf{v}$ be the respective restrictions of $\mathbf{g}$ to $\mathcal{H}$ and $\mathcal{V}$. Let us make the following assumptions:

- $\mathcal{V}$ is integrable, $\mathbf{g}$ is complete, $\mathring{\nabla}\,\mathbf{g} = 0$ and the assumptions (i)–(iv) of Section 2 hold with $m_{\mathcal{R}} > 0$.

From our investigations so far, we then know that

- $\mathcal{V}$ is a metric-preserving complement of $(M, \mathcal{H}, \mathbf{h})$; the foliation of $\mathcal{V}$ is a totally geodesic Riemannian foliation.
- the sub-Laplacian $\Delta_{\mathbf{h}}$ of $\mathcal{V}$ is symmetric with respect to the volume form vol of $\mathbf{g}$;
- $\Delta_{\mathbf{h}}$ is essentially self-adjoint on $C_c^\infty(M)$;
- $\Delta_{\mathbf{h}}$ satisfies (CD) with respect to $\mathbf{v}^*$;
- both (A) and (B) hold.

We list the results that can be deduced on such manifolds using the approach of the generalized curvature-dimension inequality. We will split the results up into two propositions.

Proposition 5.1. *Define* $\kappa = \frac{1}{2} m_{\mathcal{R}}^2 \rho_{\mathcal{H}} - \mathscr{M}_{\mathcal{H}\mathcal{V}}^2$ *and assume that* $\kappa \geq 0$. *Let* $f \in C_b^\infty(M)$ *be non-negative, not identically zero. Define*

$$N = \frac{n}{4} \frac{\left(\sqrt{2\rho_{\mathcal{H}} + \kappa} + \sqrt{\rho_{\mathcal{H}} + \kappa}\right)^2}{\kappa}, \qquad D = \frac{\sqrt{(\kappa + \rho_{\mathcal{H}})(\kappa + 2\rho_{\mathcal{H}})}}{\kappa}.$$

(a) *Assume that* $\Gamma^{\mathbf{h}^* + \mathbf{v}^*}(f) \in C_b^\infty(M)$. *Then for any* $1 < \beta < 2$, *we have*

$$\frac{\Gamma^{\mathbf{h}^*}(P_t f)}{(P_t f)^2} - \left(1 + \frac{\rho_{\mathcal{H}}}{2\kappa}\beta\right) \frac{P_t L f}{P_t f} \leq \frac{n}{4t} \left(\frac{\left(1 + \frac{\rho_{\mathcal{H}}}{2\kappa}\beta\right)^2}{(2-\beta)(\beta - 1)} \right).$$

(b) *Let* $p_t(x,y)$ *be the heat kernel of* $\frac{1}{2}\Delta_{\mathbf{h}}$ *with respect to* vol. *Then*

$$p_t(x,x) \leq \frac{1}{t^{N/2}} p_1(x,x)$$

for any $x \in M$ *and* $0 \leq t \leq 1$. *Furthermore, for any* $0 < t_0 < t_1$,

$$P_{t_0} f(x) \leq (P_{t_1} f)(y) \left(\frac{t_1}{t_0}\right)^{N/2} \exp\left(D \frac{\mathsf{d}_{\mathrm{cc}}(x,y)^2}{2(t_1 - t_0)}\right).$$

In both results, if $\kappa = 0$, *we interpret the quotient* $\kappa/\rho_{\mathcal{H}}$ *as* $\frac{1}{2} m_{\mathcal{R}}^2$.

Note that if $\mathscr{M}_{\mathcal{H}\mathcal{V}} = 0$, the constant in the above result is independent of $\rho_{\mathcal{H}}$.

Proof. From the formulas (2.3), we know that $\Delta_{\mathbf{h}}$ satisfies (CD) with $\rho_2 > 0$ and $\rho_1 \geq 0$. In particular, we can choose $c = 1/\rho_{\mathcal{H}}$ if $\rho_{\mathcal{H}} > 0$ and ∞ if $\rho_{\mathcal{H}} = 0$. This choice gives us $\rho_1 = 0$, while maximizing ρ_2. Note that if $\rho_{\mathcal{H}} = 0$, then $\mathscr{M}_{\mathcal{H}\mathcal{V}}$ must be 0 as well, since we have required $\kappa \geq 0$. □ □

Example 5.2 (Free nilpotent Lie algebra of step 2). Let $\mathfrak{h}$ be a vector space of dimension n with an inner product $\langle \cdot, \cdot \rangle$ and let $\mathfrak{k}$ denote the vector space $\bigwedge^2 \mathfrak{h}$. Define a Lie algebra $\mathfrak{g}$ as the vector space $\mathfrak{h} \oplus \mathfrak{k}$ with Lie brackets determined by $\mathfrak{k}$ being the center and for any $A, B \in \mathfrak{h}$, we have

$$[A, B] = A \wedge B \in \mathfrak{k}, .$$

This is clearly a nilpotent Lie algebra of step 2 and dimension $n(n+1)/2$.

Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Define a sub-Riemannian structure $(\mathcal{H}, \mathbf{h})$ by left translation of $\mathfrak{h}$ and its inner product. Let $A_1, \dots, A_n$ be a left invariant orthonormal basis of $\mathcal{H}$ and define $L = \sum_{i=1}^n A_i^2$. From Part I, Example 4.4, we know that L satisfies (CD) with respect to some $\mathbf{v}^*$, $n = \operatorname{rank} \mathfrak{h}$, $\rho_1 = 0$ and $\rho_2 = \frac{1}{2(n-1)}$. This choice of $\mathbf{v}^*$ also gives us a complete Riemannian metric $\mathbf{g}$ satisfying $\mathring{\nabla} \mathbf{g} = 0$ and with L being the sub-Laplacian of the volume form of $\mathbf{g}$. We then obtain that for any $0 < t_0 < t_1$ and $f \in C_b^\infty(M)$

$$P_{t_0} f(x) \leq (P_{t_1} f)(y) \left(\frac{t_1}{t_0}\right)^{N/2} \exp\left(D \frac{\mathsf{d}_{\mathrm{cc}}(x,y)^2}{2(t_1 - t_0)}\right)$$

where $N = \frac{n}{4}\left(\sqrt{4n-3} + \sqrt{2n-1}\right)^2$ and $D = \sqrt{(2n-1)(4n-3)}$.

Proposition 5.3. *Define* $\kappa = \frac{1}{2} m_{\mathcal{R}}^2 \rho_{\mathcal{H}} - \mathscr{M}_{\mathcal{H}\mathcal{V}}^2$ *and assume that* $\kappa > 0$. *Then the following statements hold.*

(a) *M is compact.*

(b) *If $f \in C^\infty(M)$ is an arbitrary function and*

$$\alpha := \left(\frac{2\kappa}{2\mathscr{M}_{\mathcal{H}\mathcal{V}} + m_{\mathcal{R}} \sqrt{2\rho_{\mathcal{H}} + 2\kappa}} \right)^2,$$

we have

$$\|\Gamma^{\mathbf{h}^*}(P_t f)\|_{L^1} \leq e^{-\alpha t} \|\Gamma^{\mathbf{h}^*}(f)\|_{L^1}, \quad \text{and} \quad \|f - f_M\|_{L^2}^2 \leq \frac{1}{\alpha} \int_M \Gamma^{\mathbf{h}^*}(f) \, d\mathrm{vol}$$

where $f_M = \mathrm{vol}(M)^{-1} \int_M f \, d\mathrm{vol}$.

(c) *Let $f \in C^\infty(M)$ be an arbitrary function. Then*

$$t\Gamma^{\mathbf{h}^*}(P_t f) \leq \left(1 + \frac{2\rho_{\mathcal{H}}}{\kappa}\right)(P_t f^2 - (P_t f)^2).$$

(d) *Let f be a strictly positive smooth function. Then for any $x \in M$,*

$$t\Gamma^{\mathbf{h}^*}(\log P_t f)(x) \leq 2\left(1 + \frac{2\rho_{\mathcal{H}}}{\kappa}\right) P_t \left(\frac{f}{f(x)} \log \frac{f}{f(x)} \right)(x).$$

Proof. From the formulas (2.3), we know that $\Delta_{\mathbf{h}}$ satisfies (CD) with $\rho_2 > 0$ and $\rho_1 > 0$.

(a) Follows directly from Theorem (4.6) (a).

