$\mathbb{Z}_2^n$-Supergeometry I
Manifolds and Morphisms

Tiffany Covolo, Janusz Grabowski, and Norbert Poncin

Abstract

In Physics and in Mathematics $\mathbb{Z}_2^n$-gradings, $n \geq 2$, do appear quite frequently. The corresponding sign rules are determined by the 'scalar product' of the involved $\mathbb{Z}_2^n$-degrees. The present paper is the first of a series on $\mathbb{Z}_2^n$-Supergeometry. The new theory exhibits challenging differences with the classical one: nonzero degree even coordinates are not nilpotent, and even (resp., odd) coordinates do not necessarily commute (resp., anticommute) pairwise (the parity is the parity of the total degree). It is based on the hierarchy: '$\mathbb{Z}_2^0$-Supergeometry (classical differential Geometry) contains the germ of $\mathbb{Z}_2^1$-Supergeometry (standard Supergeometry), which in turn contains the sprout of $\mathbb{Z}_2^2$-Supergeometry, etc.' The $\mathbb{Z}_2^n$-supergeometric viewpoint provides deeper insight and simplified solutions; interesting relations with Quantum Field Theory and Quantum Mechanics are expected. In this article, we define $\mathbb{Z}_2^n$-supermanifolds and provide examples in the atlas, the ringed space and coordinate settings. We thus show that formal series are the appropriate substitute for nilpotency. Moreover, the category of $\mathbb{Z}_2^n$-supermanifolds is closed with respect to the tangent and cotangent functors. The fundamental theorem describing supermorphisms in terms of coordinates is extended to the $\mathbb{Z}_2^n$-context.

MSC 2010: 17A70, 58A50, 13F25, 16L30

Keywords: Supersymmetry, supergeometry, superalgebra, higher grading, sign rule, ringed space, manifold, morphism, tangent space, higher vector bundle

Contents

1 Introduction

Classical Supersymmetry and Supergeometry are not sufficient to suit the current needs. In *Physics*, $\mathbb{Z}_2^n$-gradings, $n \geq 2$, are used to describe anyons and paraparticles. In *Mathematics*, there exist good examples of $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-*commutative algebras* (i.e. the superscript of -1 in the sign rule is the standard 'scalar product' of $\mathbb{Z}_2^n$): quaternions and, more generally, any Clifford algebra, the algebra of Deligne differential superforms... And there exist interesting examples of $\mathbb{Z}_2^n$-*supermanifolds*: e.g. (completions of) tangent and cotangent bundles, n-vector bundles...

Indeed, the tangent bundle of a classical $\mathbb{Z}_2^1$-supermanifold $\mathcal{M}$ is a $\mathbb{Z}_2^1$-supermanifold $\mathsf{T}[1]\mathcal{M}$ (resp., a $\mathbb{Z}_2^2$-supermanifold $\mathsf{T}\mathcal{M}$) with function sheaf the differential superforms of $\mathcal{M}$ together with the Bernstein-Leites (resp., with the Deligne) sign convention. Actually the tangent (and cotangent) bundle(s) of any $\mathbb{Z}_2^n$-supermanifold is a (are) $\mathbb{Z}_2^{n+1}$-supermanifold(s). Further, any n-vector bundle canonically provides a $\mathbb{Z}_2^n$-supermanifold as its 'superization'.

To be more precise, suppose that $\mathcal{M}$ is a supermanifold with local coordinates $(x^1, \ldots, x^p, \xi^1, \ldots, \xi^q)$, where the x^i are even and the ξ^a odd. For the tangent bundle $\mathsf{T}\mathcal{M}$, with the adapted local coordinates $(x^i, \xi^a, \dot{x}^j, \dot{\xi}^b)$, one can introduce a supermanifold structure, in principle, in two ways: declaring $\dot{x}^j$ to be even and $\dot{\xi}^b$ to be odd, or reversing these parities.

For the latter, the variables $\dot{\xi}^b$ are even, what works in principle well, but it is hard to regard them as 'true' real-valued functions because they are formal variables, so equations like $\dot{\xi}^b = 2$ do not make much sense.

On the other hand, $\mathsf{T}\mathcal{M}$, as every vector bundle, admits an $\mathbb{N}$-gradation for which $\dot{x}^j$ and $\dot{\xi}^b$ are of degree 1. Thus we have a canonical bigradation by the monoid $\mathbb{N} \times \mathbb{Z}_2$, which can be reduced to $\mathbb{Z}_2^2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. With respect to this bigradation, $(x^i, \xi^a, \dot{x}^j, \dot{\xi}^b)$ are of bidegrees $(0,0), (0,1), (1,0)$, and $(1,1)$, respectively. Now, any symmetric biadditive map $\langle -, - \rangle : \mathbb{Z}_2^2 \times \mathbb{Z}_2^2 \to \mathbb{Z}$ gives rise to a sign rule:

$$AB = (-1)^{\langle (m,n),(k,l) \rangle} BA \ ,$$

where A and B are coordinates of bidegrees (m, n) and (k, l), respectively. We get the usual sign rule when choosing $\langle (m, n), (k, l) \rangle = mk$, and obtain the sign rule for reversed parity (Berntein-Leites sign rule) choosing $\langle (m, n), (k, l) \rangle = (m+n)(k+l)$, whereas the 'scalar product' $\langle (m, n), (k, l) \rangle = mk + nl$ has been used by Deligne – see discussion in [DM99, Appendix to §1]. Note that the latter does not lead to a superalgebra, as the $\dot{\xi}^b$ are even in the sense that they commute among themselves, but anticommute with the ξ^a. Not excluding any sign rule forces us to work with this bigradation and to include Deligne's convention into the picture, which – as mentioned – does not correspond to any supermanifold. It is therefore natural to extend the notion of supermanifold admitting $\mathbb{Z}_2^n$-gradations and the corresponding sign rule

$$AB = (-1)^{\sum_{i=1}^n m_i k_i} BA \tag{1}$$

(here $m = (m_1, \dots, m_n)$ and $k = (k_1, \dots, k_n)$ are the degrees of A and B, respectively), so that the additional canonical gradings on $\mathsf{T}\mathcal{M}$ or $\mathsf{T}^*\mathcal{M}$ do not move us out of the corresponding category.

Although not universally accepted at the beginning, $\mathbb{Z}_2^n$-Supergeometry is thus a necessary and natural generalization. When defining the parity of a $\mathbb{Z}_2^n$-degree as the parity of the total degree, nonzero degree even coordinates are not nilpotent, and even (resp., odd) coordinates do not necessarily commute (resp., anticommute) pairwise. These circumstances lead to challenging differences with the classical theory. The reason for initial skepticism was Neklyudova's equivalence [Lei11]: this result states that the categories of $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-*commutative* and $\mathbb{Z}_2^n$-graded *supercommutative* algebras are equivalent. However, our previous work shows that pullbacks of '*supercommutative* concepts' to the $\mathbb{Z}_2^n$-*commutative* setting are not always as easy as expected, and do not always operate properly: Neklyudova's theorem **does not ban** studies of $\mathbb{Z}_2^n$-commutative algebras! On the other hand, $\mathbb{Z}_2^n$-commutative algebras **are sufficient**, in the sense that any sign rule, for any finite number m of coordinates, is of the form (1), for some $n \leq 2m$.

Actually $\mathbb{Z}_2^n$-Supergeometry focusses on the following hierarchy: classical differential Geometry ($\mathbb{Z}_2^0$-Supergeometry) contains the germ (differential forms) of standard Supergeometry ($\mathbb{Z}_2^1$-Supergeometry), which in turn contains the sprout (Deligne superforms) of $\mathbb{Z}_2^2$-Supergeometry...

The $\mathbb{Z}_2^n$-supergeometric viewpoint provides improved insight and simplified solutions, thus emphasizing an effect already observed for classical Supergeometry. Tight relations with diverse concepts in Quantum Field Theory and Quantum Mechanics, e.g. polarization of the vacuum and Feynman integrals, can be expected. The theory of $\mathbb{Z}_2^n$-supermanifolds is closely related to Clifford calculus. Clifford algebras have numerous applications in Physics: the use of $\mathbb{Z}_2^n$-gradings has never been studied. Our theory should lead to a novel approach to quaternionic functions: examples of application areas include thermodynamics, hydrodynamics, geophysics and structural mechanics. It is further interesting to observe the parallelism of our extension with Baez' suggestion of a common generalization – under the name of r-Geometry – of superalgebras and Clifford algebras with the goal to incorporate, besides bosons and fermions, also anyons into the picture [Bae92].

Finally, the key-concept of $\mathbb{Z}_2^n$-Superalgebra is the $\mathbb{Z}_2^n$-Berezinian. This higher Berezinian (which is tightly connected with quasi-determinants) and the corresponding (via the Liouville formula) higher trace have recently been constructed [COP12]. It provides a new solution ('different' from the Dieudonné determinant) to Cayley's challenge to build a determinant of quaternionic matrices. Hence, our $\mathbb{Z}_2^n$-Supergeometry not only includes differential but also integral calculus, whereas Molotkov – to our knowledge the only author who understood so far the necessity to define a (functorial) concept of $\mathbb{Z}_2^n$-supermanifold – mentions explicitly that he has no insight in this respect [Mol10].

The paper is organized as follows. In Section 2, we prove that any sign rule, for any finite set of variables, is of the type (1), for some $\mathbb{Z}_2^n$-grading. The necessity to consider $\mathbb{Z}_2^n$-superdomains, characterized just as in [CB89] by algebras of formal power series, is explained in Section 3. Moreover, invertibility, locality and completeness issues are addressed, and a coordinate version of the $\mathbb{Z}_2^n$-supermorphism theorem is proved. In Sections 2 and 3, we use exclusively coordinate computations, thus allowing the reader to get acquainted with the specificities and foundations of the new theory. The latter is developed in the next sections via the atlas approach, as well as, mainly, in the ringed space setting. The concept of $\mathbb{Z}_2^n$-supermanifold is introduced in Section 4. In Section 5, we detail first examples. Section 6 contains the main proofs of the present work. We show that most important results of classical Supergeometry extend to the $\mathbb{Z}_2^n$-context, although nilpotency is lost in this generalized framework – it turns out that formal series are the appropriate substitute. We prove that $\mathbb{Z}_2^n$-superfunctions project consistently to the base and that the latter actually carries a smooth manifold structure. Continuity of the pullback maps of morphisms between $\mathbb{Z}_2^n$-supermanifolds with respect to the filtration provided by the kernel of the base projection, as well as continuity of the induced maps between stalks with respect to the filtration implemented by the unique maximal homogeneous ideal – combined with an appropriate polynomial approximation of $\mathbb{Z}_2^n$-superfunctions –, allow to show that the fundamental theorem of supermorphisms extends to the $\mathbb{Z}_2^n$-setting. Complementary information can be found in the appendix-section 7.

Let us finally provide a non-exhaustive list of references on classical supermanifolds and related topics that were of importance for the present text: [Lei80], [Lei11], [Var04], [Man02], [DM99], [CCF11], [BBH91], [DSB03], [Vor12], [CR12], [BP12], [GKP09], [GKP10].

2 Sign rules

Supergeometry is commonly understood as the theory of manifold-like objects admitting anticommuting variables. This corresponds to a $\mathbb{Z}_2$-gradation in the structure sheaf of the corresponding ringed space, so that even elements are central (commute with everything) and odd elements anticommute among themselves. In particular, they are nilpotent in step 2. This means that the sign rules between generators of the algebra are completely determined by their *parity*. Why not accept arbitrary commutation rules between different generators, even with fixed parities? In principle, one can consider a general gradation by a semigroup and an arbitrary commutation factor, i.e. work with so-called *colored algebras*.

More precisely, let K be a commutative unital ring, $K^\times$ be the group of invertible elements of K, and let G be a commutative semigroup. A map $\varepsilon : G \times G \to K^\times$ is called a *commutation factor* on G if

$$\varepsilon(g,h)\varepsilon(h,g) = 1\,, \quad \varepsilon(g,g) = \pm 1\,, \quad \text{and} \quad \varepsilon(f, g+h) = \varepsilon(f,g)\varepsilon(f,h)\,, \tag{2}$$

for all $f, g, h \in G$. Note that these axioms imply that

$$\varepsilon(f+g,h) = \varepsilon(f,h)\varepsilon(g,h)$$

(which is sometimes unnecessarily assumed additionally). Indeed,

$$\varepsilon(f,h)\varepsilon(g,h) = \varepsilon(h,f)^{-1}\varepsilon(h,g)^{-1} = (\varepsilon(h,g)\varepsilon(h,f))^{-1} = \varepsilon(h,g+f)^{-1} = \varepsilon(f+g,h)\,.$$

The condition $\varepsilon(g,g) = \pm 1$ also follows automatically from the rest if only K is a field.

