On structure signatures and probability signatures of general decomposable systems

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Coherent systems: notation

- $C = \{c_1, \ldots, c_n\}$: $n$ binary components (two possible states);
- they are connected to form a system;
- Basic examples: series, parallel, bridge, $k$-out-of-$n$ systems.
Structure functions

- With each component $c_k, (k \in [n] = \{1, \ldots, n\})$, we associate a Boolean variable

$$x_k = \begin{cases} 
0 & \text{if } c_k \text{ is in a failed state} \\
1 & \text{if } c_k \text{ is in function.} 
\end{cases}$$

The Boolean vector $x = (x_1, \ldots, x_n)$ encodes the states of all components.

- We can also consider the set $A$ of components in function:
  $x = (1, 0, 1, 0, 1)$ corresponds to $A = \{1, 3, 5\}$.
  So the states are represented by $x \in \{0, 1\}^n$ or $A \subset [n] = \{1, \ldots, n\}$.

- The structure function defines the state of the system :

$$\phi : \{0, 1\}^n \to \{0, 1\} : x = (x_1, \ldots, x_n) \mapsto x_\mathcal{S} = \phi(x_1, \ldots, x_n).$$
Examples

The structure function can be defined with $\land$ and $\lor$ or as a polynomial function.

\[
\phi(x_1, \ldots, x_n) = x_1 \cdots x_n \quad \phi(x_1, \ldots, x_n) = 1 - (1 - x_1) \cdots (1 - x_n)
\]

\[
\phi(x_1, \ldots, x_5) = 1 - (1 - x_2 x_4)(1 - x_1 x_5)(1 - x_2 x_3 x_5)(1 - x_1 x_3 x_4).
\]

This function can be extended as a polynomial $\hat{\phi} : \mathbb{R}^n \to \mathbb{R}$, of degree at most 1 in each variable. This is the multilinear extension of $\phi$. 
Properties of $\phi$

- $\phi : \{0,1\}^n \rightarrow \{0,1\}$ or $\phi : \mathcal{P}([n]) \rightarrow \{0,1\}$;
- $\phi(0, \ldots, 0) = \phi(\emptyset) = 0$;
- $\phi(1, \ldots, 1) = \phi([n]) = 1$;
- $\phi$ is increasing (nondecreasing) :

$$A \subset B \Rightarrow \phi(A) \leq \phi(B).$$

Every function with these properties is the structure function of a semi-coherent system.
This system is coherent if in addition, all the variables are essential in $\phi$.

**Example** : a $k$-out-of-$n$ system is a system that fails with the $k$-th failure :

$$\phi(A) = 1 \text{ iff } |A| > n - k.$$ 

or $\phi(x) = x_{k:n}$ (series are 1-out-of-$n$, parallel are $n$-out-of-$n$).
Some notation concerning probability

1. $T_k$ : random lifetime of component $c_k$.
2. For $t > 0$, $X_k(t)$ : random state of comp. $c_k$ at time $t$ (Bernoulli var.).
3. $T_S$ : system random lifetime.
4. $X_S(t)$ : random state of the system at time $t$ (Bernoulli var.).
5. Joint cumulative distribution of component lifetimes :

$$F(t_1, \ldots, t_n) = \Pr(T_1 \leq t_1, \ldots, T_n \leq t_n).$$

So in general a system is a triple :

$$S = (n, \phi, F).$$

Classical hypotheses :

- $F$ is absolutely continuous; the lifetimes are i.i.d.
- or the lifetimes are exchangeable;
- or ties have null probability (no ties) :

$$\Pr(T_k = T_\ell) = 0, \quad \text{when } k \neq \ell.$$
Structure Signature

The random variables $T_1, \ldots, T_n$ induce the order statistics $T_{1:n}, \ldots, T_{n:n}$, such that (when there are no ties)

$$T_{1:n} < \ldots < T_{n:n}.$$

Consider a system $S = (n, \phi, F)$, where $F$ is absolutely continuous i.i.d.