(b) We use Propositions 3.6 (a) and 3.8 (b). With our assumption of $\mathring{\nabla}\mathbf{v} = 0$, the formulas (2.2) show that we can choose $\rho_{2,1} = 0$ and both ρ_1 and ρ_2 strictly positive, since $\kappa > 0$. The result follows by maximizing $\frac{\rho_2 \rho_1}{\rho_2 + 1}$ with respect to c.

(c) We use Proposition 3.6 (c) and using (2.3) with $c = 1/\rho_{\mathcal{H}}$.

(d) Similar to the proof of (c), only using Corollary 4.4 (a) instead. □

□

Example 5.4. Let $\mathfrak{g}$ be a compact semisimple Lie algebra with bi-invariant metric

$$\langle A, B \rangle = -\frac{1}{4\rho} \operatorname{tr} \operatorname{ad}(A) \operatorname{ad}(B), \quad \rho > 0.$$

Let G be a (compact) Lie group with Lie algebra G and with metric $\breve{\mathbf{g}}$ given by left (or right) translation of the above inner product. Then $\rho > 0$ is the lower Ricci bound of G.

Let $\mathfrak{h}$ be the subspace of the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ consisting of elements on the form $(A, 2A)$, $A \in \mathfrak{g}$. Define the subbundle $\mathcal{H}$ on $G \times G$ by left translation of $\mathfrak{h}$. If we use the same symbol for an element in the Lie algebra and the corresponding left invariant vector field, we define a metric $\mathbf{h}$ on $\mathcal{H}$ by

$$\mathbf{h}((A, 2A), (A, 2A)) = \langle A, A \rangle.$$

Define $\pi : G \times G \to G$ as projection on the second coordinate with vertical bundle $\mathcal{V} = \ker \pi_*$ and give this bundle a metric $\mathbf{v}$ determined by

$$\|(A, 0)\|_{\mathbf{v}}^2 = \frac{1}{4\rho} \langle A, A \rangle.$$

If we then define $\mathring{\nabla}$ relative to $\mathcal{H} \oplus \mathcal{V}$ and $\mathbf{g} = \mathrm{pr}_{\mathcal{H}}^* \mathbf{h} + \mathrm{pr}_{\mathcal{V}}^* \mathbf{v}$, then $\mathring{\nabla} \mathbf{g} = 0$. Let $\Delta_{\mathbf{h}}$ be the sub-Laplacian with respect to $\mathcal{V}$, which coincides with the sub-Laplacian of the volume form vol of $\mathbf{g}$. We showed in Part I, Example 4.6 that this satisfies (CD) with respect to $\mathbf{v}^*$, $n = \dim G$, $\rho_1 = \rho_{\mathcal{H}} = 4\rho$ and $\rho_2 = \frac{1}{2} m_{\mathcal{R}}^2 = 1/4$.

We then have that for any $f \in C^\infty(G \times G)$,

$$\|f - f_{G\times G}\|_{L^2}^2 = \frac{5}{4\rho} \int_M \Gamma^{\mathbf{h}^*}(f) \, d\mathrm{vol}$$

where $f_{G\times G} = \mathrm{vol}(G \times G)^{-1} \int_{G \times G} f \, d\mathrm{vol}$.

5.2. **Comparison to Riemannian Ricci curvature.** Let us consider a sub-Riemannian manifold such as in Section 5.1. Given the results of Proposition 5.1 and Proposition 5.3, it seems reasonable to consider sub-Riemannian manifolds with $\kappa \geq 0$ or $\kappa > 0$ as the analogue of Riemannian manifolds with respectively non-negative and positive Ricci curvature. However, given the extra structure in the choice of $\mathbf{v}$ on $\mathcal{V}$, it is natural to ask how these sub-Riemannian results compare to the Ricci curvature of the metric $\mathbf{g} = \mathrm{pr}_{\mathcal{H}}^* \mathbf{h} + \mathrm{pr}_{\mathcal{V}}^* \mathbf{v}$. We give the comparison here.

Introduce the following symmetric 2-tensor

$$\mathrm{Ric}_{\mathcal{V}}(Y,Z) = \mathrm{tr} \left(V \mapsto \mathrm{pr}_{\mathcal{V}} R^{\mathring{\nabla}}(V,Y)Z \right).$$

Then the Ricci curvature of $\mathbf{g}$ can be written in the following way.

Proposition 5.5. *The Ricci curvature* $\mathrm{Ric}_{\mathbf{g}}$ *of* $\mathbf{g}$ *satisfies*

$$\begin{aligned}\mathrm{Ric}_{\mathbf{g}}(Y,Y) &= \mathrm{Ric}_{\mathcal{H}}(Y,Y) + \mathrm{Ric}_{\mathcal{H}\mathcal{V}}(Y,Y) + \frac{1}{2} \| \mathbf{g}(Y, \mathcal{R}(\,\cdot\,,\,\cdot\,)) \|_{\wedge^2 \mathbf{g}^*}^2 \\ &\quad + \mathrm{Ric}_{\mathcal{V}}(Y,Y) - \frac{3}{4} \| \mathcal{R}(Y,\,\cdot\,) \|_{\mathbf{g}^* \otimes \mathbf{g}}^2. \end{aligned} \tag{5.1}$$

Before we get to the proof, let us note the consequences of this result. If $\kappa = \frac{1}{2} \rho_{\mathcal{H}} m_{\mathcal{R}}^2 - \mathscr{M}_{\mathcal{H}\mathcal{V}}^2$ is respectively non-negative or positive, this ensures that the first line of (5.1) has respectively a non-negative or positive lower bound. Furthermore, note that this part is independent of any covariant derivative of vertical vector fields.

of Proposition 5.5. Let ∇ be the Levi-Civita connection of $\mathbf{g}$. Define a two tensor $\mathcal{B}(A,Z) = \nabla_A Z - \mathring{\nabla}_A Z$. Then it is clear that

$$\begin{aligned} R^{\nabla}(A,Y)Z &= R^{\mathring{\nabla}}(A,Y)Z + (\mathring{\nabla}_A \mathcal{B})(Y,Z) - (\mathring{\nabla}_Y \mathcal{B})(A,Z) \\ &\quad + \mathcal{B}(\mathcal{B}(Y,A),Z) + \mathcal{B}(A, \mathcal{B}(Y,Z)) - \mathcal{B}(\mathcal{B}(A,Y),Z) - \mathcal{B}(Y, \mathcal{B}(A,Z)). \end{aligned}$$

Furthermore, it is simple to verify that

$$\mathcal{B}(A,Z) = \frac{1}{2} \mathcal{R}(A,Z) - \frac{1}{2} \sharp \mathbf{g}(A, \mathcal{R}(Z,\,\cdot\,)) - \frac{1}{2} \sharp \mathbf{g}(Z, \mathcal{R}(A,\,\cdot\,)).$$

Let $A_1, \dots, A_n$ and $V_1, \dots, V_\nu$ be local orthonormal bases of respectively $\mathcal{H}$ and $\mathcal{V}$. Then

$$\begin{aligned} &\sum_{i=1}^n \mathbf{g}(A_i, R^{\nabla}(A_i,Z)Z - R^{\mathring{\nabla}}(A_i,Z)Z) \\ &= \sum_{i=1}^n \mathbf{g}(Z, (\mathring{\nabla}_{A_i} \mathcal{R})(A_i,Z)) - \frac{3}{4} \| \mathcal{R}(Z,\,\cdot\,) \|_{\mathbf{g}^* \otimes \mathbf{g}}^2 + \frac{1}{2} \| \mathbf{g}(Z, \mathcal{R}(\,\cdot\,,\,\cdot\,)) \|_{\wedge^2 \mathbf{g}^*}^2. \end{aligned}$$

Similarly, $\sum_{s=1}^{\nu} \mathbf{g}(R^{\nabla}(V_s, Z)Z - R^{\mathring{\nabla}}(V_s, Z)Z, V_s) = 0$. □ □

5.3. **Generalizations to equiregular submanifolds of steps greater than two.** Many of the results in Section 3 and Section 4 depend on the condition $\rho_2 > 0$. A necessary condition for this to hold is that our sub-Riemannian manifold is bracket-generating of step 2. Let us note some of the difficulties in generalizing the approach of this paper to sub-Riemannian manifolds $(M, \mathcal{H}, \mathbf{h})$ of higher steps.