Let $\mathcal{A}$ be a G-graded K-algebra $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}^g$. Elements x from $\mathcal{A}^g$ are called *G-homogeneous of degree* or *weight* $g =: \deg(x)$. The algebra $\mathcal{A}$ is said to be *ε-commutative* if

$$ab = \varepsilon(\deg(a), \deg(b))ba\,, \tag{3}$$

for all G-homogeneous elements $a, b \in \mathcal{A}$. Homogeneous elements x with $p(\deg(x)) = p(g) := \varepsilon(g,g) = -1$ are *odd*, the other homogeneous elements are *even*.

In what follows, K will be $\mathbb{R}$ and ε will take the form

$$\varepsilon(g,h) = (-1)^{\langle g,h\rangle}\,,$$

for a 'scalar product' $\langle -,-\rangle : G \times G \to \mathbb{Z}$. This means that we use the *commutation factor* as a *sign rule*. In this note we confine ourselves to $G = \mathbb{Z}_2^n$ and the standard 'scalar product' of $\mathbb{Z}_2^n$, what will lead to $\mathbb{Z}_2^n$-Supergeometry with nicer categorical properties than standard Supergeometry. More precisely, we propose a generalization of differential $\mathbb{Z}_2^1$-Supergeometry to the case of a $\mathbb{Z}_2^n$-gradation in the structure sheaf.

Indeed, we will show that any sign rule, for any finite number of coordinates, can be obtained from the 'scalar product'

$$\langle (i_1, \ldots, i_n), (j_1, \ldots, j_n)\rangle_n = i_1 j_1 + \ldots + i_n j_n \tag{4}$$

on $\mathbb{Z}_2^n$ for a sufficiently big n. In other words, any algebra that is finitely generated by some generators satisfying certain sign rules can be viewed as a $\mathbb{Z}_2^n$-graded associative algebra $\mathcal{A} = \bigoplus_{i\in\mathbb{Z}_2^n} \mathcal{A}^i$, $\mathcal{A}^i\mathcal{A}^j \subset \mathcal{A}^{i+j}$, which is $\mathbb{Z}_2^n$-commutative in the sense that

$$y^i y^j = (-1)^{\langle i,j\rangle_n} y^j y^i \ ,$$

for all $y^i \in \mathcal{A}^i$, $y^j \in \mathcal{A}^j$. We simply refer to such algebras as $\mathbb{Z}_2^n$-*commutative associative algebras*. Let us mention that a similar theorem was proved independently in [MGO10] for group gradations.

Let now S be a finite set, say $S = \{1, \dots, m\}$, and let $\varepsilon : S \times S \to \{\pm 1\}$ be any symmetric function. We can understand ε as a sign rule for an associative algebra generated by elements y^i, $i = 1, \dots, m$, i.e.

$$y^i y^j = \varepsilon(i,j) y^j y^i \ .$$

We then have the

Theorem 2.1. *There is $n \leq 2m$ and a map $\sigma : S \to \mathbb{Z}_2^n$, $i \mapsto \sigma_i$, such that*

$$\varepsilon(i,j) = (-1)^{\langle \sigma_i, \sigma_j\rangle_n} \ . \tag{5}$$

Proof. We interpret $\mathbb{Z}_2^{2m}$ as the set of functions $\{\pm 1, \dots, \pm m\} \to \mathbb{Z}_2$, and denote by $p(i,j) \in \{0,1\}$ the *parity* of $\varepsilon(i,j)$: $(-1)^{p(i,j)} = \varepsilon(i,j)$.

First, define $\sigma_1 \in \mathbb{Z}_2^{2m}$ by $\sigma_1(1) = 1$, $\sigma_1(-1) = 1 + p(1,1) \in \mathbb{Z}_2$, and $\sigma_1(k) = 0$ for $|k| > 1$. Then, for $j = 2, \dots, m$, define $\sigma_j(1) = p(j,1)$ and $\sigma_j(-1) = 0$. Independently of the definition of the remaining values of σ_j, Condition (5) is valid for $i = 1$ and all $j = 1, \dots, m$, since $\sigma_1(k) = 0$ for $|k| > 1$.

Assume inductively that we have fixed $\sigma_1, \dots, \sigma_r$, with $\sigma_j(k) = 0$ for $|k| > j$, as well as the values $\sigma_j(k)$, for $j = r+1, \dots, m$ and $|k| \leq r$, so that (5) is valid for $i = 1, \dots, r$ and all j.

Define:

$$\sigma_{r+1}(r+1) = 1\,, \quad \sigma_{r+1}(-r-1) = 1 + \sum_{|k|=1}^{r} \sigma_{r+1}(k) + p(r+1, r+1)\,, \quad \sigma_{r+1}(k) = 0 \text{ for } |k| > r+1\,.$$

Then, (5) is valid also for $i = j = r+1$. Putting now $\sigma_j(-r-1) = 0$ and

$$\sigma_j(r+1) = \sum_{|k|=1}^{r} \sigma_j(k)\sigma_{r+1}(k) + p(j, r+1)$$

for $j = r+2, \dots, m$, we finish with fixed $\sigma_1, \dots, \sigma_{r+1}$, with $\sigma_j(k) = 0$ for $|k| > j$, and the values $\sigma_j(k)$, for $j = r+2, \dots, m$ and $|k| \leq r+1$, so that (5) is valid for $i = 1, \dots, r+1$ and all j. This proves the inductive step and the theorem follows. □

Let now $y^1, \dots, y^m$ be 'variables' with $\mathbb{Z}_2^n$-degrees fixed by a map $\sigma : \{1, \dots, m\} \to \mathbb{Z}_2^n$. We can consider $\mathbb{R}[y^1, \dots, y^m]_\sigma$, which is the free graded tensor algebra over reals generated by variables $y^1, \dots, y^m$ modulo the commutation relations described by σ,

$$y^i y^j = (-1)^{\langle \sigma_i, \sigma_j\rangle_n} y^j y^i \tag{6}$$

[CM14]. This algebra is referred to as the *free σ-commutative associative $\mathbb{R}$-algebra*, or, if σ is fixed, the *free $\mathbb{Z}_2^n$-commutative associative $\mathbb{R}$-algebra* in m generators. Moreover, if n is fixed, we usually omit subscript n in $\langle -, - \rangle_n$. The variable y^i is even (resp., odd) if $p(y^i) := |\sigma_i| := \sigma_i(1) + \ldots + \sigma_i(n) \in \mathbb{Z}_2$ is 0 (resp., 1). We can write every element of $\mathbb{R}[y^1, \ldots, y^m]_\sigma$ uniquely as a polynomial

$$f(y) = \sum_{|\mu|=0}^{N_f} f_{\mu_1 \ldots \mu_m} (y^1)^{\mu_1} \ldots (y^m)^{\mu_m} = \sum_{|\mu|=0}^{N_f} f_\mu y^\mu \,, \tag{7}$$

where $|\mu| = \mu_1 + \ldots + \mu_m$.

3 $\mathbb{Z}_2^n$-superdomains and their morphisms

To develop a generalization of Supergeometry, we wish to distinguish coordinates $x^1, \ldots, x^p$ of degree $0 := (0, \ldots, 0) \in \mathbb{Z}_2^n$ and view them as local coordinates on a standard manifold. The remaining coordinates $\xi^1, \ldots, \xi^q$ have nontrivial degrees $\sigma_1, \ldots, \sigma_q \in \mathbb{Z}_2^n \setminus \{0\}$ determined by a fixed map $\sigma : \{1, \ldots, q\} \to \mathbb{Z}_2^n \setminus \{0\}$. We will call them *formal variables.*

3.1 Sheaf of polynomials

The first idea would be to define a σ-superdomain or $\mathbb{Z}_2^n$-superdomain as a *ringed space* $\mathfrak{U} = (U, \mathfrak{O}_{U,\sigma})$, where $U \subset \mathbb{R}^p$ is an open subset and the structure sheaf is given by

$$\mathfrak{O}_{U,\sigma}(-) := C_U^\infty(-)[\xi^1, \ldots, \xi^q]_\sigma \,. \tag{8}$$

Here $\xi^1, \ldots, \xi^q$ is a sequence of variables of $\mathbb{Z}_2^n$-degrees σ_a, i.e. commuting according to

$$\xi^a \xi^b = (-1)^{\langle \sigma_a, \sigma_b \rangle_n} \xi^b \xi^a \,. \tag{9}$$

As already mentioned above, we omit in the sequel subscript σ, since this map is fixed. Thus, on $V \subset U$, our algebra of superfunctions would be the $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebra

$$\mathfrak{O}_U(V) = C_U^\infty(V)[\xi^1, \ldots, \xi^q] \tag{10}$$

of polynomials

$$f(x, \xi) = \sum_{|\mu|=0}^{N_f} f_{\mu_1 \ldots \mu_q}(x)\; (\xi^1)^{\mu_1} \ldots (\xi^q)^{\mu_q} = \sum_{|\mu|=0}^{N_f} f_\mu(x) \xi^\mu \tag{11}$$

in the variables ξ^a and with coefficients in the ring $C^\infty(V)$, whose multiplication is subject to the sign rules determined by (9). Note that we omit subscripts like U, whenever we do not wish to stress the (Hausdorff, second-countable) topological space over which the considered sheaf is defined. Of course, those ξ^a which are odd, $p(\xi^a) = 1$, appear in the polynomials with exponents ≤ 1.

Morphisms $\mathfrak{O}(W) \to \mathfrak{O}(V)$ of $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebras (in particular changes of coordinates) should preserve the grading, so have the form

$$\begin{aligned} x'^i &= \varphi^i(x) + \sum_{\deg(\xi^\mu)=0} f_\mu^i(x) \xi^\mu \,, \\ \xi'^a &= \sum_{\deg(\xi^\mu)=\sigma_a} f_\mu^a(x) \xi^\mu \,, \end{aligned} \tag{12}$$

where the functions $f_\mu : V \to \mathbb{R}^p$ and the map $\varphi : V \to W$ are smooth, and the sums are finite.

It is easy to see that the ideal $\mathfrak{J}(V) \subset \mathfrak{O}(V)$ generated by the formal variables is respected by morphisms and that the projection

$$\mathfrak{p}_V : \mathfrak{O}(V) \to \mathfrak{O}(V)/\mathfrak{J}(V) \simeq C^\infty(V)$$

is covariantly defined (we come back to this and similar points later on).

However, this approach has clear shortcomings.

First, as we allow formal variables which are even, the ideal $\mathfrak{J}(V)$ is not nilpotent, in general, so superfunctions f with invertible 'body' $\mathfrak{p}_V(f)$ need not to be invertible in the ring $\mathfrak{O}(V)$. Formal inverting of polynomials requires using formal power series.

Second, for a proper development of differential calculus, we should be able to compose elements of degree 0, see (12), with arbitrary differentiable functions and not only polynomials. But what is $F(x+\xi^2)$ for a differentiable F and formal even variable ξ? Since ξ is not nilpotent, the Taylor formula (proceed as in standard Supergeometry) leads again to a formal power series.

3.2 Sheaf of formal power series

A consistent differential calculus for $\mathbb{Z}_2^n$-superdomains forces us to complete the structure sheaf to formal power series in the formal variables. In this respect our definition of $\mathbb{Z}_2^n$-superdomains, and the below definition of $\mathbb{Z}_2^n$-supermanifolds, are similar to Choquet-Bruhat's definition of standard Rogers-De Witt supermanifolds [CB89]. Of course, odd variables will appear only with power 1.

In the following, we consider the n-tuples of $\mathbb{Z}_2^n$ as ordered lexicographically.

Definition 3.1. Let $n, p, q \in \mathbb{N}$ and let $\sigma : \{1, \ldots, q\} \to \mathbb{Z}_2^n \setminus \{0\}$. Denote by $q_k \in \mathbb{N}$, $k \in \{1, \ldots, 2^n - 1\}$, the number of degrees σ_a that coincide with the k-th element of $\mathbb{Z}_2^n \setminus \{0\}$ and set $\mathbf{q} = (q_1, \ldots, q_{2^n-1})$. A *$\sigma$-superdomain* or *$\mathbb{Z}_2^n$-superdomain* of dimension $p|\mathbf{q}$ is a *ringed space* $\mathcal{U}^{p|\mathbf{q}} = (U, \mathcal{O}_{U,\sigma})$, where $U \subset \mathbb{R}^p$ is an open subset and the structure sheaf is the sheaf

$$\mathcal{O}_{U,\sigma}(-) := C^\infty_U(-)[[\xi^1, \ldots, \xi^q]]_\sigma \ . \tag{13}$$

Over $V \subset U$, the algebra of $\mathbb{Z}_2^n$-functions is the $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebra

$$\mathcal{O}_U(V) = C^\infty_U(V)[[\xi^1, \ldots, \xi^q]] \tag{14}$$

of formal power series

$$f(x, \xi) = \sum_{|\mu|=0}^{\infty} f_{\mu_1 \ldots \mu_q}(x)(\xi^1)^{\mu_1} \ldots (\xi^q)^{\mu_q} = \sum_{|\mu|=0}^{\infty} f_\mu(x)\xi^\mu \tag{15}$$

in formal variables $\xi^1, \ldots, \xi^q$ of degrees $\sigma_1, \ldots, \sigma_q$ commuting according to (9), and with coefficients in $C^\infty(V)$.