**Definition (Samaniego (1985))**

The *structure signature* of the system (or *Samaniego signature*) is the $n$-tuple

$$s = (s_1, \ldots, s_n)$$

where

$$s_k = \Pr(T_S = T_{k:n}).$$

**Remark** : it is not the Barlow index $(b_1, \ldots, b_n)$ where

$$b_k = \Pr(T_S = T_k).$$

**Theorem** : the signature $s$ does not depend on $F$ (in the i.i.d. situation), but only on the structure. It is a combinatorial object.
Examples

For a series system: \( s = (1, 0, \ldots, 0) \).
For a parallel system: \( s = (0, \ldots, 0, 1) \).
The signature of a more complex system:

\[
\begin{array}{c}
\text{\(c_1\)} \\
\downarrow \\
\text{\(c_2\)} \\
\downarrow \\
\text{\(c_3\)} \\
\downarrow \\
\text{\(c_4\)} \\
\text{\(c_5\)} \\
\end{array}
\]

\[ s = \left( \frac{48}{120}, \frac{36}{120}, \frac{36}{120}, 0, 0 \right) \]

Why? All the events

\[
E_\sigma = \left( T_{\sigma(1)} < T_{\sigma(2)} < T_{\sigma(3)} < T_{\sigma(4)} < T_{\sigma(5)} \right)
\]

where \( \sigma \) is a permutation of \( \{1, 2, 3, 4, 5\} \) are equally likely. Since there are no ties:

\[
\Pr( T_{\sigma(1)} < T_{\sigma(2)} < T_{\sigma(3)} < T_{\sigma(4)} < T_{\sigma(5)} ) = \frac{1}{120}.
\]

We just have to count the number of these events that correspond to the event \( (T_S = T_{k:5}) \).

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How to compute: Boland’s formula

**Proposition (Boland (2001))**

*If the components have i.i.d. lifetimes, we have for $k \leq n$*

$$s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{|A|=n-k+1} \phi(A) - \frac{1}{\binom{n}{n-k}} \sum_{|A|=n-k} \phi(A)$$

*where for $A \subseteq [n]$, $|A|$ is the cardinality of $A$.*

Both terms that appear in the formula have a meaning:

$$\overline{S}_k = \frac{1}{\binom{n}{n-k}} \sum_{|A|=n-k} \phi(A) = \sum_{i=k+1}^{n} s_i = \Pr(T_S > T_{k:n}).$$

It is the $k$th component of the *tail structure signature.*

It can be extended (for convenience) with $\overline{S}_0 = 1$ and $\overline{S}_n = 0$. We then have

$$s_k = \overline{S}_{k-1} - \overline{S}_k, \quad \forall k : 1 \leq k \leq n.$$
Some more examples

We have $\overline{S}_0 = 1$, $\overline{S}_1 = \frac{3}{5}$, $\overline{S}_2 = \frac{6}{20}$, $\overline{S}_3 = \overline{S}_4 = \overline{S}_5 = 0$, so

$s = (1 - \frac{3}{5}, \frac{3}{5} - \frac{6}{20}, \frac{6}{20} - 0, 0 - 0, 0 - 0) = (\frac{48}{120}, \frac{36}{120}, \frac{36}{120}, 0, 0)$

We have $\overline{S}_0 = 1$, $\overline{S}_1 = \frac{2}{3}$, $\overline{S}_2 = \frac{8}{30}$, $\overline{S}_3 = \overline{S}_4 = \overline{S}_5 = \overline{S}_6 = 0$. 

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Use: Samaniego’s decomposition of reliability

The reliability is $R(t) = \Pr(T_S > t)$. Set $R_{k:n}(t) = \Pr(T_{k:n} > t)$.

**Proposition (Samaniego (1985))**

If $F$ is absolutely continuous and i.i.d., then

$$R_S(t) = \sum_{k=1}^{n} s_k R_{k:n}(t),$$

for all $t > 0$, and every coherent system $S = (n, \phi, F)$.