As usual, we require $\mathcal{H}$ bracket-generating. Assume also, for the sake of simplicity, that $\mathcal{H}$ is equiregular, i.e. there exists a flag of subbundles $\mathcal{H} = \mathcal{H}^1 \subseteq \mathcal{H}^2 \subseteq \mathcal{H}^3 \subseteq \cdots \subseteq \mathcal{H}^r$ such that

$$\mathcal{H}^{k+1}_x = \operatorname{span}\left\{ Z|_x, [A, Z]|_x : \, Z \in \Gamma(\mathcal{H}^k), \; A \in \Gamma(\mathcal{H}) \right\}, \quad x \in M.$$

Choose a metric tensor $\mathbf{v}$ on $\mathcal{V}$ and let $\mathbf{g} = \operatorname{pr}_{\mathcal{H}}^* \mathbf{h} + \operatorname{pr}_{\mathcal{V}}^* \mathbf{v}$ be the corresponding Riemannian metric. Let $\mathcal{V}_k$ be the orthogonal complement of $\mathcal{H}^k$ in $\mathcal{H}^{k+1}$. Let $\operatorname{pr}_{\mathcal{V}_k}$ be the projection to $\mathcal{V}_k$ relative to the splitting $\mathcal{H} \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_{r-1}$. Define $\mathbf{v}_k = \mathbf{v}|_{\mathcal{V}_k}$ and let $\mathbf{v}_k^*$ be the corresponding co-metric. We could attempt to construct a curvature-dimension inequality with $\mathsf{\Gamma}^{\mathbf{h}^*}(f), \mathsf{\Gamma}^{\mathbf{v}_1^*}(f), \dots, \mathsf{\Gamma}^{\mathbf{v}_{r-1}^*}(f)$. However, a condition similar to (B) could never hold in this case, i.e. $\mathsf{\Gamma}^{\mathbf{h}^*}(f, \mathsf{\Gamma}^{\mathbf{v}_k^*}(f)) = \mathsf{\Gamma}^{\mathbf{v}_k^*}(f, \mathsf{\Gamma}^{\mathbf{h}^*}(f))$ cannot hold for any $k \leq r-2$.

To see this let α and β be forms that only are non-vanishing on respectively $\mathcal{V}_k$ and $\mathcal{V}_{k+1}$ for $k \leq r-2$. Then $\mathsf{\Gamma}^{\mathbf{h}^*}(f, \mathsf{\Gamma}^{\mathbf{v}_k^*}(f)) = \mathsf{\Gamma}^{\mathbf{v}_k^*}(f, \mathsf{\Gamma}^{\mathbf{h}^*}(f))$ holds if and only if $\mathring{\nabla}\mathbf{h}^* = 0$ and $\mathring{\nabla}_A \mathbf{v}_k^* = 0$ for any $A \in \Gamma(\mathcal{H})$. Hence we obtain

$$0 = (\mathring{\nabla}_A \mathbf{v}_k^*)(\alpha, \beta) = \beta([A, \sharp^{\mathbf{v}_k^*}\alpha]).$$

However, this is a contradiction, since by our construction, $\mathcal{V}_{k+1}$ must be spanned by orthogonal projections of brackets on the form $[A, Z]$, $A \in \Gamma(\mathcal{H})$, $Z \in \Gamma(\mathcal{V}_k)$.

Appendix A. Graded analysis on forms

A.1. **Graded analysis on forms.** Let $(M, \mathcal{H}, \mathbf{h})$ be a sub-Riemannian manifold with an integrable complement $\mathcal{V}$, and let $\mathbf{v}$ be a chosen positive definite metric tensor on $\mathcal{V}$. Let $\mathbf{g} = \operatorname{pr}_{\mathcal{H}}^* \mathbf{h} + \operatorname{pr}_{\mathcal{V}}^* \mathbf{v}$ be the corresponding Riemannian metric. The subbundle $\mathcal{V}$ gives us a foliation of M, and corresponding to this foliation we have a grading on forms, see e.g. [2, 1]. Let $\Omega(M)$ be the algebra of differential forms on M. Let $\operatorname{Ann}(\mathcal{H})$ and $\operatorname{Ann}(\mathcal{V})$ be the subbundles of T^*M of elements vanishing on respectively $\mathcal{H}$ and $\mathcal{V}$. If either a or b is a negative integer, then $\eta \in \Omega(M)$ is a homogeneous element of degree (a, b) if and only if $\eta = 0$. Otherwise, for nonnegative integers a and b, η is a homogeneous element of degree (a, b), if it is a sum of elements which can be written as

$$\alpha \wedge \beta, \quad \alpha \in \Gamma(\textstyle\bigwedge^a \operatorname{Ann}(\mathcal{V})), \; \beta \in \Gamma(\textstyle\bigwedge^b \operatorname{Ann}(\mathcal{H})).$$

Relative to this grading, we can split the exterior differential d into graded components

$$d = d^{1,0} + d^{0,1} + d^{2,-1}.$$

The same is true for its formal dual

$$\delta = \delta^{-1,0} + \delta^{0,-1} + \delta^{-2,1},$$

i.e. the dual with respect to the inner product on forms of compact support α, β, defined by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star \beta, \qquad \alpha, \beta \text{ of compact support,} \tag{A.1}$$

where $\star$ is the Hodge star operator defined relative to $\mathbf{g}$. Note that $\delta^{-a,-b}$ is the formal dual of $d^{a,b}$ from our assumptions that $\mathcal{H}$ and $\mathcal{V}$ are orthogonal. We will give formulas for each graded component.

A.2. **Metric-preserving complement and local representation.** We will use $\flat : TM \to T^*M$ for the map $v \mapsto \mathbf{g}(v, \cdot)$ with inverse $\sharp$. Let $\mathring{\nabla}$ be defined as in (2.1) relative to $\mathbf{g}$ and the splitting $TM = \mathcal{H} \oplus \mathcal{V}$. If α is a one-form and $A_1, \dots, A_n$ and $V_1, \dots, V_\nu$ are respective local orthonormal bases of $\mathcal{H}$ and $\mathcal{V}$, then locally

$$d\alpha = \sum_{i=1}^n \flat A_i \wedge \mathring{\nabla}_{A_i} \alpha + \sum_{s=1}^\nu \flat V_s \wedge \mathring{\nabla}_{V_s} \alpha - \alpha \circ \mathcal{R},$$

and hence each of the three terms are local representations of respectively $d^{1,0}\alpha$, $d^{0,1}\alpha$ and $d^{2,-1}\alpha$. Local representations on forms of all orders follow.

Assume now that $\mathring{\nabla} \mathbf{g} = 0$, i.e. $\mathcal{V}$ is a metric-preserving compliment of $(\mathcal{H}, \mathbf{h})$ with a metric tensor $\mathbf{v}$ satisfying $\mathring{\nabla} \mathbf{v}^* = 0$. From the formula of $d^{1,0}\eta = \sum_{i=1}^n \flat A_i \wedge \mathring{\nabla}_{A_i} \eta$, we obtain $\delta^{-1,0}\eta = -\sum_{i=1}^n \iota_{A_i} \mathring{\nabla}_{A_i} \eta$ for any form η. Let $\Delta_{\mathbf{h}}$ be the sub-Laplacian of $\mathcal{V}$ or equivalently vol. Let Δ be the Laplacian of $\mathbf{g}$. Then it is clear that for any $f \in C^\infty(M)$, we have

$$\Delta f = -\delta d f, \qquad \Delta_{\mathbf{h}} f = -\delta^{-1,0} d^{1,0} f.$$

Lemma A.1. *For any form $\eta \in \Omega(M)$, we have*

$$\delta^{-1,0} d^{0,1} \alpha = -d^{0,1} \delta^{-1,0} \alpha, \quad \delta^{0,-1} d^{1,0} \alpha = -d^{1,0} \delta^{0,-1} \alpha. \tag{A.2}$$

As a consequence, for any $f \in C^\infty(M)$ we have $\Delta_{\mathbf{h}} \Delta f = \Delta \Delta_{\mathbf{h}} f$.

The following result is helpful for our computation in Part I, Lemma 3.3 (b) and Corollary 3.11.