We refer to a ringed space of $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebras as a $\mathbb{Z}_2^n$-*ringed space.*

Example 3.2. Consider the case $n = 2$ and $p|q_1|q_2|q_3 = 1|1|1|1$, write for simplicity $(x, \xi, \eta, \vartheta)$ instead of (x, ξ^1, ξ^2, ξ^3), and choose $\sigma_\xi = (0,1), \sigma_\eta = (1,0)$, and $\sigma_\vartheta = (1,1)$. A $\mathbb{Z}_2^2$-function is then of the form

$$f(x,\xi,\eta,\vartheta) = \sum_{r\geq 0} f_r(x)\vartheta^{2r} + \sum_{r\geq 0} g_r(x)\vartheta^{2r+1}\xi\eta + \sum_{r\geq 0} h_r(x)\vartheta^{2r}\xi + \sum_{r\geq 0} k_r(x)\vartheta^{2r+1}\eta$$
$$+ \sum_{r\geq 0} \ell_r(x)\vartheta^{2r}\eta + \sum_{r\geq 0} m_r(x)\vartheta^{2r+1}\xi + \sum_{r\geq 0} n_r(x)\vartheta^{2r+1} + \sum_{r\geq 0} p_r(x)\vartheta^{2r}\xi\eta\ , \tag{16}$$

where the sums are formal series and the functions in x are smooth. Note that the first (resp., second, third, fourth) two sums contain terms of $\mathbb{Z}_2^2$-degree $(0,0)$ (resp., $(0,1)$, $(1,0)$, and $(1,1)$).

3.3 Locality of $\mathbb{Z}_2^n$-superdomains

In classical Supergeometry a (super) ringed space is called a *space* if all its stalks are local rings, i.e. rings that have a unique maximal homogeneous ideal. Such ringed spaces are referred to as *locally ringed spaces*. Further, a ringed space is a *supermanifold* if it is a space that is locally modelled on a superdomain. Superdomains are thus 'trivial' locally ringed spaces. Of course, one has to verify that the stalks of a superdomain are local rings.

To show that $\mathbb{Z}_2^n$-superdomains are *locally $\mathbb{Z}_2^n$-ringed spaces*, we need two lemmas.

Let R be a commutative unital ring and let $(\xi^1, \dots, \xi^q)$ be a finite number of $\mathbb{Z}_2^n \setminus \{0\}$-graded parameters, which satisfy

$$\xi^i\xi^j = (-1)^{\langle \deg(\xi^i), \deg(\xi^j)\rangle}\xi^j\xi^i$$

(the scalars R are assumed to be central). We denote by $R[[\xi^1, \dots, \xi^q]]$ the $\mathbb{Z}_2^n$-commutative associative unital R-algebra of formal series in the ξ^a with coefficients in R.

Lemma 3.3. *Any series $1 - v$, where $v = \sum_{|\mu|>0} v_\mu \xi^\mu$ has no independent term, is invertible, with inverse $v^{-1} = \sum_{k\geq 0} v^k$.*

Proof. Observe first that, for any $k \in \mathbb{N}$,

$$v^k = \sum_{|\nu|\geq k} \left(\sum_{\mu_1+\ldots+\mu_k=\nu} \pm v_{\mu_1} \cdots v_{\mu_k} \right) \xi^\nu\ ,$$

where the $\mu_i \in \mathbb{N}^q$ are of course multi-indices. It follows that the coefficients of $v^{-1} := \sum_{k\geq 0} v^k$ are finite sums in R, so that $v^{-1} \in R[[\xi^1, \dots, \xi^q]]$. It suffices now to observe that

$$(1 - v)\sum_{k\geq 0} v^k = \sum_{k\geq 0} v^k - \sum_{k\geq 1} v^k = 1\ .$$

□

Lemma 3.4. *A series $w \in R[[\xi^1, \dots, \xi^q]]$ is invertible if and only if its independent term w_0 is invertible in R.*

Proof. Necessity directly follows from the definition of the multiplication in $R[[\xi^1, \ldots, \xi^q]]$. Conversely, consider $w \in R[[\xi^1, \ldots, \xi^q]]$ with w_0 invertible: $w = w_0(1 - v)$. In view of the preceding lemma, we then have $w^{-1} = w_0^{-1} \sum_{k \geq 0} v^k$. □

We are now prepared to prove the

Proposition 3.5. *Any $\mathbb{Z}_2^n$-superdomain $(U, C_U^\infty[[\xi^1, \ldots, \xi^q]])$ is a locally $\mathbb{Z}_2^n$-ringed space, i.e. for any $x \in U$, the stalk $C_{U,x}^\infty[[\xi^1, \ldots, \xi^q]]$ has a unique maximal homogeneous ideal*

$$\mathfrak{m}_x = \{[f]_x : f_0(x) = 0\} .$$

Proof. Set $S_x = C_{U,x}^\infty[[\xi^1, \ldots, \xi^q]]$. In view of Lemma 3.4, a series $[f]_x \in S_x$ is invertible if and only if $[f_0]_x \in C_{U,x}^\infty$ is invertible, i.e. if and only if $f_0(x) \neq 0$:

$$S_x \setminus S_x^\times = \{[f]_x : f_0(x) = 0\} .$$

The latter is clearly a proper homogeneous ideal. Let I_x be any proper homogeneous ideal. If it strictly contains $S_x \setminus S_x^\times$, it contains an invertible element of S_x and can thus not be proper: the homogeneous ideal $\mathfrak{m}_x := S_x \setminus S_x^\times$ is maximal. If I_x is another maximal homogeneous ideal, it does not contain any invertible element: $I_x \subset \mathfrak{m}_x \subset S_x$ – a contradiction. □

Moreover, Lemma 3.4 has the following

Corollary 3.6. *For any open $V \subset U$, a $\mathbb{Z}_2^n$-function $f \in \mathcal{O}_U(V) = C_U^\infty(V)[[\xi^1, \ldots, \xi^q]]$ is invertible in $\mathcal{O}_U(V)$ if and only if its independent term f_0 is invertible in $C_U^\infty(V)$.*

This corollary guarantees that a number of results of classical Supergeometry still hold in $\mathbb{Z}_2^n$-Supergeometry, although formal variables are no longer necessarily nilpotent.

3.4 Completeness of $\mathbb{Z}_2^n$-function algebras

The algebra $\mathcal{O}(V) = C^\infty(V)[[\xi^1, \ldots, \xi^q]]$ of formal power series is the completion of the algebra $\mathfrak{O}(V) = C^\infty(V)[\xi^1, \ldots, \xi^q]$ of polynomials. Moreover,

Proposition 3.7. *The algebra $\mathcal{O}(V) = C^\infty(V)[[\xi^1, \ldots, \xi^q]]$ of $\mathbb{Z}_2^n$-functions on V is Hausdorff-complete (in the sense of Section 7.1).*

Proof. Consider a $\mathbb{Z}_2^n$-superdomain with $\mathbb{Z}_2^n$-functions

$$f(x, \xi) = \sum_{|\mu|=0}^{\infty} f_\mu(x) \xi^\mu \in \mathcal{O}(V) = C^\infty(V)[[\xi^1, \ldots, \xi^q]] .$$

The number $k := |\mu|$ of generators defines an $\mathbb{N}$-grading in $\mathcal{O}(V)$ that induces a decreasing filtration $\mathcal{O}_\ell(V) = C^\infty(V)[[\xi^1, \ldots, \xi^q]]_{\geq \ell}$, where subscript $\geq \ell$ means that we consider only series whose terms contain at least ℓ parameters ξ^a (in the following we omit V if no confusion arises). Of course $J = J^1 := \mathcal{O}_1$ – the kernel of the projection of $\mathbb{Z}_2^n$- onto base-functions – is an ideal of $\mathcal{O}$ and $J^\ell = \mathcal{O}_\ell$: $\mathcal{O} \supset J \supset J^2 \supset \ldots$ The sequence $\mathcal{O}/J \leftarrow \mathcal{O}/J^2 \leftarrow \mathcal{O}/J^3 \leftarrow \ldots$, which can be identified with the sequence $C^\infty \leftarrow C^\infty[[\xi^1,\dots,\xi^q]]_{\le 1} \leftarrow C^\infty(V)[[\xi^1,\dots,\xi^q]]_{\le 2} \leftarrow \dots$, is an inverse system, whose limit is

$$\varprojlim_\ell \mathcal{O}/J^\ell = \mathcal{O} \; . \tag{17}$$

This means that $\mathcal{O}$ is Hausdorff-complete, see Section 7.1. □

- It is well known that Equation (17) means that $\mathcal{O}$ is a complete topological algebra with respect to the topology in $\mathcal{O}$ defined by the filtration J^ℓ, $\ell \ge 1$, viewed as a basis of neighborhoods of 0.

- Remark that also the sequence $\mathfrak{O}/\mathfrak{J} \leftarrow \mathfrak{O}/\mathfrak{J}^2 \leftarrow \mathfrak{O}/\mathfrak{J}^3 \leftarrow \dots$ can be identified with $C^\infty \leftarrow C^\infty[[\xi^1,\dots,\xi^q]]_{\le 1} \leftarrow C^\infty(V)[[\xi^1,\dots,\xi^q]]_{\le 2} \leftarrow \dots$ It follows that

$$\varprojlim_\ell \mathfrak{O}/\mathfrak{J}^\ell = \mathcal{O} \; , \tag{18}$$

  so that $\mathcal{O}$ is actually the completion $\widehat{\mathfrak{O}}$ of $\mathfrak{O}$ with respect to the filtration implemented by $\mathfrak{J}$ (as well as, in view of (17), its own completion with respect to J).

3.5 Morphisms of $\mathbb{Z}_2^n$-superdomains

The following remark shows that morphisms of $\mathbb{Z}_2^n$-superdomains can be viewed as in classical differential Geometry. It will be formulated more rigorously in the case of general $\mathbb{Z}_2^n$-supermanifolds.

Consider two $\mathbb{Z}_2^n$-superdomains of dimension $p|\mathbf{q}$ and $p'|\mathbf{q}'$ over open subsets $U \subset \mathbb{R}^p$ and $U' \subset \mathbb{R}^{p'}$, respectively. Roughly, $\mathbb{Z}_2^n$-morphisms between these $\mathbb{Z}_2^n$-superdomains correspond to graded unital $\mathbb{R}$-algebra morphisms

$$\phi^* : C^\infty(V')[[\xi'^1,\dots,\xi'^{q'}]] \to C^\infty(V)[[\xi^1,\dots,\xi^q]]$$

and are determined by their coordinate form

$$\begin{aligned} x'^i &= \varphi^i(x) + \sum_{\sigma(\mu)=0} f^i_\mu(x)\xi^\mu \, , \\ \xi'^a &= \sum_{\sigma(\mu)=\sigma_a} f^a_\mu(x)\xi^\mu \, , \end{aligned} \tag{19}$$

where the sums are formal series with coefficients in smooth functions and where $\varphi : V \ni (x^1,\dots,x^p) \mapsto (x'^1,\dots,x'^{p'}) \in V'$ is a smooth map.

Example 3.8. In the case of $\mathbb{Z}_2^2$-superdomains of dimension $1|1|1|1$ with variables (x,ξ,η,ϑ) (resp., (y,α,β,γ)) of $\mathbb{Z}_2^2$-degrees $((0,0),(0,1),(1,0),(1,1))$, a $\mathbb{Z}_2^2$-morphism can be viewed as usual:

$$\begin{cases} y = \sum_r f_r^y(x)\vartheta^{2r} + \sum_r g_r^y(x)\vartheta^{2r+1}\xi\eta \, , \\ \alpha = \sum_r f_r^\alpha(x)\vartheta^{2r}\xi + \sum_r g_r^\alpha(x)\vartheta^{2r+1}\eta \, , \\ \beta = \sum_r f_r^\beta(x)\vartheta^{2r}\eta + \sum_r g_r^\beta(x)\vartheta^{2r+1}\xi \, , \\ \gamma = \sum_r f_r^\gamma(x)\vartheta^{2r+1} + \sum_r g_r^\gamma(x)\vartheta^{2r}\xi\eta \, . \end{cases} \tag{20}$$

To explain the above claim, we have to prove that any $\mathbb{Z}_2^n$-morphism has a coordinate form of the announced type (what is almost obvious), and that, conversely, any pullbacks $\phi^*(x'^i)$ ($\simeq x'^i$) and $\phi^*(\xi'^a)$ ($\simeq \xi'^a$) of the form (19) uniquely extend to a $\mathbb{Z}_2^n$-morphism. We will show here that such a $\mathbb{Z}_2^n$-morphism does exist. Uniqueness (and other details) will be proven independently in the more general case of $\mathbb{Z}_2^n$-morphisms of $\mathbb{Z}_2^n$-supermanifolds.