Lemma A.2. (a) *For any horizontal $A \in \Gamma(\mathcal{H})$, a vertical $V \in \Gamma(\mathcal{V})$ and arbitrary vector filed $Z \in \Gamma(TM)$, we have*

$$\mathbf{g}(R^{\mathring{\nabla}}(A, V)Z, A) = 0.$$

(b) *If $\mathring{\nabla} \mathbf{g} = 0$, then for every point x_0, there exist local orthonormal bases $A_1, \dots, A_n$ and $V_1, \dots V_\nu$, defined in a neighborhood of x_0, such that for any $Y \in \Gamma(TM)$,*

$$\mathring{\nabla}_Z A_i |_{x_0} = \frac{1}{2} \sharp\, \mathbf{g}(Z, \mathcal{R}(A_i, \cdot))|_{x_0}, \qquad \mathring{\nabla}_Z V_s |_{x_0} = 0.$$

of Lemma A.1. It is sufficient to show one of the identities in (A.2), since $\delta^{-1,0} d^{0,1}$ is the formal dual of $\delta^{0,-1} d^{1,0}$. From Lemma A.2 (a), any $A \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$ satisfy

$$\iota_A \mathring{\nabla}_V \mathring{\nabla}_A \alpha = \iota_A \mathring{\nabla}_A \mathring{\nabla}_V \alpha + \iota_A \mathring{\nabla}_{[V,A]} \alpha.$$

From the definition of $\mathring{\nabla}$, it follows that $T^{\mathring{\nabla}}(A, V) = 0$, where $T^{\mathring{\nabla}}$ is the torsion of $\mathring{\nabla}$. For a given point $x_0 \in M$, let $A_1, \dots, A_n$ and $V_1, \dots, V_\nu$ be as in Lemma A.2 (b).

All terms below are evaluated at the point x_0, giving us

$$\begin{aligned}
d^{0,1}\delta^{-1,0}\alpha &= -\sum_{s=1}^{\nu}\sum_{i=1}^{n} \flat V_s \wedge \mathring{\nabla}_{V_s}\iota_{A_i}\mathring{\nabla}_{A_i}\alpha \\
&= -\sum_{s=1}^{\nu}\sum_{i=1}^{n} \flat V_s \wedge \iota_{\mathring{\nabla}_{V_s}A_i}\mathring{\nabla}_{A_i}\alpha - \sum_{s=1}^{\nu}\sum_{i=1}^{n} \flat V_s \wedge \iota_{A_i}\mathring{\nabla}_{V_s}\mathring{\nabla}_{A_i}\alpha \\
&= -\frac{1}{2}\sum_{s=1}^{\nu}\sum_{i,j=1}^{n} \mathbf{g}(V_s, \mathcal{R}(A_i, A_j))\flat V_s \wedge \iota_{A_j}\mathring{\nabla}_{A_i}\alpha \\
&\quad - \sum_{s=1}^{\nu}\sum_{i=1}^{n} \flat V_s \wedge \iota_{A_i}\mathring{\nabla}_{A_i}\mathring{\nabla}_{V_s}\alpha - \sum_{s=1}^{\nu}\sum_{i=1}^{n} \flat V_s \wedge \iota_{A_i}\mathring{\nabla}_{\mathring{\nabla}_{V_s}A_i - \mathring{\nabla}_{A_i}V_s}\alpha \\
&= \sum_{s=1}^{\nu}\sum_{i,j=1}^{n} \iota_{A_i}\left(\flat V_s \wedge \mathring{\nabla}_{A_i}\mathring{\nabla}_{V_s}\right)\alpha \\
&= \sum_{s=1}^{\nu}\sum_{i,j=1}^{n} \iota_{A_i}\mathring{\nabla}_{A_i}\left(\flat V_s \wedge \mathring{\nabla}_{V_s}\right)\alpha = -\delta^{-1,0}d^{1,0}\alpha.
\end{aligned}$$

Next, we prove the identity $[\Delta_{\mathbf{h}}, \Delta]f = 0$. If we consider the degree $(1,1)$-part of $d^2 = 0$, we get

$$d^{0,1}d^{1,0} + d^{1,0}d^{0,1} = 0.$$

The same relation will then hold for their formal duals. Since $\Delta f = \Delta_{\mathbf{h}} f - \delta^{0,1}d^{0,1}f$, it is sufficient to show that $\Delta_{\mathbf{h}}\delta^{0,-1}d^{0,1}f = \delta^{0,-1}d^{0,1}\Delta_{\mathbf{h}}f$. This gives us the result

$$\Delta_{\mathbf{h}}\delta^{0,-1}d^{0,1}f = -\delta^{-1,0}d^{1,0}\delta^{0,-1}d^{0,1}f = -\delta^{0,-1}d^{0,1}\delta^{-1,0}d^{1,0}f = \delta^{0,-1}d^{0,1}\Delta_{\mathbf{h}}f,$$

since we have to do an even number of permutations. □ □

A.3. **Spectral theory of the sub-Laplacian.** Let L be a self-adjoint operator on $L^2(M, \mathrm{vol})$ with domain $\mathrm{Dom}(L)$. Define $\|f\|^2_{\mathrm{Dom}(L)} = \|f\|^2_{L^2} + \|Lf\|^2_{L^2}$. Write the spectral decomposition of L as $L = \int_{-\infty}^{\infty} \lambda dE_\lambda$ with respect to the corresponding projector valued spectral measure E_λ. For any Borel measurable function $\varphi\colon \mathbb{R} \to \mathbb{R}$, we write $\varphi(L)$ for the operator $\varphi(L) := \int_{-\infty}^{\infty} \varphi(\lambda) dE_\lambda$ which is self adjoint on its domain

$$\mathrm{Dom}(\varphi(L)) = \left\{ f \in L^2(M, \mathrm{vol}) : \int_{-\infty}^{\infty} \varphi(\lambda)^2 d\langle E_\lambda f, f\rangle \right\}.$$

In particular, if φ is bounded, $\varphi(L)$ is defined on the entire of $L^2(M, \mathrm{vol})$. See [15, Ch VIII.3] for details.

Let $(M, \mathcal{H}, \mathbf{h})$ be a sub-Riemannian manifold with sub-Laplacian $\Delta_{\mathbf{h}}$ defined relative to a volume form vol. Assume that $\mathcal{H}$ is bracket-generating and that (M, d_{cc}) is complete metric space, where d_{cc} is the Carnot-Carathéodory metric of $(\mathcal{H}, \mathbf{h})$. Then

$$\int_M f\Delta_{\mathbf{h}} g \; d\mathrm{vol} = \int_M g\Delta_{\mathbf{h}} f \; d\mathrm{vol} \quad \text{and} \quad \int f\Delta_{\mathbf{h}} f \, d\mathrm{vol} \leq 0.$$

From [17, Section 12], we have that $\Delta_{\mathbf{h}}$ is a an essentially self adjoint operator on $C_c^\infty(M)$. We denote its unique self-adjoint extension by $\Delta_{\mathbf{h}}$ as well with domain $\mathrm{Dom}(\Delta_{\mathbf{h}}) \subseteq L^2(M, \mathrm{vol})$.

Since $\Delta_{\mathbf{h}}$ is non-positive and the maps $\lambda \mapsto e^{t\lambda/2}$ and $\lambda \mapsto \lambda^j e^{t\lambda/2}$ are bounded on $(-\infty, 0]$ for $t > 0$, $j > 0$, we have that $f \mapsto e^{t/2\Delta_{\mathbf{h}}} f$ is a map from $L^2(M, \mathrm{vol})$ into $\cap_{j=1}^{\infty} \mathrm{Dom}(\Delta_{\mathbf{h}}^j)$. Define $P_t f$ as in Section 3.1 with respect to $\frac{1}{2}\Delta_{\mathbf{h}}$-diffusions for bounded measurable functions f. Then clearly $\|P_t f\|_{L^\infty} \leq \|f\|_{L^\infty}$. Since $\Delta_{\mathbf{h}}$ is symmetric with respect to vol and $P_t 1 \leq 1$, we obtain $\|P_t f\|_{L^1} \leq \|f\|_{L^1}$ as well. The Riesz-Thorin theorem then ensures that $\|P_t f\|_{L^p} \leq \|f\|_{L^p}$ for any $1 \leq p \leq \infty$. In particular, $P_t f$ is in $L^2(M, \mathrm{vol})$ whenever f is in $L^2(M, \mathrm{vol})$. This implies that $P_t f = e^{t/2\Delta_{\mathbf{h}}} f$ for any bounded $f \in L^2(M, \mathrm{vol})$ by the following result.

Lemma A.3 ([14, Prop], [8, Prop 4.1]). *Let L be equal to the Laplacian Δ or sub-Laplacian $\Delta_{\mathbf{h}}$ defined relative to a complete Riemannian or sub-Riemannian metric, respectively. Let $u_t(x)$ be a solution in $L^2(M, \mathrm{vol})$ of the heat equation*

$$(\partial_t - L)u_t = 0, \quad u_0 = f,$$

for a function $f \in L^2(M, \mathrm{vol})$. Then $u_t(x)$ is the unique solution of this equation in $L^2(M, \mathrm{vol})$.

Hence, we will from now on just write $P_t = e^{t/2\Delta_{\mathbf{h}}}$ without much abuse of notation.

A.3.1. *Global bounds using spectral theory.* We now introduce some additional assumptions. Assume that $\mathbf{g}$ is a complete Riemannian metric with volume form vol, such that $\mathbf{g}\,|_{\mathcal{H}} = \mathbf{h}, \mathcal{H}^\perp = \mathcal{V}$ and $\mathbf{g}\,|_{\mathcal{V}} = \mathbf{v}$. Let Δ be the Laplace-Beltrami operator of $\mathbf{g}$ and write $\Delta f = \Delta_{\mathbf{h}} f + \Delta_{\mathbf{v}} f$ where $\Delta_{\mathbf{v}} f = \mathrm{div}\, \sharp^{\mathbf{v}^*} df$. Since $\mathbf{g}$ is complete, Δ is also essentially self-adjoint on $C_c^\infty(M)$ by [16] and we will also denote its unique self-adjoint extension by the same symbol.

Assume that $\mathring{\nabla} \mathbf{g} = 0$ where $\mathring{\nabla}$ is defined as in (2.1). Recall that $\Delta_{\mathbf{h}}$ and Δ commute on $C_c^\infty(M)$ by Lemma A.1.

Lemma A.4.

(a) *The operators $\Delta_{\mathbf{h}}$ and Δ spectrally commute, i.e. for any bounded Borel function $\varphi : \mathbb{R} \to \mathbb{R}$ and $f \in L^2(M, \mathrm{vol})$,*

$$\varphi(\Delta_{\mathbf{h}})\varphi(\Delta) f = \varphi(\Delta)\varphi(\Delta_{\mathbf{h}}) f.$$

Also $\mathrm{Dom}(\Delta) \subseteq \mathrm{Dom}(\Delta_{\mathbf{h}})$.

(b) *Assume that $\Delta_{\mathbf{h}}$ satisfies the assumptions of Theorem 2.2 with $m_{\mathcal{R}} > 0$. Then there exist a constant $C = C(\rho_1, \rho_2)$ such that for any $f \in C^\infty(M) \cap \mathrm{Dom}(\Delta_{\mathbf{h}}^2)$,*

$$C\|f\|^2_{\mathrm{Dom}(\Delta_{\mathbf{h}}^2)} = C\left(\|f\|^2_{L^2} + \|\Delta_{\mathbf{h}}^2 f\|^2_{L^2}\right),$$

is an upper bound for

$$\int_M \Gamma^{\mathbf{h}^*}(f)\, d\mathrm{vol}, \quad \int_M \Gamma_2^{\mathbf{h}^*}(f)\, d\mathrm{vol}, \quad \int_M \Gamma^{\mathbf{v}^*}(f)\, d\mathrm{vol} \quad \text{and} \quad \int_M \Gamma_2^{\mathbf{v}^*}(f)\, d\mathrm{vol}.$$

Proof. (a) Note first that for any $f \in C_c^\infty(M)$, using Lemma A.1 and the inner product (A.1)

$$\int_M \Delta_{\mathbf{v}} f \Delta_{\mathbf{h}} f\, d\mathrm{vol} = \langle \delta^{0,-1} d^{0,1} f, \delta^{-1,0} d^{1,0} f\rangle = \langle d^{0,1} f, d^{0,1}\delta^{-1,0} d^{1,0} f\rangle$$
$$= -\langle d^{0,1} f, \delta^{-1,0} d^{0,1} d^{1,0} f\rangle = \langle d^{1,0} d^{0,1} f, d^{1,0} d^{0,1} f\rangle \geq 0.$$

Hence

$$\int_M (\Delta_{\mathbf{h}} f)^2 \, d\mathrm{vol} \leq \int_M ((\Delta_{\mathbf{h}} + \Delta_{\mathbf{v}}) f)^2 \, d\mathrm{vol} = \int_M (\Delta f)^2 \, d\mathrm{vol},$$

and hence $\|\Delta_{\mathbf{h}} f\|_{L^2} \leq \|\Delta f\|_{L^2}$ is true for any $f \in \mathrm{Dom}(\Delta)$. We conclude that $\mathrm{Dom}(\Delta) \subseteq \mathrm{Dom}(\Delta_{\mathbf{h}})$. Define $Q_t = e^{t/2\Delta}$. It follows that, for any $f \in \mathrm{Dom}(\Delta_{\mathbf{h}})$, $u_t = \Delta_{\mathbf{h}} Q_t f$ is an $L^2(M, \mathrm{vol})$ solution of

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \Delta \right) u_t = 0, \quad u_0 = \Delta_{\mathbf{h}} f.$$

In conclusion, by Lemma A.3 we obtain $\Delta_{\mathbf{h}} Q_t f = Q_t \Delta_{\mathbf{h}} f$.

For any $s > 0$ and $f \in L^2(M, \mathrm{vol})$, we know that $Q_s f \in \mathrm{Dom}(\Delta) \subseteq \mathrm{Dom}(\Delta_{\mathbf{h}})$, and since

$$(\partial_t - \frac{1}{2} \Delta_{\mathbf{h}}) Q_s P_t f = 0,$$

it again follows from Lemma A.3 that $P_t Q_s f = Q_s P_t f$ for any $s, t \geq 0$ and $f \in L^2(M, \mathrm{vol})$. It follows that the operators spectrally commute, see [15, Chapter VIII.5].

(b) From Theorem 2.2, we know that $\Delta_{\mathbf{h}}$ satisfies (CD) with $\rho_2 > 0$ and an appropriately chosen value of c. The proof is otherwise identical to [8, Lemma 3.4 & Prop 3.6] and is therefore omitted. □

□

A.3.2. *Proof of Theorem* 3.4. We are going to prove that (A) holds without using stochastic analysis. We therefore need the following lemma.

Lemma A.5 ([8, Prop 4.2]). *Assume that $(M, \mathbf{g})$ is a complete Riemannian manifold. For any $T > 0$, let $u, v \in C^\infty(M \times [0, T])$, $(x, t) \mapsto u_t(x)$, $(x, t) \mapsto v_t(x)$ be smooth functions satisfying the following conditions:*

(i) *For any $t \in [0, T]$, $u_t \in L^2(M, \mathrm{vol})$ and $\int_0^T \|u_t\|_{L^2} dt < \infty$.*
(ii) *For some $1 \leq p \leq \infty$, $\int_0^T \|\Gamma^{\mathbf{h}^*}(u_t)^{1/2}\|_{L^p} \, d\mathrm{vol} < \infty$.*
(iii) *For any $t \in [0, T]$, $v_t \in L^q(M, \mathrm{vol})$ and $\int_0^T \|v_t\|_{L^q} dt < \infty$ for some $1 \leq q \leq \infty$.*

Then, if $(L + \frac{\partial}{\partial t}) u \geq v$ holds on $M \times [0, T]$, we have

$$P_T u_T \geq u_0 + \int_0^t P_t v_t dt.$$

Let $P_t = e^{t/2\Delta_{\mathbf{h}}}$. For given compactly supported $f \in C_c^\infty(M)$ and $T > 0$, define function

$$z_{t,\varepsilon} = \left(\Gamma^{\mathbf{v}^*}(P_{T-t} f) + \varepsilon^2 \right)^{1/2} - \varepsilon, \tag{A.3}$$

with $\varepsilon > 0, t \in [0, T]$. Since $P_t f \in \mathrm{Dom}(\Delta_{\mathbf{h}}^2)$, Lemma A.4 (b) tells us that,

$$\|z_{t,\varepsilon}\|_{L^2} \leq \int_M \Gamma^{\mathbf{v}^*}(P_{T-t}) \, \mathrm{vol} \leq C \|P_{T-t} f\|_{\mathrm{Dom}(\Delta_{\mathbf{h}}^2)} < \infty,$$

so that $z_{t,\varepsilon} \in L^2(M, \mathrm{vol})$. By Proposition 2.3,

$$\Gamma^{\mathbf{h}^*}(z_{t,\varepsilon}) \leq \frac{\Gamma^{\mathbf{h}^*}(\Gamma^{\mathbf{v}^*}(P_{T-t} f))}{4 z_{t,\varepsilon} + \varepsilon} \leq \Gamma_2^{\mathbf{v}^*}(P_{T-t} f). \tag{A.4}$$