In the sequel we write $\phi^*(x'^i) = \varphi^i(x) + j^i(x, \xi)$, with $j^i(x, \xi) = \sum_{\sigma(\mu)=0} f^i_\mu(x)\xi^\mu \in J$. For any

$$g(x', \xi') = \sum_{|\nu|\geq 0} g_\nu(x')\xi'^\nu \in C^\infty(V')[[\xi'^1, \dots, \xi'^{q'}]] \; ,$$

we set

$$(\phi^*(g))(x, \xi) = \sum_{|\nu|\geq 0} \phi^*(g_\nu(x'))(\phi^*(\xi'))^\nu \; , \tag{21}$$

where

$$\phi^*(g_\nu(x')) = g_\nu(\phi^*(x')) = g_\nu(\varphi(x) + j(x, \xi)) = \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} \, (\partial^\alpha_{x'} g_\nu)(\varphi(x)) \, j^\alpha(x, \xi) \tag{22}$$

is a formal Taylor expansion; we use here the multiindex notation: $j^\alpha = (j^1)^{\alpha^1} \dots (j^{p'})^{\alpha^{p'}} \in J^{|\alpha|}$. In fact the RHS of (22) is a series of series and it could lead to rearranged series with non-converging series of $C^\infty(V)$-coefficients. However, any type of monomial in the formal variables ξ^a contains a fixed number N of parameters. As the terms indexed by $|\alpha| > N$ contain at least $N + 1$ parameters, they not contribute to the considered monomial. The coefficient of the latter is therefore a finite sum in $C^\infty(V)$, so that the RHS of (22) is actually a series in $C^\infty(V)[[\xi^1, \dots, \xi^q]]$. The same argument can be used for the RHS of (21).

It is quite easily seen that the thus defined pullback map ϕ^* is a unital (obvious) graded $\mathbb{R}$-algebra morphism. As for the degree of ϕ^*, note that j^i is of degree 0, so that $\phi^*(g_\nu(x'))$ has $\mathbb{Z}_2^n$-degree 0; Equation (21) allows now to see that ϕ^* is of degree 0. To prove that ϕ^* is an algebra morphism, we first show that its restriction (22) respects multiplication. If $g_\nu, h_\rho \in C^\infty(V')$, we get

$$\begin{aligned}
\phi^*(g_\nu h_\rho) &= \sum_\alpha \frac{1}{\alpha!} \, \partial^\alpha_{x'}(g_\nu h_\rho) \, j^\alpha \\
&= \sum_\alpha \sum_{\beta+\gamma=\alpha} \frac{1}{\alpha!} \frac{\alpha!}{\beta!\gamma!} \, \partial^\beta_{x'} g_\nu \, \partial^\gamma_{x'} h_\rho \, j^{\beta+\gamma} \\
&= \sum_\beta \sum_\gamma \frac{1}{\beta!\gamma!} \, \partial^\beta_{x'} g_\nu \, \partial^\gamma_{x'} h_\rho \, j^\beta j^\gamma \\
&= \phi^*(g_\nu) \, \phi^*(h_\rho) \; ,
\end{aligned}$$

where we omitted for simplicity the evaluation at $\varphi(x)$, as well as the variables of j (remember that j^i is of degree 0). Let now $g = \sum_\nu g_\nu \xi'^\nu$ and $h = \sum_\rho h_\rho \xi'^\rho$ be two arbitrary $\mathbb{Z}_2^n$-functions:

$$gh = \sum_\alpha \sum_{\nu+\rho=\alpha} \pm g_\nu h_\rho \xi'^\alpha \; ,$$

where the sign is due to commutation of components of ξ'. Thus

$$\phi^*(gh) = \sum_\alpha \phi^* \left(\sum_{\nu+\rho=\alpha} \pm g_\nu h_\rho \right) (\phi^*(\xi'))^\alpha$$

$$= \sum_\alpha \sum_{\nu+\rho=\alpha} \phi^*(g_\nu)\, \phi^*(h_\rho)\, (\phi^*(\xi'))^\nu\, (\phi^*(\xi'))^\rho\ ,$$

where the sign disappears as ϕ^* is of degree 0. The conclusion follows.

As mentioned above, the precise definition of a $\mathbb{Z}_2^n$-morphism as a morphism of locally $\mathbb{Z}_2^n$-ringed spaces will be given in Section 4, and the preceding explanation will be completed and generalized.

4 $\mathbb{Z}_2^n$-Supermanifolds

4.1 Definitions

A $\mathbb{Z}_2^n$-supermanifold is a locally $\mathbb{Z}_2^n$-ringed space (LZRS) that is locally modelled on a $\mathbb{Z}_2^n$-superdomain. For details on the category `LZRS` of LZRS, we encourage the reader to have a look at Section 7.2. In the following, the elements of $\mathbb{Z}_2^n$, $n \in \mathbb{N}$, are considered as ordered lexicographically.

Definition 4.1 (Ringed space definition)**.** A (smooth) $\mathbb{Z}_2^n$*-supermanifold* $\mathcal{M}$ of dimension $p|\mathbf{q}$, $p \in \mathbb{N}$, $\mathbf{q} = (q_1, \ldots, q_{2^n-1}) \in \mathbb{N}^{2^n-1}$, is a LZRS $(M, \mathcal{A}_M)$ that is locally isomorphic to a $\mathbb{Z}_2^n$-superdomain $C^\infty_{\mathbb{R}^p}[[\xi^1, \ldots, \xi^q]]$, where $q = |\mathbf{q}|$ and where $\xi^1, \ldots, \xi^q$ are formal variables of which q_k have the k-th degree in $\mathbb{Z}_2^n \setminus \{0\}$.

The mentioned isomorphisms are of course invertible morphisms in `LZRS`.

Many geometric concepts can be glued from local pieces: they can be defined via a cover by coordinate systems, with specific coordinate transformations that satisfy the usual cocycle condition. The same holds for $\mathbb{Z}_2^n$-supermanifolds. Roughly, a $\mathbb{Z}_2^n$-supermanifold of dimension $p|\mathbf{q}$ can be viewed as a second-countable Hausdorff topological space M surrounded by 'a cloud of formal stuff', which is locally (with respect to the topology of M) described by coordinate systems (x, ξ), where $x = (x^1, \ldots, x^p) \in U \subset \mathbb{R}^p$ is of degree 0 (and can be viewed as a homeomorphism $x(m) \rightleftarrows m(x)$ between U an open subset of M – which is often also denoted by U) and $\xi = (\xi^1, \ldots, \xi^q)$ are formal variables as in Definition 4.1; further, the coordinate transformations respect the $\mathbb{Z}_2^n$-degree and satisfy the cocycle condition.

The rigorous alternative definition of $\mathbb{Z}_2^n$-supermanifolds follows naturally from this idea. It is similar to the atlas description of a supermanifold [Lei80].

Definition 4.2. A *chart* (or coordinate system) over a (second-countable Hausdorff) topological space M is a LZRS

$$\mathcal{U} = (U, C^\infty_U[[\xi^1, \ldots, \xi^q]]),\ U \subset \mathbb{R}^p, p, q \in \mathbb{N}\ ,$$

together with a homeomorphism $\psi : U \to \psi(U)$, where $\psi(U)$ is an open subset of M.

Given two charts $(\mathcal{U}_\alpha, \psi_\alpha)$ and $(\mathcal{U}_\beta, \psi_\beta)$ over M, we will denote by $\psi_{\beta\alpha}$ the homeomorphism

$$\psi_{\beta\alpha} := \psi_\beta^{-1}\psi_\alpha : V_{\beta\alpha} := \psi_\alpha^{-1}(\psi_\alpha(U_\alpha) \cap \psi_\beta(U_\beta)) \to V_{\alpha\beta} := \psi_\beta^{-1}(\psi_\alpha(U_\alpha) \cap \psi_\beta(U_\beta)) \, .$$

Whereas in classical Differential Geometry the coordinate transformations are completely defined by the coordinate systems, in ($\mathbb{Z}_2^n$-)Supergeometry, they have to be specified separately.

Definition 4.3. A *coordinate transformation* between two charts $(\mathcal{U}_\alpha, \psi_\alpha)$ and $(\mathcal{U}_\beta, \psi_\beta)$ over M is an isomorphism of LZRS $\Psi_{\beta\alpha} = (\psi_{\beta\alpha}, \psi^*_{\beta\alpha}) : \mathcal{U}_\alpha|_{V_{\beta\alpha}} \to \mathcal{U}_\beta|_{V_{\alpha\beta}}$, where the source and target are restrictions of 'sheaves' (note that the underlying homeomorphism is $\psi_{\beta\alpha}$).

An *atlas* over M is a covering $(\mathcal{U}_\alpha, \psi_\alpha)_\alpha$ by charts together with a coordinate transformation for each pair of charts, such that the usual cocycle condition $\Psi_{\beta\gamma}\Psi_{\gamma\alpha} = \Psi_{\beta\alpha}$ holds (appropriate restrictions are understood).

Definition 4.4 (Atlas definition)**.** A (smooth) $\mathbb{Z}_2^n$*-supermanifold* $\mathcal{M}$ is a second-countable Hausdorff topological space M together with a preferred atlas $(\mathcal{U}_\alpha, \psi_\alpha)_\alpha$ over it.

4.2 Rationale

Let us consider the case $n = 2$, $p|q_1|q_2|q_3 = 1|1|1|1$, and assume for simplicity that the underlying topological space M carries a smooth manifold structure (we prove later that the underlying topological space of any $\mathbb{Z}_2^n$-supermanifold carries a smooth structure). We use notation from Examples 3.2 and 3.8; in particular $(x, \xi, \eta, \vartheta)$ are of degree $((0,0),(0,1),(1,0),(1,1))$. A coordinate transformation $(x, \xi, \eta, \vartheta) \rightleftarrows (y, \alpha, \beta, \gamma)$ is then of the form

$$(a)\begin{cases} y = \sum_r f_r^y(x)\vartheta^{2r} + \sum_r g_r^y(x)\vartheta^{2r+1}\xi\eta \\ \alpha = \sum_r f_r^\alpha(x)\vartheta^{2r}\xi + \sum_r g_r^\alpha(x)\vartheta^{2r+1}\eta \\ \beta = \sum_r f_r^\beta(x)\vartheta^{2r}\eta + \sum_r g_r^\beta(x)\vartheta^{2r+1}\xi \\ \gamma = \sum_r f_r^\gamma(x)\vartheta^{2r+1} + \sum_r g_r^\gamma(x)\vartheta^{2r}\xi\eta \end{cases} \quad (b)\begin{cases} x = \sum_r F_r^x(y)\gamma^{2r} + \sum_r G_r^x(y)\gamma^{2r+1}\alpha\beta \\ \xi = \sum_r F_r^\xi(y)\gamma^{2r}\alpha + \sum_r G_r^\xi(y)\gamma^{2r+1}\beta \\ \eta = \sum_r F_r^\eta(y)\gamma^{2r}\beta + \sum_r G_r^\eta(y)\gamma^{2r+1}\alpha \\ \vartheta = \sum_r F_r^\vartheta(y)\gamma^{2r+1} + \sum_r G_r^\vartheta(y)\gamma^{2r}\alpha\beta \end{cases} \tag{23}$$

where the functions in x and y are smooth.