From Lemma A.4 (b) it follows that both (i) and (ii) of Lemma A.5 is satisfied. Hence, using that from Proposition 2.3

$$\begin{aligned}(A.5)\qquad &\left(\partial_t - \frac{1}{2}\Delta_{\mathbf{h}}\right) z_{t,\varepsilon} \\ &= \frac{1}{2(z_{t,\varepsilon}+\varepsilon)^3}\left(\Gamma^{\mathbf{v}^*}(P_{T-t}f)\Gamma_2^{\mathbf{v}^*}(P_{T-t}f) - \frac{1}{4}\Gamma^{\mathbf{h}^*}(\Gamma^{\mathbf{v}^*}(P_{T-t}f))\right) \geq 0,\end{aligned}$$

we get $P_T z_{T,\varepsilon} = P_T(\Gamma^{\mathbf{v}^*}(f)+\varepsilon^2)^{1/2} - P_T\varepsilon \geq z_{0,\varepsilon} = (\Gamma^{\mathbf{v}^*}(P_T f)+\varepsilon^2)^{1/2} - \varepsilon$. By letting ε tend to 0, we obtain

$$\sqrt{\Gamma^{\mathbf{v}^*}(P_T f)} \leq P_T\sqrt{\Gamma^{\mathbf{v}^*}(f)}. \tag{A.6}$$

Next, let $y_{t,\varepsilon} = \left(\Gamma^{\mathbf{h}^*}(P_{T-t}f)+\varepsilon^2\right)^{1/2} - \varepsilon$, choose any $\alpha > \max\{-\rho_{\mathcal{H}}, \mathscr{M}^2_{\mathcal{HV}}\} \geq 0$ and define

$$u_{t,\varepsilon} = e^{-\alpha/2(T-t)}\big(y_{t,\varepsilon} + \ell\Gamma^{\mathbf{v}^*}(P_{T-t}(f))\big).$$

Note first that

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta_{\mathbf{h}}\right) u_{t,\varepsilon} \\ &= \frac{e^{-\alpha/2(T-t)}}{2y_{t,\varepsilon}+2\varepsilon}\left(\Gamma_2^{\mathbf{h}^*}(P_{T-t}f) + \ell y_{t,\varepsilon}\Gamma_2^{\mathbf{v}^*}(P_{T-t}f) - \frac{1}{4y_{t,\varepsilon}^2}\Gamma^{\mathbf{h}^*}(\Gamma^{\mathbf{h}^*}(P_{T-t}f))\right) \\ &\quad + \frac{\alpha e^{-\alpha/2(T-t)}}{2y_{t,\varepsilon}+2\varepsilon}\left(\Gamma^{\mathbf{h}^*}(P_{T-t}f) + \varepsilon + \ell y_{t,s}\Gamma^{\mathbf{v}^*}(P_{T-t}f)\right).\end{aligned}$$

We use Proposition 2.3 with ℓ replaced by $\ell y_{t,\varepsilon}$ to get

$$\begin{aligned}\frac{1}{4y_{t,\varepsilon}^2}\Gamma^{\mathbf{h}^*}(\Gamma^{\mathbf{h}^*}(P_{T-t}f)) &\leq \Gamma_2^{\mathbf{h}^*}(f) - (\rho_{\mathcal{H}} - c^{-1} - \ell^{-1}y_{t,\varepsilon}^{-1})\Gamma^{\mathbf{h}^*}(f) \\ &\quad + \ell y_{t,\varepsilon}\Gamma_2^{\mathbf{v}^*}(f) - c\mathscr{M}^2_{\mathcal{HV}}\Gamma^{\mathbf{v}^*}(f).\end{aligned}$$

As a result, for any $c > 0$, $\left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta_{\mathbf{h}}\right) u_{t,\varepsilon}$ has lower bound

$$\begin{aligned}&\frac{e^{-\alpha/2(T-t)}}{2y_{t,\varepsilon}+2\varepsilon}\left((\rho_{\mathcal{H}} - c^{-1} - \ell^{-1}y_{t,\varepsilon}^{-1})\Gamma^{\mathbf{h}^*}(P_{T-t}f) - c\mathscr{M}^2_{\mathcal{HV}}\Gamma^{\mathbf{v}^*}(P_{T-t}f)\right) \\ &\quad + \frac{\alpha e^{-\alpha/2(T-t)}}{2y_{t,\varepsilon}+2\varepsilon}\left(\Gamma^{\mathbf{h}^*}(P_{T-t}f) + \ell y_{t,\varepsilon}\Gamma^{\mathbf{v}^*}(P_{T-t}f)\right).\end{aligned}$$

Since it is true for any value of $c > 0$, it remains true for $c = \ell y_{t,\varepsilon}$, and hence

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta_{\mathbf{h}}\right) u_{t,\varepsilon} \geq -\frac{e^{-\alpha/2(T-t)}}{\ell}.$$

In a similar way as before, we can verify that the conditions of Lemma A.5 hold by using Lemma A.4. We can hence conclude that

$$\begin{aligned}u_{0,\varepsilon} &= e^{-\alpha T/2}(y_{0,\varepsilon} + \ell\Gamma^{\mathbf{v}^*}(P_T f)) \\ &\leq P_T u_{T,\varepsilon} + \int_0^T P_t\frac{e^{-\alpha(T-t)/2}}{\ell}\,dt \\ &\leq P_T\left(y_{T,\varepsilon} + \ell\Gamma^{\mathbf{v}^*}(f)\right) + \frac{2}{\alpha\ell}\left(1 - e^{-\alpha T/2}\right).\end{aligned}$$

Multiplying with $e^{\alpha T/2}$ on both sides, letting $\varepsilon \to 0$ and $\alpha \to k := \max\{-\rho_{\mathcal{H}}, \mathscr{M}_{\mathcal{R}}\}$, we finally get that for any $\ell > 0$,

$$\text{(A.7)} \quad \sqrt{\Gamma^{\mathbf{h}^*}(P_T f)} + \ell \Gamma^{\mathbf{v}^*}(P_T f) \leq e^{kT/2} P_T \left(\sqrt{\Gamma^{\mathbf{h}^*}(f)} + \ell \Gamma^{\mathbf{v}^*}(f) \right) + \ell^{-1} F_k(T),$$

where

$$F_k(t) = \begin{cases} \frac{2}{k}(e^{kt/2} - 1) & \text{if } k > 0, \\ t & \text{if } k = 0. \end{cases}$$

Since this estimate holds pointwise, it holds for $\ell = \left(P_T \Gamma^{\mathbf{v}^*}(f) - \Gamma^{\mathbf{v}^*}(P_T f) \right)^{-1/2}$ or $\ell = \infty$ at points where $P_T \Gamma^{\mathbf{v}^*}(f) - \Gamma^{\mathbf{v}^*}(P_T f) = 0$. The resulting inequality is

$$\text{(A.8)} \quad \sqrt{\Gamma^{\mathbf{h}^*}(P_T f)} \leq e^{kT/2} P_T \sqrt{\Gamma^{\mathbf{h}^*}(f)} + (e^{kT/2} + F_k(T)) \sqrt{P_T \Gamma^{\mathbf{v}^*}(f) - \Gamma^{\mathbf{v}^*}(P_t f)}.$$

We will now show how this inequality implies (A). Since $\mathbf{g}$ is complete, there exist a sequence of compactly supported functions $g_n \in C^\infty(M)$ satisfying $g_n \uparrow 1$ pointwise and $\|\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(g_n)\|_{L^\infty} \to 0$. It follows from equation (A.6) and (A.8) that

$$\lim_{n\to\infty} \|\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(P_t g_n)\|_{L^\infty} \to 0$$

as well. Hence, since $P_t g_n \to P_t 1$ and $\|dP_t g_n\|_{\mathbf{g}^*}$ approach 0 uniformly, we have that $\Gamma^{\mathbf{g}^*}(P_t 1) = 0$. It follows that $P_t 1 = 1$.