The substitution of (23)(a) in a local function

$$f(y, \alpha, \beta, \gamma) = \sum_{i,j\in\{0,1\},\ r\in\mathbb{N}} f_{ijr}(y)\alpha^i\beta^j\gamma^r \tag{24}$$

leads to a function $g(x, \xi, \eta, \vartheta)$ in the initial variables – the pullback of f. As mentioned before, to transform

$$f_{ijr}\left(\sum_r f_r^y(x)\vartheta^{2r} + \sum_r g_r^y(x)\vartheta^{2r+1}\xi\eta\right) \, , \tag{25}$$

we detach the independent term $f_0^y(x)$ from the series $j(x, \xi, \eta, \vartheta)$ of all the remaining terms and set

$$f_{ijr}\left(f_0^y(x) + j(x, \xi, \eta, \vartheta)\right) = \sum_n \frac{1}{n!}\frac{\mathrm{d}^n f_{ijr}}{\mathrm{d}\, y^n}(f_0^y(x))\; j^n(x, \xi, \eta, \vartheta) \; . \tag{26}$$

It is now quite obvious that a coordinate transformation (23) in a $\mathbb{Z}_2^n$-supermanifold induces a coordinate transformation $y = f_0^y(x)$, $x = F_0^x(y)$ in the base manifold. Indeed, since the transformations (23) are inverse, we get x when substituting (23)(a) in

$$x = \sum_r F_r^x(y)\gamma^{2r} + \sum_r G_r^x(y)\gamma^{2r+1}\alpha\beta \ ,$$

i.e. all the terms of the RHS that contain, after substitution, at least one parameter cancel, whereas the unique parameter independent term $F_0^x(f_0^y(x))$ coincides with x. Similarly, $f_0^y(F_0^x(y)) = y$.

From the 'atlas standpoint', a global $\mathbb{Z}_2^2$-superfunction $f \in \mathcal{A}_M^{(i,j)}(M)$ of degree $(i,j) \in \mathbb{Z}_2^2$, is a family $f(y,\alpha,\beta,\gamma)$, over all coordinate systems (y,α,β,γ), of local functions of degree (i,j), such that, when substituting (23)(a) in $f(y,\alpha,\beta,\gamma)$, we get the function $f(x,\xi,\eta,\vartheta)$ associated to the coordinate system (x,ξ,η,ϑ) – just as a global function $g \in C_M^\infty(M)$ is a family $g(y)$, over the induced coordinate systems (y), such that when substituting $y = f_0^y(x)$ in $g(y)$, we get $g(x)$. The degree (i,j) is compatible with the coordinate transformations as the latter respect the degrees.

For any global $\mathbb{Z}_2^2$-superfunction in $\mathcal{A}_M(M)$, i.e. any family of 'glueable' local functions $f(y,\alpha,\beta,\gamma)$, see (24), the induced family $f_{000}(y)$ defines a global base function in $C_M^\infty(M)$. Indeed, in view of what has been said, it is easily checked that the gluing property of the family $f(y,\alpha,\beta,\gamma)$ entails that $f_{000}(f_0^y(x)) = f_{000}(x)$.

Remark 4.5. This means that the canonical projections of the local expressions of a global function glue to give a global base function. In particular projection commutes with restriction. Therefore, projection is an algebra morphism.

5 Examples

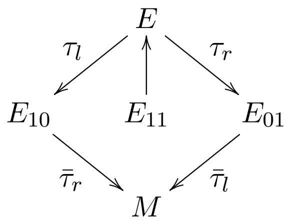
Example 5.1. For $n = 1$, we recover classical supermanifolds. Indeed, in this case there are no formal variables that bear powers higher than 1 and formal series are thus just polynomials.

Example 5.2. We already mentioned that the tangent bundle $\mathsf{T}\mathcal{M}$ to a $\mathbb{Z}_2$-supermanifold $\mathcal{M} = (M, \mathcal{A}_M)$ gives rise to a $\mathbb{Z}_2^2$-supermanifold. Indeed, the parities of local coordinates (x^i, ξ^a) on $\mathcal{M}$ induce canonically parities of the adapted system of coordinates $(x^i, \xi^a, \dot{x}^j, \dot{\xi}^b)$ on $\mathsf{T}\mathcal{M}$ in which $(x^i, \dot{x}^j)$ are even and $(\xi^a, \dot{\xi}^b)$ are odd. But $\mathsf{T}\mathcal{M}$ is also a vector bundle what induces an additional $\mathbb{N}$-gradation in which $(\dot{x}^j, \dot{\xi}^b)$ are of degree 1. Using the canonical monoid homomorphism from $\mathbb{N}$ to $\mathbb{Z}_2$, we get a $\mathbb{Z}_2^2$-gradation in which $(x^i, \xi^a, \dot{x}^j, \dot{\xi}^b)$ have the bidegrees $((0,0),(0,1),(1,0),(1,1))$. We can find an atlas whose coordinate changes respect the bidegrees; hence, we deal with a $\mathbb{Z}_2^2$-supermanifold. As the changes of coordinates are linear in $(\dot{x}^j, \dot{\xi}^b)$, the algebra of $\mathbb{Z}_2^2$-superfunctions which are polynomial in the latter variables is well-defined. It can be identified with the algebra $\Omega_D(\mathcal{M})$ of Deligne super differential forms on $\mathcal{M}$. Since it is dense in the whole algebra of $\mathbb{Z}_2^2$-superfunctions on $\mathsf{T}\mathcal{M}$, the latter can be identified with the corresponding completion

$$\widehat{\Omega}_D(\mathcal{M}) = \prod_{k\geq 0} \wedge^k \Omega^1 \ .$$

Proposition 5.3. *The tangent bundle* $\mathsf{T}\mathcal{M}$ *of a* $\mathbb{Z}_2$*-supermanifold* $\mathcal{M} = (M, \mathcal{A}_M)$*, interpreted as ringed space* $(M, \widehat{\Omega}_D(\mathcal{M}))$*, is a* $\mathbb{Z}_2^2$*-supermanifold.*

Example 5.4. Let now



(27)

be a double vector bundle with the side bundles E_{01}, E_{10} and the core bundle E_{11}. This corresponds to a choice of two commuting Euler vector fields ∇_1, ∇_2 on E [GR09]. We can choose an atlas with bihomogeneous local coordinates, say $(x, \xi, \eta, \vartheta)$ of bidegrees $(0,0), (0,1), (1,0), (1,1)$, respectively. Moreover, all coordinate changes have the form

$$\begin{aligned} x' &= \varphi(x)\,, \\ \xi' &= a(x)\xi\,, \\ \eta' &= b(x)\eta\,, \\ \vartheta' &= c(x)\vartheta + d(x)\xi\eta\,, \end{aligned}$$

and thus respect the bigradation. We can now 'superize' assuming that these coordinates satisfy the sign rules of the 'scalar product' in $\mathbb{Z}_2^2$. As the coordinate changes respect the bidegrees, this is consistent and leads to a $\mathbb{Z}_2^2$-supermanifold ΠE. In the super case, we have to fix the ordering, as its change may result in changing the sign (see discussion in [GR09]).

All this can be generalized to n-tuple vector bundles if we fix a lexicographic ordering in $\mathbb{Z}_2^n$ relative to an ordering of the corresponding Euler vector fields.

Proposition 5.5. *The superization of an* n*-vector bundle,* $n \geq 1$*, is a* $\mathbb{Z}_2^n$*-supermanifold.*

Note that certain superizations of n-tuple vector bundles have been considered also by Voronov.

6 Morphisms of $\mathbb{Z}_2^n$-supermanifolds

6.1 Embedding of the smooth base manifold

We already mentioned that global $\mathbb{Z}_2^n$-functions project consistently to the base, see Remark 4.5. In the present section, we make this observation more precise.

Proposition 6.1. *The base topological space* M *of any* $\mathbb{Z}_2^n$*-supermanifold* $\mathcal{M} = (M, \mathcal{A}_M)$ *of dimension* $p|\mathbf{q}$ *carries a smooth manifold structure of dimension* p*, and there exists a short exact sequence of sheaves*

$$0 \to \mathcal{J}_M \to \mathcal{A}_M \overset{\varepsilon}{\to} C^\infty_M \to 0\,.$$

Proof. Let $V \subset \mathbb{R}^p$ be open, let $f \in \mathcal{O}(V) = C^\infty(V)[[\xi^1, \dots, \xi^q]]$, let $x \in V$ and $k \in \mathbb{R}$. In view of Corollary 3.6, the $\mathbb{Z}_2^n$-function $f - k$ is not invertible, in any neighborhood of x in V, if and only if its independent term $f_0 - k$ is not invertible, in any neighborhood of x, i.e. if and only if $k = f_0(x)$. Hence, for any $V \subset \mathbb{R}^p$, any $f \in \mathcal{O}(V)$ and any $x \in V$, there exists a unique $k \in \mathbb{R}$, such that $f - k$ is not invertible, in any neighborhood of x in V. Since $\mathcal{A}_M$ is locally isomorphic to $\mathcal{O}_{\mathbb{R}^p}$, the same property holds in $\mathcal{A}_M$. For any open $U \subset M$, for any $f \in \mathcal{A}(U)$ and any $m \in U$, the unique $k \in \mathbb{R}$ such that $f - k$ is not invertible, in any neighborhood of m, is denoted by $\varepsilon_U(f)(m)$. If m runs through U, we obtain a function $\varepsilon_U(f) : U \to \mathbb{R}$, and if f runs through $\mathcal{A}(U)$, we get a map $\varepsilon_U : \mathcal{A}(U) \to \mathcal{F}(U)$, where $\mathcal{F}(U) = \operatorname{im} \varepsilon_U$ is the algebra of these functions. Actually ε_U is a surjective algebra morphism and the short sequence of algebras

$$0 \to \mathcal{J}(U) \to \mathcal{A}(U) \to \mathcal{F}(U) \to 0 \ ,$$

where $\mathcal{J}(U) = \ker \varepsilon_U$, is exact. In fact $\mathcal{J}_M : U \to \mathcal{J}(U)$ is a subsheaf of $\mathcal{A}_M$. On the other hand, it is clear that the presheaf $\mathcal{F}_M$ is locally isomorphic to $C^\infty_{\mathbb{R}^p}$ and is thus locally a sheaf. Hence, $\mathcal{F}_M$ generates a sheaf $\mathfrak{F}_M$ which is locally isomorphic to $C^\infty_{\mathbb{R}^p}$ and thus implements a p-dimensional differential manifold structure on M such that $C^\infty_M \simeq \mathfrak{F}_M$. Since the sequence of sheaves

$$0 \to \mathcal{J}_M \to \mathcal{A}_M \xrightarrow{\varepsilon} C^\infty_M \to 0$$

is exact, we have $\mathcal{A}_M / \mathcal{J}_M \simeq C^\infty_M$. For details on sheaves, we refer the interested reader to Section 7.3. See also [Var04]. □

6.2 Continuity of morphisms

In the 'ringed space definition' of $\mathbb{Z}_2^n$-supermanifolds the requirement that $(M, \mathcal{A}_M)$ be local is actually redundant – in view of the local model. The unique maximal homogeneous ideal of $\mathcal{A}_m$, $m \in M$, will be denoted by $\mathfrak{m}_m$.

The key-fact about morphisms of $\mathbb{Z}_2^n$-supermanifolds is a generalization of Section 3.5, see below. This result can be proved due to the continuity of morphisms with respect to the topologies induced by the ideals $\mathcal{J}(U) \subset \mathcal{A}(U)$ and $\mathfrak{m}_m \subset \mathcal{A}_m$.

In this section, we prove these continuities.

Definition 6.2. A *morphism of $\mathbb{Z}_2^n$-supermanifolds* or *$\mathbb{Z}_2^n$-morphism* is a morphism of the underlying locally $\mathbb{Z}_2^n$-ringed spaces.

This means that the category `ZSMan` of $\mathbb{Z}_2^n$-supermanifolds is a full subcategory of the category `LZRS`, see Section 7.2.

We first show that $\mathbb{Z}_2^n$-morphisms commute with the projections ε onto the bases:

Proposition 6.3. *Let*

$$\Psi = (\psi, \psi^*) : \mathcal{M} = (M, \mathcal{A}_M) \to \mathcal{N} = (N, \mathcal{B}_N)$$

be a morphism of $\mathbb{Z}_2^n$-supermanifolds, let $V \subset N$ be an open subset, and $U = \psi^{-1}(V)$. Then,

$$\varepsilon_U \circ \psi^*_V = \psi^*_V \circ \varepsilon_V \ , \tag{28}$$

where the LHS pullback of $\mathbb{Z}_2^n$-functions is given by the second component of Ψ and where the RHS pullback of classical functions is equal to $- \circ \psi|_U$ and thus given by the first component of Ψ.

Proof. Let $t \in \mathcal{B}(V)$ and $m \in U$. If we set $s = \psi_V^*(t) \in \mathcal{A}(U)$, we have to show that

$$\varepsilon_U(s)(m) = \varepsilon_V(t)(\psi(m)) .$$

The RHS of this equation is, by definition, the unique $k \in \mathbb{R}$ such that $t - k$ is not invertible, in any neighborhood of $\psi(m)$. It suffices thus to prove that the LHS has this property. Suppose that $t - \varepsilon_U(s)(m)$ is invertible in some neighborhood of $\psi(m)$. Then, since ψ_V^* is a unital $\mathbb{R}$-algebra morphism,

$$\psi_V^* \left(t - \varepsilon_U(s)(m)\right) = \psi_V^*(t) - \varepsilon_U(s)(m)\ \psi_V^*(1) = s - \varepsilon_U(s)(m)$$

is invertible in some neighborhood of m – a contradiction. □

Corollary 6.4. *For any $\mathbb{Z}_2^n$-supermanifold $\mathcal{M} = (M, \mathcal{A}_M)$ and any point $m \in M$, the unique maximal homogeneous ideal $\mathfrak{m}_m$ of $\mathcal{A}_m$ is given by*

$$\mathfrak{m}_m = \{[f]_m : (\varepsilon f)(m) = 0\} . \tag{29}$$

Proof. If

$$\Phi = (\phi, \phi^*) : (U, \mathcal{A}_M|_U) \xrightarrow{\sim} (V, C^\infty_{\mathbb{R}^p}|_V[[\xi^1, \ldots, \xi^q]])$$

denotes an isomorphism in LZRS, with $m \in U$, we have $\mathfrak{m}_m = \phi_m^* \mathfrak{m}_x$, where $x = \phi(m)$. It now suffices to apply Proposition 6.3. □

As mentioned above, we need not assume that $(M, \mathcal{A}_M)$ is local. Then Φ is only an isomorphism in ZRS and we cannot ask that ϕ_m^* respects maximal ideals. However, since ϕ_m^* is an isomorphism of graded unital $\mathbb{R}$-algebras, $\phi_m^* \mathfrak{m}_x$ *is* the unique maximal homogeneous ideal of $\mathcal{A}_m$.

The next result is the announced $\mathcal{J}$- and $\mathfrak{m}$-continuity theorem for $\mathbb{Z}_2^n$-morphisms. It shows in particular that $\mathbb{Z}_2^n$-morphisms automatically respect maximal ideals, so that this requirement is actually redundant in the definition of $\mathbb{Z}_2^n$-morphisms.

Corollary 6.5. *Any $\mathbb{Z}_2^n$-morphism $\Psi = (\psi, \psi^*) : \mathcal{M} = (M, \mathcal{A}_M) \to \mathcal{N} = (N, \mathcal{B}_N)$ is continuous with respect to $\mathcal{J}$ and $\mathfrak{m}$, i.e., for any open $V \subset N$ and any $m \in M$, we have*

$$\psi_V^* \left(\mathcal{J}_N(V)\right) \subset \mathcal{J}_M(\psi^{-1}(V)) \ \text{ and } \ \psi_m^* \left(\mathfrak{m}_{\psi(m)}\right) \subset \mathfrak{m}_m .$$

Proof. This result is a direct consequence of the definition $\mathcal{J} = \ker \varepsilon$, Equation (29), and Proposition 6.3. □

Corollary 6.6. *The base map $\psi : M \to N$ of any $\mathbb{Z}_2^n$-morphism $\Psi : (M, \mathcal{A}_M) \to (N, \mathcal{B}_N)$ is smooth.*

Proof. Let $m \in M$, let $(V, y = (y^1, \ldots, y^u))$ be a classical chart of N around $\psi(m)$, and set $U = \psi^{-1}(V)$. For any $g \in C^\infty(V)$, there is $t \in \mathcal{B}(V)$ (just restrict V), such that

$$g \circ \psi = \psi^* (\varepsilon_V(t)) = \varepsilon_U (\psi^*(t)) \in C^\infty(U) \; .$$

In particular, for $g = y^j$, we get $\psi^j = y^j \circ \psi \; \in C^\infty(U)$, so that $\psi \in C^\infty(U, N)$ and, since U is a neighborhood of an arbitrary point $m \in M$, $\psi \in C^\infty(M, N)$. □

6.3 Completeness of the $\mathbb{Z}_2^n$-function sheaf and the $\mathbb{Z}_2^n$-function algebras

In standard Supergeometry, the decreasing filtration $\mathcal{A} \supset \mathcal{J} \supset \mathcal{J}^2 \supset \ldots$ of the structure sheaf $\mathcal{A}$ of a supermanifold $\mathcal{M} = (M, \mathcal{A})$ by the sheafs of ideals $\mathcal{J}^k$ [Var04], induces embeddings

$$M \hookrightarrow \mathcal{M}^2 \hookrightarrow \mathcal{M}^3 \hookrightarrow \ldots \hookrightarrow \mathcal{M} \; ,$$

where $\mathcal{M}^k = (M, \mathcal{A}/\mathcal{J}^k)$ is the superspace characterized by the sheaf $\mathcal{A}/\mathcal{J}^k$ [Man02] (see also Section 7.3).

Let now $\mathcal{M} = (M, \mathcal{A})$ be a $\mathbb{Z}_2^n$-supermanifold. The decreasing filtration $\mathcal{A} \supset \mathcal{J} \supset \mathcal{J}^2 \supset \ldots$ gives rise to an inverse system

$$\mathcal{A}/\mathcal{J} \leftarrow \mathcal{A}/\mathcal{J}^2 \leftarrow \mathcal{A}/\mathcal{J}^3 \leftarrow \ldots$$

of sheaves of algebras (we prove at the end of this subsection that the quotient presheaves $\mathcal{A}/\mathcal{J}^k$ are actually sheaves). Since a limit is a universal cone, there is a sheaf morphism $\varprojlim_k \mathcal{A}/\mathcal{J}^k \leftarrow \mathcal{A}$. Moreover, as a limit in a category of sheaves is just the corresponding limit in the category of presheaves (which is computed objectwise), we get, for any $\mathbb{Z}_2^n$-superchart domain U_α,

$$\left(\varprojlim_k \mathcal{A}/\mathcal{J}^k\right)(U_\alpha) = \varprojlim_k \mathcal{A}(U_\alpha)/\mathcal{J}^k(U_\alpha) \simeq \mathcal{A}(U_\alpha) \; ,$$

see Equation (17). It follows that

$$\varprojlim_k \mathcal{A}/\mathcal{J}^k \simeq \mathcal{A}$$

in the category of sheaves, so that the structure sheaf $\mathcal{A}$ is complete with respect to the filtration implemented by $\mathcal{J}$. Since isomorphic sheaves have isomorphic sections, we thus obtain, for any $U \subset M$,

$$\varprojlim_k \mathcal{A}(U)/\mathcal{J}^k(U) \simeq \mathcal{A}(U) \; ,$$

i.e. all function algebras $\mathcal{A}(U)$, $U \subset M$, are Hausdorff-complete with respect to the filtration induced by $\mathcal{J}(U)$, see Section 7.1.

Proposition 6.7. *The $\mathbb{Z}_2^n$-function sheaf $\mathcal{A}_M$ (resp., the $\mathbb{Z}_2^n$-function algebra $\mathcal{A}_M(U)$, $U \subset M$) of a $\mathbb{Z}_2^n$-supermanifold $(M, \mathcal{A}_M)$ is Hausdorff-complete with respect to the $\mathcal{J}_M$-adic (resp., $\mathcal{J}_M(U)$-adic) topology.*

Proof. It suffices to show that the presheaves $\mathcal{A}/\mathcal{J}^k$, $k \geq 1$, are in fact sheaves.

Let $s, t \in \mathcal{A}(U)/\mathcal{J}^k(U)$, $U \subset M$, coincide on an open cover $(V_\alpha)_\alpha$ of U. Then $s|_{V_\alpha} = t|_{V_\alpha} + j^k_{V_\alpha}$, with $j^k_{V_\alpha} \in \mathcal{J}^k(V_\alpha)$. The latter thus coincide on the intersections $V_\alpha \cap V_\beta$. Their gluing provides $j^k \in \mathcal{J}^k(U)$ and $s|_{V_\alpha} = t|_{V_\alpha} + j^k|_{V_\alpha}$. Since $\mathcal{A}$ is a separated presheaf (even a sheaf), we get $s = t + j^k$ in $\mathcal{A}(U)$ and $s = t$ in $\mathcal{A}(U)/\mathcal{J}^k(U)$.

As for the second sheaf property, consider a family $s_\alpha \in \mathcal{A}(V_\alpha)/\mathcal{J}^k(V_\alpha)$ such that $s_\alpha = s_\beta$ on $V_\alpha \cap V_\beta$. Let π_α be a partition of unity of $\mathcal{M}$ that is subordinated to V_α (see Section 7.4). Any product $\pi_\alpha s_\alpha$ of s_α by $\pi_\alpha \in \mathcal{A}^0(U)$ ($\operatorname{supp} \pi_\alpha \subset V_\alpha$) is well-defined in $\mathcal{A}(U)/\mathcal{J}^k(U)$ ($\operatorname{supp}(\pi_\alpha s_\alpha) \subset V_\alpha$). Then $s = \sum_\alpha \pi_\alpha s_\alpha \in \mathcal{A}(U)/\mathcal{J}^k(U)$ and $s|_{V_\alpha} = s_\alpha$. □

6.4 Fundamental theorem of $\mathbb{Z}_2^n$-morphisms

In this section we prove an extension of the main statement of Section 3.5.

6.4.1 Statement

Let

$$\Psi = (\psi, \psi^*) : \mathcal{M} = (M, \mathcal{A}_M) \to \mathcal{V}^{u|\mathbf{v}} = (V, C^\infty_V[[\eta^1, \ldots, \eta^v]])$$

be a $\mathbb{Z}_2^n$-morphism valued in a $\mathbb{Z}_2^n$-superdomain. Denote by $y = (y^1, \ldots, y^u)$ the coordinates of V and by $\tau_b \in \mathbb{Z}_2^n \setminus \{0\}$ the degree of η^b. Then the functions

$$s^j := \psi_V^* y^j \text{ and } \zeta^b := \psi_V^* \eta^b$$

(for all j and b) satisfy

$$s^j \in \mathcal{A}^0_M(M)\ , \zeta^b \in \mathcal{A}^{\tau_b}_M(M) \text{ and } \left(\varepsilon s^1, \ldots, \varepsilon s^u\right)(M) \subset V\ . \tag{30}$$

Actually the pullbacks (s^j, ζ^b) (all j and b) of the coordinate functions (y^j, η^b) completely determine the $\mathbb{Z}_2^n$-morphism. More precisely:

Theorem 6.8 (Fundamental theorem of $\mathbb{Z}_2^n$-morphisms)**.** *If $\mathcal{M} = (M, \mathcal{A}_M)$ is a $\mathbb{Z}_2^n$-supermanifold, $\mathcal{V}^{u|\mathbf{v}}$ a $\mathbb{Z}_2^n$-superdomain as above, and if (s^j, ζ^b) is an $(u+v)$-tuple of $\mathbb{Z}_2^n$-functions in $\mathcal{A}_M(M)$ that satisfy the conditions (30), there exists a unique morphism of $\mathbb{Z}_2^n$-supermanifolds $\Psi = (\psi, \psi^*) : \mathcal{M} \to \mathcal{V}^{u|\mathbf{v}}$, such that $s^j = \psi_V^* y^j$ and $\zeta^b = \psi_V^* \eta^b$.*

The proof requires some preparatory work.

6.4.2 Polynomial approximation of $\mathbb{Z}_2^n$-functions

We first describe the $\mathfrak{m}_m$-adic topology of $\mathcal{A}_m$, $m \in M$.

When taking an interest in the stalks $\mathcal{A}_m$ of the function sheaf of a $\mathbb{Z}_2^n$-supermanifold $(M, \mathcal{A}_M)$ of dimension $p|\mathbf{q}$, we can choose a centered chart (x, ξ) around m and work in a $\mathbb{Z}_2^n$-superdomain $\mathcal{U}^{p|\mathbf{q}}$ associated with a convex open subset $U \subset \mathbb{R}^p$, in which $m \simeq x = 0$. Since $\mathfrak{m}_m = \{[f]_m : \varepsilon(f)(m) = 0\}$, a Taylor expansion (with remainder) around $m \simeq x = 0$ of the coordinate form of $\varepsilon(f)$ shows that

$$\mathfrak{m}_m \simeq \{[f]_0 : f(x, \xi) = 0(x) + \sum_{|\mu|>0} f_\mu(x) \xi^\mu\}\ ,$$

where $0(x)$ are terms of degree at least 1 in x.

Lemma 6.9. *For any $m \in M$, the basis of neighborhoods of 0 in the $\mathfrak{m}_m$-adic topology of $\mathcal{A}_m$ is given by*

$$\mathfrak{m}_m^{k+1} = \{[f]_0 : f(x,\xi) = \sum_{0 \le |\mu| \le k} 0_\mu(x^{k-|\mu|+1})\xi^\mu + \sum_{|\mu|>k} f_\mu(x)\xi^\mu\} \quad (k \ge 0)\ , \tag{31}$$

where notation is the same as above.