To finish the proof, consider a smooth function $f \in C^\infty(M)$ with $\|f\|_{L^\infty} < \infty$ and $\|\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(f)\|_{L^\infty} < \infty$. Define $f_n = g_n f \in C^\infty(M)$. Then $P_T f_n \to P_T f$ pointwise. It follows that

$$\text{(A.9)} \qquad \int_a^b dP_T f(\dot\gamma(t))\, dt = \lim_{n\to\infty} \int_a^b dP_T f_n\,(\dot\gamma(t))\; dt$$

for any smooth curve $\gamma : [a,b] \to M$. We want to use the dominated convergence theorem to show that the integral sign and limit on the right side of (A.9) can be interchanged.

Without loss of generality, we may assume that $\|\Gamma^{\mathbf{h}^*}(g_n)\|_{L^\infty} < 1$ for any n. We then note that

$$\left\| \sqrt{\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(f_n)} \right\|_{L^\infty} \leq \|f\|_{L^\infty} + \left\| \sqrt{\Gamma^{\mathbf{h}^*}(f)} \right\|_{L^\infty} =: K < \infty.$$

This relation, combined with (A.6) and (A.8), gives us

$$\| \sqrt{\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(P_T f_n)} \|_{L^\infty} \leq \left(2e^{kT/2} + F_k(T) + 1 \right) K.$$

Furthermore, the dominated convergence theorem tells us that both $P_T \Gamma^{\mathbf{v}^*}(f_m - f_n)$ and $\lim_{n\to\infty} P_T \Gamma^{\mathbf{h}^*}(f_n - f_m)$ approach 0 pointwise as $n, m \to \infty$. By inserting $f_n - f_m$ into (A.6) and (A.8), we see that $\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(P_T f_n)$ at any fixed point is a Cauchy sequence and hence convergent. We conclude that

$$\int_a^b dP_T f(\dot\gamma(t)) dt = \int_a^b \left(\lim_{n\to\infty} dP_T f_n \right) (\dot\gamma(t)) dt.$$

It follows that $dP_t f - \lim_{n\to\infty} dP_t f$ vanishes outside a set of measure zero along any curve, so

$$\|\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(P_T f)\|_{L^\infty} = \lim_{n\to\infty} \|\Gamma^{\mathbf{h}^*+\mathbf{v}^*}(P_T f_n)\|_{L^\infty} < \infty.$$

In conclusion, we have proven that condition (A) holds. Without any loss of generality we can put $\ell = 1$, since we can obtain all the other inequalities by replacing f with ℓf. □

Remark A.6. If we know that any $\frac{1}{2}\Delta_{\mathbf{h}}$-diffusion starting at a point has infinite lifetime then using Lemma A.4, we can actually make a probabilistic proof. We outline the proof here. We will only prove the inequality (A.6) as the proof of (A.7) is similar.

We will again use $z_{t,\varepsilon}$ as in (A.3). Let $X = X(x)$ be an $\frac{1}{2}\Delta_{\mathbf{h}}$-diffusion with $X_0(x) = x \in M$. We define Z^ε by $Z^\varepsilon_t = z_{t,\varepsilon} \circ X_t$. Then Z^ε is a local submartingale by (A.5). By using the Burkholder-Davis-Gundy inequality, there exist a constant B such that

$$\mathbb{E}\left[\sup_{0\leq s\leq t} Z^\varepsilon_s\right] \leq B\mathbb{E}\left[\sqrt{\langle Z^\varepsilon\rangle_t}\right] + z_{0,\varepsilon}(x) + \mathbb{E}\left[\int_0^t (\partial_s - \frac{1}{2}\Delta_{\mathbf{h}}) z_{s,\varepsilon} \circ X_s \, ds\right]$$

where $\langle Z^\varepsilon\rangle_t = \int_0^t \Gamma^{\mathbf{h}^*}(z_{s,\varepsilon}) \circ X_s \, ds$ is the quadratic variation of Z^ε. By the Cauchy-Schwartz inequality and the bound (A.4), we get the conclusion

$$\mathbb{E}\left[\sqrt{\langle Z^\varepsilon\rangle_t}\right] \leq p_{2t}(x,x)\sqrt{\int_0^t \|\Gamma_2^{\mathbf{v}^*}(P_{T-s}f)\|_{L^1}\, dt} < \infty$$

which means that $\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau} Z^\varepsilon_s\right] < \infty$. Hence, Z^ε is a true submartingale, giving us (A.6).

A.4. **Interpretation of** $\mathrm{Ric}_{\mathcal{H}\mathcal{V}}$**.** Let $\mathcal{V}$ be any integrable subbundle. Choose a subbundle $\mathcal{H}$ such that $TM = \mathcal{H} \oplus \mathcal{V}$. Any such choice of $\mathcal{H}$ correspond uniquely to a constant rank endomorphism $\mathrm{pr} = \mathrm{pr}_{\mathcal{V}} : TM \to \mathcal{V} \subseteq TM$. This can be considered as a splitting of the short exact sequence $\mathcal{V} \to TM \xrightarrow{F} TM/\mathcal{V}$.

Let $\Omega(M)$ be the the exterior algebra of M with $\mathbb{Z}\times\mathbb{Z}$-grading of Section A.1. Choose nondegenerate metric tensors

$$\mathbf{v} \in \Gamma(\mathrm{Sym}^2 \mathcal{V}^*) \text{ and } \breve{\mathbf{g}} \in \Gamma(\mathrm{Sym}^2(TM/\mathcal{V})^*)$$

on $\mathcal{V}$ and $TM/\mathcal{V}$. Since $\bigwedge^\nu \mathcal{V}^* \oplus \bigwedge^n (TM/\mathcal{V})^*$ is canonically isomorphic to $\bigwedge^{n+\nu} T^*M$, the choices of $\mathbf{v}$ and $\breve{\mathbf{g}}$ gives us a volume form vol on M.

We also have an energy functional defined on projections to $\mathcal{V}$. Relative to pr, define a Riemannian metric $\mathbf{g}_{\mathrm{pr}} = F^*\breve{\mathbf{g}} + \mathrm{pr}^*\mathbf{v}$. We introduce a functional E on the space of projections pr by

$$E(\mathrm{pr}) = \int_M \|\mathcal{R}_{\mathrm{pr}}\|^2_{\wedge^2 \mathbf{g}^*_{\mathrm{pr}} \otimes \mathbf{g}_{\mathrm{pr}}} \, d\mathrm{vol}$$

where $\mathcal{R}_{\mathrm{pr}}$ is the curvature of $\mathcal{H} = \ker \mathrm{pr}$. We can only be sure that the integral is finite if M is compact, so we will assume this, and consider our calculations as purely formal when this is not the case.

Let $\nabla = \nabla^{\mathrm{pr}}$ be the restriction of the Levi-Civita connection of $\mathbf{g}_{\mathrm{pr}}$ to $\mathcal{V}$. Introduce a exterior covariant derivative of d_∇ on $\mathcal{V}$-valued forms in the usual way, i.e. for any section $V \in \Gamma(\mathcal{V})$, we have $d_\nabla V = \nabla_{\bullet} V$ and if α is a $\mathcal{V}$-valued k-form, while μ is a form in the usual sense, then

$$d_\nabla(\alpha \wedge \mu) = (d_\nabla \alpha) \wedge \mu + (-1)^k \alpha \wedge d\mu.$$

We can split this operator into graded components $d_\nabla = d_\nabla^{1,0} + d_\nabla^{0,1} + d_\nabla^{2,-1}$ and do the same with its formal dual $\delta_\nabla = \delta_\nabla^{-1,0} + \delta_\nabla^{0,-1} + \delta_\nabla^{2,-1}$.

Proposition A.7. *The endomorphism* pr *is a critical value of* E *if and only if* $\delta_{\nabla}^{-1,0}\mathcal{R} = 0$. *In particular, if* $\mathbf{g}$ *satisfies*

$$\operatorname{tr}_{\mathcal{V}}(\mathcal{L}_A \mathbf{g})(\times, \times) = 0, \quad \textit{for any } A \in \Gamma(\mathcal{H}), \tag{A.10}$$

then pr *is a critical value if and only if* $\operatorname{Ric}_{\mathcal{H}\mathcal{V}} = 0$.