Proof. The inclusion $\subset$ is obvious. For $\supset$, note that all the terms whose number $|\mu|$ of generators is $\le k$ belong to $\mathfrak{m}_m^{k+1}$; as for the series over $|\mu| > k$, write it in the form

$$\sum \xi^{i_1} \dots \xi^{i_k} \sum_{|\mu|>k} f_\mu(x)\, \xi^{\mu - e_{i_1} - \dots - e_{i_k}}\ ,$$

where e_a is the canonical basis of $\mathbb{R}^q$ and where the sum is over all combinations (with repetitions) of k of the q elements ξ^a. Each term of this finite sum is in $\mathfrak{m}_m^{k+1}$ and so is the sum itself. □

Roughly speaking, since ψ_V^* is an algebra morphism, the data $\psi_V^* y^j = s^j$ and $\psi_V^* \eta^b = \zeta^b$ uniquely determine the pullback $\psi_V^* P$ of any section $P \in \mathrm{Pol}_V(V)[[\eta^1, \dots, \eta^v]]$ with polynomial coefficients. Hence the quest for an appropriate polynomial approximation of an arbitrary section. Let us emphasize that here and in the following, the term 'polynomial section' refers to a formal series $\sum_{|\mu|\ge 0} P_\mu(x)\xi^\mu$ in the parameters ξ^a with coefficients $P_\mu(x) \in \mathrm{Pol}_V(V)$ that are polynomial in the base variables x^i.

Theorem 6.10 (Polynomial approximation)**.** *Let $m \in M$ be a base point of a $\mathbb{Z}_2^n$-supermanifold $(M, \mathcal{A}_M)$ and let $f \in \mathcal{A}(U)$ be a $\mathbb{Z}_2^n$-function defined in a neighborhood U of m. For any fixed degree of approximation $k \in \mathbb{N} \setminus \{0\}$, there exists a polynomial $P = P(x,\xi)$ such that*

$$[f]_m - [P]_m \in \mathfrak{m}_m^k\ .$$

In this statement the polynomial P depends on m, f, and k, and the variables (x,ξ) are (pullbacks of) coordinates centered at m.

Proof. When writing $f = f(x,\xi) = \sum_{|\mu|\ge 0} f_\mu(x)\xi^\mu$ and using a Taylor expansion of the $f_\mu(x)$ at $m \simeq x = 0$, we get

$$f(x,\xi) = \sum_{|\mu|\ge 0} P_\mu(x)\xi^\mu + \sum_{|\mu|\ge 0} 0_\mu(x^k)\xi^\mu,$$

where the first sum of the RHS is the searched polynomial $P = P(x,\xi)$ and where the (germ of the) second sum belongs to $\mathfrak{m}_m^k$. □

6.4.3 Proof of the Morphism Theorem

1. Uniqueness: if the searched $\mathbb{Z}_2^n$-morphism $\Psi = (\psi, \psi^*)$ exists, it is necessarily unique. Indeed, let $\Psi_1 = (\psi_1, \psi_1^*)$ and $\Psi_2 = (\psi_2, \psi_2^*)$ be two $\mathbb{Z}_2^n$-morphisms defined by the same (s^j, ζ^b).

Note first that $\psi_1 = \psi_2$: if we denote the coordinates of $\mathcal{V}^{p|\mathbf{q}}$ by (y^j, η^b), commutation of the pullback maps with the projections to the base entails that, for all j,

$$\psi_1^j = y^j \circ \psi_1 = \varepsilon \psi_1^* y^j = \varepsilon s^j = \varepsilon \psi_2^* y^j = y^j \circ \psi_2 = \psi_2^j.$$

As for the comparison of ψ_1^* and ψ_2^*, we first show (rigorously) that they coincide on polynomial sections (using continuity of the pullbacks of sections with respect to the $\mathcal{J}$-adic topology), then we show that they coincide on arbitrary section (using the polynomial approximation and continuity of the pullbacks of germs with respect to the $\mathfrak{m}$-adic topology).

Let $W \subset V$ be open, set $\mathcal{O}(W) := C^\infty(W)[[\eta^1, \ldots, \eta^v]]$, $\psi := \psi_1 = \psi_2$ and $U := \psi^{-1}(W)$, and take

$$f(y,\eta) = \sum_{|\mu|\geq 0} f_\mu(y)\eta^\mu \in \mathcal{O}(W) \ .$$

The sequence

$$\sum_{k=0}^{n} \sum_{|\mu|=k} \psi_i^* \left(f_\mu(y)\right) (\psi_i^* \eta)^\mu \in \mathcal{A}(U)$$

is Cauchy, see 7.1. Indeed,

$$\psi_i^* \sum_{|\mu|=k} f_\mu(y)\eta^\mu \in \mathcal{J}^k(U) \ ,$$

in view of the $\mathcal{J}$-continuity of ψ_i^*. Since $\mathcal{A}(U)$ is Hausdorff-complete with respect to the $\mathcal{J}(U)$-topology, the considered sequence has a unique limit

$$\sum_{|\mu|\geq 0} \psi_i^* \left(f_\mu(y)\right) (\psi_i^* \eta)^\mu \in \mathcal{A}(U) \ .$$

It is easily seen that this limit is given by $\psi_i^* f \in \mathcal{A}(U)$: the difference of the latter and the n-th term of the sequence equals

$$\psi_i^* \sum_{|\mu|>n} f_\mu(y)\eta^\mu \in \mathcal{J}^{n+1}(U) \ .$$

If the function $f =: P$ has polynomial coefficients $f_\mu(y) =: p_\mu(y)$, we get

$$\psi_i^* P = \sum_{|\mu|\geq 0} \psi_i^* \left(p_\mu(y)\right) (\psi_i^* \eta)^\mu = \sum_{|\mu|\geq 0} p_\mu(\psi_i^* y) \, (\psi_i^* \eta)^\mu = \sum_{|\mu|\geq 0} p_\mu(s)\zeta^\mu \ ,$$

so that ψ_1^* and ψ_2^* coincide on polynomial functions $P \in \mathcal{O}(W)$.

Consider now an arbitrary function $f \in \mathcal{O}(W)$, any point $m_0 \in U$, as well as a $\mathbb{Z}_2^n$-superchart (U_α, Φ_α) around m_0. For every $m \in U_\alpha$, we have $\psi(m) =: n \in W$. In view of the polynomial approximation theorem 6.10, there is, for any k, a polynomial P such that $\psi_i^* \left([f]_n - [P]_n\right) \in \mathfrak{m}_m^{k+1}$. Hence,

$$[\psi_1^* f - \psi_2^* f]_m = [\psi_1^* f]_m - [\psi_1^* P]_m - [\psi_2^* f]_m + [\psi_2^* P]_m \in \mathfrak{m}_m^{k+1} \ .$$

Since k is arbitrary, it follows from Lemma 6.9 that any coefficient of $\psi_1^* f - \psi_2^* f$ vanishes at m. As m is arbitrary in U_α, we see that $(\psi_1^* f - \psi_2^* f)\,|_{U_\alpha} = 0$, i.e. that $\psi_1^* f - \psi_2^* f$ vanishes on an open cover of U: $\psi_1^* f = \psi_2^* f$ for any f, so $\psi_1^* = \psi_2^*$. This completes the proof of uniqueness.

2. Existence: The base map ψ is defined by $\psi := (\varepsilon s^1, \ldots, \varepsilon s^u) \in C^\infty(M, V)$. As for the pullback ψ^*, let $W \subset V$ be open. To construct the graded unital $\mathbb{R}$-algebra morphism

$$\psi_W^* : \mathcal{O}(W) \to \mathcal{A}\left(\psi^{-1}(W)\right) \ ,$$

we cover the open subset $\psi^{-1}(W) \subset M$ by $\mathbb{Z}_2^n$-superchart domains U_α. In view of uniqueness, we can take $\psi^{-1}(W) = U_\alpha$ and build $\psi_W^* : \mathcal{O}(W) \to \mathcal{A}(U_\alpha)$. This construction is the same as the one described in Section 3.5 and similar to the proof that holds in the super-case [Var04].

7 Appendix

7.1 Completeness

7.1.1 Completion of an Abelian group

Let G be an Abelian group together with a decreasing filtration by Abelian subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots$$

The sequence

$$G/G_0 = \{0\} \leftarrow G/G_1 \leftarrow G/G_2 \leftarrow \ldots ,$$

where the morphism $f_{k-1,k} : G/G_k \to G/G_{k-1}$ associates to any coset $g + G_k$, $g \in G$, the coset $g + G_{k-1}$, is an inverse system in the category $\texttt{AbGr}$ of Abelian groups. The limit

$$\widehat{G} := \varprojlim_k G/G_k \in \texttt{AbGr}$$

is the *completion* of G with respect to the considered decreasing filtration. The completion $\widehat{G}$ is in fact a complete topological group. Indeed, the cosets $g + G_k$, $g \in G$, $k \in \mathbb{N}$, are a basis of a topology of G called *topology associated to the filtration* G_k. This topology turns G into a topological group and $\widehat{G}$ into a complete topological group.

If G has additional structure, e.g. is a not necessarily commutative ring (resp., a module over a ring), and if the decreasing filtration of subgroups is compatible, i.e. $G_k \cdot G_\ell \subset G_{k+\ell}$ (resp., G_k is a submodule), then the completion is itself a ring (resp., a module). If the filtration of a ring G is implemented by an ideal I, i.e. if $G_k = I^k := I \cdots I$, the associated topology is also referred to as the *I-adic topology.*

7.1.2 Hausdorff-completeness

Let κ be a commutative unital ring and let

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots$$

be a decreasing filtration of a κ-module M by κ-submodules M_k. The sequence of κ-modules

$$0 \to \cap_k M_k \stackrel{i}{\longrightarrow} M \stackrel{p}{\longrightarrow} \varprojlim_k M/M_k$$

is exact. Indeed, the map p associates to $m \in M$ the sequence $(m + M_1, m + M_2, \ldots) \in \varprojlim_k M/M_k$; its kernel is $\ker p = \cap_k M_k$, what explains exactness.

Definition 7.1. The module M is *complete* (resp., *Hausdorff*, *Hausdorff-complete*) with respect to the considered filtration M_k if and only if p is surjective (resp., p is injective, p is bijective).

We will consider sequences of partial sums in M, i.e. sequences of the type $\sum_{k=0}^n m_k$, $m_k \in M$, $n \in \mathbb{N}$.

Definition 7.2. A sequence $\sum_{k=0}^{n} m_k$ of partial sums in M is a *Cauchy sequence* if $m_k \in M_k$, for all k.

Indeed, in the topology with basis $m + M_k$, $m \in M$, $k \in \mathbb{N}$, the sequence $\sum_{k=r}^{s} m_k \in M_r$ converges to 0 if $r, s \to \infty$.

Proposition 7.3. *If M is Hausdorff (resp., complete, Hausdorff-complete), any sequence (resp., any Cauchy sequence) has at most one (resp., at least one, a unique) limit.*

Proof. If M is Hausdorff, any sequence $(m(n))_n$ has at most one limit. Assume that $(m(n))_n$ converges to m' and m''. Then, for all k, $m' - m'' = (m' - m(n)) - (m'' - m(n)) \in M_k$, so that $m' = m''$.

Further, if M is complete, any Cauchy sequence $\sum_{k=0}^{n} m_k$, $m_k \in M_k$, $n \in \mathbb{N}$, has a limit. Indeed, the sequence $\sum_{k=0}^{n} m_k + M_{n+1}$, $n \in \mathbb{N}$, is an element of the limit $\varprojlim_k M/M_k$ (the map $f_{n+1,n+2}$ sends $\sum_{k=0}^{n+1} m_k + M_{n+2}$ to

$$\sum_{k=0}^{n+1} m_k + M_{n+1} \simeq \sum_{k=0}^{n} m_k + M_{n+1}\ .)$$

Hence, it is an image by p, i.e. there is $m \in M$ such that $\sum_{k=0}^{n} m_k + M_{n+1} = m + M_{n+1}$, for all n. In other words, $m - \sum_{k=0}^{n} m_k \in M_{n+1}$, or, still, the Cauchy sequence of partial sums converges to m.

Eventually, if M is Hausdorff-complete, any Cauchy sequence of partial sums has a unique limit. □

7.2 Category of locally $\mathbb{Z}_2^n$-ringed spaces

Definition 7.4. A *locally $\mathbb{Z}_2^n$-ringed space* (LZRS) is a pair $(M, \mathcal{A}_M)$ made of a topological space M (Hausdorff and second-countable) and a sheaf $\mathcal{A}_M$ of $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebras, such that the stalks $\mathcal{A}_m$, $m \in M$, are local.

Morphisms of LZRS are maps that respect all data, i.e. they are made of a continuous base map and a sheaf morphism that respects the maximal ideals:

Definition 7.5. A *morphism* of LZRS is a map Φ between two LZRS $(M, \mathcal{A}_M)$ and $(N, \mathcal{B}_N)$, made of

1. a continuous map $\phi : M \to N$,
2. a family, indexed by the open subsets $V \subset N$, of graded unital $\mathbb{R}$-algebra morphisms

$$\phi_V^* : \mathcal{B}_N(V) \to \mathcal{A}_M\left(\phi^{-1}(V)\right)\ ,$$

   called *pullback maps*,

   - which commute with the restriction maps of the sheaves $\mathcal{A}_M$ and $\mathcal{B}_N$,

- and are such that the naturally induced graded unital $\mathbb{R}$-algebra morphisms

$$\phi_m^* : \mathcal{B}_{\phi(m)} \to \mathcal{A}_m \ ,$$

$m \in M$, respect the maximal ideals, i.e. satisfy

$$\phi_m^*(\mathfrak{m}_{\phi(m)}) \subset \mathfrak{m}_m \ . \tag{32}$$

Locally $\mathbb{Z}_2^n$-ringed spaces form a category LZRS with obvious identity morphisms and composition.

7.3 Extension and gluing of sheaves

7.3.1 Extension of a sheaf on a basis

It is a matter of common knowledge that a sheaf over a topological space is completely defined by its definition on a basis of the topology: if B is a basis of a topological space M, there is a 1:1 correspondence $\mathrm{Sh}(M) \xrightarrow{\sim} \mathrm{Sh}(B)$ between the category of sheaves on M and the category of sheaves on B.

Sheaves on B, or B-*sheaves*, are defined exactly as sheaves on M, except that only open subsets in B are considered. For instance, for the assignment $\mathcal{F} : U \mapsto \mathcal{F}(U)$, $U \in B$, the gluing condition reads: For any $U \in B$, any cover $(U_i)_i$, $U_i \in B$, of U, and any family $(f_i)_i, f_i \in \mathcal{F}(U_i)$, such that $f_i|_V = f_j|_V$, for any $V \in B$, $V \subset U_i \cap U_j$, there is a unique $f \in \mathcal{F}(U)$, such that $f|_{U_i} = f_i$. Similarly, a morphism of B-sheaves is defined exactly as a morphism of sheaves.

The functor $\mathrm{Sh}(M) \to \mathrm{Sh}(B)$ is just the forgetful functor. Conversely, any B-sheaf $\mathcal{F}$ and any B-sheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$ uniquely extend to a sheaf $\overline{\mathcal{F}}$ and a sheaf morphism $\overline{\varphi} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$, respectively. For instance, the extension $\overline{\mathcal{F}}$ is (up to isomorphism) given, for any open $W \subset M$, by

$$\overline{\mathcal{F}}(W) = \{(f_a)_a : f_a \in \mathcal{F}(U_a), U_a \in B, U_a \subset W \text{ and } f_a|_V = f_b|_V, V \in B, V \subset U_a \cap U_b\} \ .$$

7.3.2 Gluing of sheaves

Sheaves can be glued.

More precisely, if $(U_i)_i$ is an open cover of a topological space M, if $\mathcal{F}_i$ is a sheaf on U_i, and if $\varphi_{ji} : \mathcal{F}_i|_{U_i\cap U_j} \to \mathcal{F}_j|_{U_i\cap U_j}$ is a sheaf isomorphism such that the usual cocycle condition $\varphi_{kj}\,\varphi_{ji} = \varphi_{ki}$ holds, then there is a unique sheaf $\mathcal{F}$ on M, such that $\mathcal{F}|_{U_i} \simeq \mathcal{F}_i$.

Actually the sheaf isomorphisms $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ satisfy $\varphi_{ji}\,\psi_i|_{Ui\cap U_j} = \psi_j|_{U_i\cap U_j}$. Uniqueness means that if there is another sheaf $\mathcal{F}'$ on M with sheaf isomorphisms $\psi_i' : \mathcal{F}'|_{U_i} \to \mathcal{F}_i$ that satisfy the same property, then there exists a unique sheaf isomorphism $\varphi : \mathcal{F} \to \mathcal{F}'$ such that $\psi_i'\,\varphi|_{U_i} = \psi_i$.

The glued sheaf $\mathcal{F}$ is defined, for $U \subset M$, by

$$\mathcal{F}(U) = \{(f_i)_i \in \prod_i \mathcal{F}_i(U \cap U_i) : \varphi_{ji}(f_i|_{U\cap U_i\cap U_j}) = f_j|_{U\cap U_i\cap U_j}\} \ .$$

7.4 Partition of unity

The present section should be read after Section 6.2.

Proposition 7.6. *For any $\mathbb{Z}_2^n$-supermanifold $\mathcal{M} = (M, \mathcal{A}_M)$ and any open cover $(U_i)_i$ of M, there is a partition of unity of $\mathcal{M}$ subordinated to $(U_i)_i$.*

More precisely, there exists a locally finite cover $(V_j)_j$ of M, which is subordinated to $(U_i)_i$, and a family $(s_j)_j \in \mathcal{A}_M^0(M)$, such that the projections $\gamma_j := \varepsilon(s_j) \in C_M^\infty(M)$ are nonnegative, the support $\operatorname{supp}(s_j)$ is compact in V_j, and $\sum_j s_j = 1$. The support of a $\mathbb{Z}_2^n$-function $s \in \mathcal{A}_M(U)$, $U \subset M$, is defined as usual as the complementary of the open subset of identical zeros of s in U.

Proof. Let $(V_j, \gamma_j)_j$ be a partition of unity of the classical smooth manifold M that is subordinated to $(U_i)_i$: $\operatorname{supp}(\gamma_j)$ is compact in V_j and $\sum_j \gamma_j = 1$. We may of course assume that the V_j are $\mathbb{Z}_2^n$-superchart domains: on V_j, the $\mathbb{Z}_2^n$-supermanifold $(M, \mathcal{A}_M)$ is isomorphic to a $\mathbb{Z}_2^n$-superdomain. If $\Phi = (\phi, \phi^*)$ is this isomorphism, set $f_j = \phi^*(\gamma_j \circ \phi^{-1}) \in \mathcal{A}_M^0(V_j)$. It is clear that $\varepsilon(f_j) = \gamma_j$ and easily seen that $\operatorname{supp}(f_j) = \operatorname{supp}(\gamma_j) \subset V_j$. Extend now f_j by 0, so that $f_j \in \mathcal{A}_M^0(M)$, and set $f = \sum_j f_j$. This sum is well-defined in $\mathcal{A}_M^0(M)$, due to local finiteness of $(V_j)_j$, and $\varepsilon(f) = \sum_j \varepsilon(f_j) = \sum_j \gamma_j = 1$. The latter implies that $f|_{V_k}$ is invertible for all k. When gluing these local inverses, we get a global inverse $f^{-1} \in \mathcal{A}_M^0(M)$. It now suffices to set $s_j = f^{-1} f_j \in \mathcal{A}_M^0(M)$. □

8 Acknowledgements

T. Covolo thanks the Luxembourgian NRF for support via AFR grant 2010-1, 786207. The research of J. Grabowski has been founded by the Polish National Science Centre grant under the contract number DEC-2012/06/A/ST1/00256. The research of N. Poncin has been supported by the Grant GeoAlgPhys 2011-2014 awarded by the University of Luxembourg. Moreover, the authors are grateful to Sophie Morier-Genoud and Valentin Ovsienko, whose work on $\mathbb{Z}_2^n$-gradings [MGO09], [MGO10] was the starting point of this paper. They thank Dimitry Leites for his suggestions and valuable support. N. Poncin appreciated Pierre Schapira's explanations on sheaves.

References

[Bae92] J. Baez. *R*-commutative geometry and quantization of Poisson algebras. Adv. Math., 95(1), 1992, 6191.

[BBH91] C. Bartocci, U. Bruzzo and D. Hernández-Ruipérez. The geometry of supermanifolds. Mathematics and its applications, 76, Kluwer Academic Publishers, Springer, 1991.

[BP12] G. Bonavolontà and N. Poncin. On the category of Lie n-algebroids. J. Geo. Phys., 73, 2013, 70-90.

[CR12] D. Carchedi, D. Roytenberg. On Theories of Superalgebras of Differentiable Functions. Theo. Appl. Cat., 28(30), 2013, 10221098.

[CCF11] C. Carmeli, L. Caston, and R. Fioresi. Mathematical foundations of supersymmetry. EMS Series of Lectures in Mathematics, 2011, 287 pp.

[CB89] Y. Choquet-Bruhat. Graded bundles and supermanifolds. Monographs and textbooks in physical science, lecture notes, Bibliopolis, 1989, 94 pp.

[CM14] T. Covolo, J.-P. Michel. Determinants over graded-commutative algebras, a categorical viewpoint. `arXiv:1403.7474`.

[COP12] T. Covolo, V. Ovsienko, and N. Poncin. Higher Trace and Berezinian of Matrices over a Clifford Algebra. J. Geo. Phys., 62, 2012, 2294-2319.

[DM99] P. Deligne and J. Morgan. Notes on Supersymmetry (following Joseph Bernstein). In: Quantum Fields and Strings: A Course for Mathematicians. AMS, Institute for Advanced Study, 1999, ISBN 0-8218-1198-3.

[DSB03] S. Duplij, W. Siegel and J. Bagger. Concise Encyclopedia of Supersymmetry and Noncommutative Structures in Mathematics and Physics. Springer, 2003, ISBN 1-4020-1338-8.

[Gaw77] K. Gawedzki. Supersymmetries-mathematics of supergeometry. Ann. I.H.P., A, 27(4), 1977, 335-366.

[GKP09] J. Grabowski, A. Kotov and N. Poncin. The Lie Superalgebra of a Supermanifold. J. Lie Theo., 20, 2010, 739 - 749.

[GKP10] J. Grabowski, A. Kotov and N. Poncin. Lie superalgebras of differential operators. J. Lie Theo., 23(1), 2013, 035-054.

[GKP12] J. Grabowski, D. Khudaverdyan and N. Poncin. The Supergeometry of Loday Algebroids. J. Geo. Mech., 5(2), 2013, 185-213.

[GR09] J. Grabowski and M. Rotkiewicz. Higher vector bundles and multi-graded symplectic manifolds. J. Geo. Phys., 59, 2009, 1285-1305.

[Lei80] D. Leites. Introduction to the Theory of Supermanifolds. Russian Math. Surveys, 35 (1980), no.1, 1-64.

[Lei11] D. Leites (ed.). Seminar on supersymmetry (v. 1. Algebra and calculus: Main chapters) (J. Bernstein, D. Leites, V. Molotkov, V. Shander). MCCME, Moscow, 2011, 410 pp. (in Russian; an English version is in preparation but available for perusal).

[Man02] Y. Manin. Gauge Field Theory and Complex Geometry. A series of comprehensive studies in mathematics, 289, 2nd edition, Springer, ISBN 970-3-540-61378-7.

[Mol10] V. Molotkov. Infinite-dimensional and colored supermanifolds. J. Nonlin. Math. Phys., 17(1), 2010, 375-446.

[MGO09] S. Morier-Genoud and V. Ovsienko. Well, Papa, can you multiply triplets? Math. Intell., 31, 2009, 1-2.

[MGO10] S. Morier-Genoud and V. Ovsienko. Simple graded commutative algebras. J. Alg., 323, 2010, 1649-1664.

[Sch97] T. Schmitt. Supergeometry and quantum field theory, or: what is a classical configuration? Rev. Math. Phys., 9(8), 1997, 993-1052.

[Var04] V.S. Varadarajan. Supersymmetry for Mathematicians: An Introduction. Courant Lecture Notes, 11. American Mathematical Society, Courant Institute of Mathematical Sciences, 2004.

[Vor12] T. Voronov. Q-manifolds and Mackenzie theory. `arXiv:1206.3622`.

Tiffany COVOLO
University of Luxembourg
Campus Kirchberg, Mathematics Research Unit
6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg City, Grand-Duchy of Luxembourg
Email: tiffany.covolo@uni.lu

Janusz GRABOWSKI
Polish Academy of Sciences
Institute of Mathematics
Śniadeckich 8, P.O. Box 21, 00-656 Warsaw, Poland
Email: jagrab@impan.pl

Norbert PONCIN
University of Luxembourg
Campus Kirchberg, Mathematics Research Unit
6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg City, Grand-Duchy of Luxembourg
Email: norbert.poncin@uni.lu