Recall from Part I, Section 2.4 that condition (A.10) is equivalent to the leafs of the foliation of $\mathcal{F}$ being minimal submanifolds. If $\mathcal{V}$ is the vertical bundle of a submersion $\pi \colon M \to B$, then we can identify $TM/\mathcal{V}$ with $\pi^* TB$. In this case, a critical value of E can be considered as an optimal way of choosing an Ehresmann connection on π.

Proof. We write $\mathrm{id} := \mathrm{id}_{TM}$ for the identity on TM. Let pr be a projection to $\mathcal{V}$ and $\alpha : TM \to \mathcal{V}$ be any $\mathcal{V}$-values one-from with $\mathcal{V} \subseteq \ker \alpha$. Define a curve in the space projections $\mathrm{pr}_t = \mathrm{pr} + t\alpha$. Then

$$\mathbf{g}_t(v, v) := \mathbf{g}_{\mathrm{pr}_t}(v, v) = \mathbf{g}_{\mathrm{pr}}(v, v) + 2t\mathbf{v}(\alpha v, \mathrm{pr}\, v) + t^2 \mathbf{v}(\alpha v, \alpha v).$$

Let $\mathcal{R}_t$ be the curvature of pr_t. Then

$$\begin{aligned}\mathcal{R}_t(A, Z) = \mathcal{R}(A, Z) &+ t\alpha[(\mathrm{id} - \mathrm{pr})A, (\mathrm{id} - \mathrm{pr})Z] \\ &- t\,(\mathrm{pr}[\alpha A, (\mathrm{id} - \mathrm{pr})Z] + \mathrm{pr}[(\mathrm{id} - \mathrm{pr})A, \alpha Z]) + O(t^2).\end{aligned}$$

If $\nabla^t = \nabla^{\mathrm{pr}_t}$, then

$$\begin{aligned}\nabla_A^t V &= \nabla_A V + \frac{1}{2} t d_{\nabla}\alpha(A, V) - \frac{1}{2} t \sharp^{\mathbf{v}^*} \mathbf{g}(d_{\nabla}\alpha(A, \centerdot), V), \quad \text{and} \\ d_{\nabla^t} \mathrm{pr}_t &= d_{\nabla} \mathrm{pr} + \frac{1}{2} t (d_{\nabla}\alpha)_{1,1} - \frac{1}{2} t (d_{\nabla}\alpha)_{1,1}^{\top} + t d_{\nabla}\alpha + O(t^2),\end{aligned}$$

where $(d_{\nabla}\alpha)_{1,1}$ is the (1,1)-graded component of $d_{\nabla}\alpha$ and

$$\mathbf{v}((d_{\nabla}\alpha)_{1,1}^{\top}(A, V_1), V_2) = \mathbf{v}((d_{\nabla}\alpha)_{1,1}(A, V_1), V_2).$$

Since $\mathcal{R}_t = -d_{\nabla}^{2,-1} \mathrm{pr}_t$, we get

$$\begin{aligned}\frac{d}{dt} E(\mathrm{pr}_t)\bigg|_{t=0} &= \int_M (\wedge^2 \mathbf{g}_{\mathrm{pr}}^* \otimes \mathbf{g}_{\mathrm{pr}})(d_{\nabla}^{2,-1} \mathrm{pr}, d_{\nabla}^{1,0}\alpha)\, d\mathrm{vol} \\ &= -\int_M (\mathbf{h}^* \otimes \mathbf{v})(\delta_{\nabla}^{-1,0}\mathcal{R}, \alpha)\, d\mathrm{vol}\,.\end{aligned}$$

Hence, pr is a critical value if and only if $\delta_{\nabla}^{-1,0}\mathcal{R} = 0$.

We give a local expression for this identity. Let $A_1, \dots, A_n$ be a local orthonormal basis of $\mathcal{H}$. Then

$$\delta_{\nabla}^{-1,0}\mathcal{R} = -\sum_{k=1}^n (\mathring{\nabla}_{A_k}\mathcal{R})(A_k, \centerdot) + \mathcal{R}(N, \centerdot)$$

where N is defined by $\mathbf{g}_{\mathrm{pr}}(A, N) = -\frac{1}{2} \operatorname{tr}_{\mathcal{V}}(\mathcal{L}_{\mathrm{pr}_{\mathcal{H}} A}\, \mathbf{g})(\times, \times)$ and $\mathring{\nabla}$ is the (0,0)-degree component of the Levi-Civita connection, i.e.

$$\mathring{\nabla}_A Z = \mathrm{pr}_{\mathcal{H}} \nabla_A \mathrm{pr}_{\mathcal{H}} Z + \mathrm{pr}_{\mathcal{V}} \nabla_A \mathrm{pr}_{\mathcal{V}} Z.$$

This coincides with $\operatorname{Ric}_{\mathcal{H}\mathcal{V}}$ when (A.10) holds. □ □

A.5. **If $\mathcal{V}$ is not integrable.** Let $(M, \mathbf{g})$ be a complete Riemannian manifold and let $\mathcal{H}$ be a bracket-generating subbundle of TM with orthogonal complement $\mathcal{V}$. Define $\mathring{\nabla}$ as in (2.1) with respect to $\mathbf{g}$ and the splitting $TM = \mathcal{H} \oplus \mathcal{V}$ and assume that $\mathring{\nabla}\, \mathbf{g} = 0$. Let $\Delta_{\mathbf{h}}$ be the sub-Laplacian defined relative to $\mathcal{V}$ or equivalently to the volume form of $\mathbf{g}$. Then it may happen that (CD) holds for $\Delta_{\mathbf{h}}$ even without assuming that $\mathcal{V}$ is integrable. More precisely, we will need the condition

$$\operatorname{tr} \overline{\mathcal{R}}(v, \mathcal{R}(v, \centerdot)) = 0, \qquad v \in TM, \tag{A.11}$$

$$\mathcal{R}(A, Z) = \operatorname{pr}_{\mathcal{V}}[\operatorname{pr}_{\mathcal{H}} A, \operatorname{pr}_{\mathcal{H}} Z], \quad \overline{\mathcal{R}}(A, Z) = \operatorname{pr}_{\mathcal{H}}[\operatorname{pr}_{\mathcal{V}} A, \operatorname{pr}_{\mathcal{V}} Z], \quad A, Z \in \Gamma(TM).$$

We refer to $\mathcal{R}$ and $\overline{\mathcal{R}}$ as respectively the curvature and the co-curvature of $\mathcal{H}$.

In Part I, Section 3.8, we showed that Theorem 2.2 and Proposition 2.3 hold with the same definitions and with $\mathcal{V}$ not integrable, as long as (A.11) also holds. The same is true for Theorem 3.4. We give some brief details regarding this.

First of all, in Section A.1, the exterior derivative d now also has a part of degree $(-1, 2)$, determined by

$$d^{-1,2} f = 0, \quad d^{-1,2}\alpha = -\alpha \circ \overline{\mathcal{R}}, \qquad f \in C^\infty(M),\ \alpha \in \Gamma(T^*M),$$

and hence, the co-differential has a degree $(1, -2)$-part. However, these do not have any significance for our calculations. More troubling is the fact that both Lemma A.2 (a) and the formula for $\mathring{\nabla}_Z V_s|_{x_0}$ in Lemma A.2 (b) are false when $\mathcal{V}$ is not integrable. However, (A.11) ensures that

$$\sum_{i=1}^n \mathbf{g}(R^{\mathring{\nabla}}(A_i, V)Z, A_i) = 0$$

for any orthonormal basis $A_1, \dots, A_n$ of $\mathcal{H}$ and vertical vector field V, which is all we need for the proof of Lemma A.1. Furthermore the same proof is still holds even if now $\mathring{\nabla}_Z V_s|_{x_0} = \frac{1}{2} \sharp \overline{\mathcal{R}}(Z, \centerdot)|_{x_0}$ in Lemma A.2 (b), as the extra terms cancel out.

Once Lemma A.1 holds, there is no problem with the rest of the proof of Theorem 3.4. See Part I, Section 4.6 for an example where this theorem holds.

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