- Use: Comparison of systems built with the same components.
- Proofs: Samaniego used probabilities, but it’s also simple algebra.
- Marichal, M., Waldhauser (2011): This decomposition holds for every coherent structure $\phi$ if and only if the state variables $X_1(t), \ldots, X_n(t)$ are exchangeable.
The general (non i.i.d.) setting

For a system $S = (n, \phi, F)$ such that $F$ has no ties, we can define

- The structure signature $s$ (through Boland's formula);
- The probability signature $p$, defined as above (Navarro et al. 2010):

$$p = (p_1, \ldots, p_n),$$

where

$$p_k = \Pr(T_S = T_{k:n}).$$

- The probability signature may depend both on $F$ and $\phi$.

Can we find an explicit expression of $p_k$ in terms of $F$ et $\phi$, for instance by generalizing Boland’s formula?
Expression of $p$ and the relative quality function

The relative quality function is defined by

$$
q : \mathcal{P}([n]) \to \mathbb{R} : A \mapsto q(A) = \Pr \left( \max_{k \not\in A} T_k < \min_{j \in A} T_j \right),
$$

and $q(\emptyset) = q([n]) = 1$. So $q(A)$ measures the quality of elements of $A$.

Proposition (Marichal, M. (2011))

If $F$ has no ties, then the probability signature is given by

$$
p_k = \sum_{|A| = n-k+1} q(A) \phi(A) - \sum_{|A| = n-k} q(A) \phi(A).
$$

Here again both terms have a direct meaning

$$
\overline{P}_k = \sum_{|A| = n-k} q(A) \phi(A) = \sum_{i=k+1}^{n} p_i = \Pr(T_S > T_{k:n})
$$

is the $k$-th coordinate of the tail probability signature.

When $F$ is i.i.d. we have $q(A) = \frac{1}{\binom{n}{|A|}}$. 

Modular decomposition of the structure

A modular decomposition of \( ([n], \phi, F) \) into \( r \) disjoint modules is

- a partition \( C = \{ C_1, \ldots, C_r \} \) of \([n]\);
- for every \( j \), a semi-coherent system \( M_j = (C_j, \chi_j, G_j) \), where \( G_j \) is the marginal distribution of the components in \( C_j \);
- the modules are connected according to a structure function \( \psi: \{0,1\}^r \rightarrow \{0,1\} \) such that

\[
\phi(A) = \psi(\chi_1(A \cap C_1), \ldots, \chi_r(A \cap C_r)), \quad A \subseteq C, \quad (2)
\]

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Modular decomposition of the structure signature $s$

**Question**: does the structure signature of a modular system decompose in terms of the signatures of the modules and the organizing structure $\psi$?

- A known result, expressed in terms of the tail signature:

**Theorem (Gertsbakh, Shpungin, Spizzichino (2011))**

For two modules $C_1$ and $C_2$ of size $n_1$ and $n_2$ connected in series, we have

$$
\overline{S}_{n-k} = \sum_{0 \leq a_1 \leq n_1, \ 0 \leq a_2 \leq n_2} \frac{\binom{n_1}{a_1} \binom{n_2}{a_2}}{\binom{n}{k}} \overline{S}_{n_1-a_1}^{1} \overline{S}_{n_2-a_2}^{2}.
$$

- The same kind of formula holds for a parallel of two modules;
- Independent results in the same direction by Da, Zheng and Hu (2012);
Modular decomposition of $s$ : general result

For $r$ modules of $C_1, \ldots, C_r$ of size $n_1, \ldots, n_r$ we set
\[
T_k = \{ a = (a_1, \ldots, a_r) \in \mathbb{N}^r : 0 \leq a_j \leq n_j \text{ for } j = 1, \ldots, r \text{ and } \sum_{j=1}^r a_j = k \}.
\]
and for every $a \in T_k$ :
\[
c_0(a_1, \ldots, a_r) = \frac{(n_1)^{a_1} \cdots (n_r)^{a_r}}{(n)^k}.
\]

Theorem (Marichal, M., Spizzichino (submitted))

For every semi-coherent system $(C, \phi)$ with a modular decomposition into $r$ disjoint modules $(C_j, \chi_j)$, $j = 1, \ldots, r$, connected according to a semi-coherent structure $\psi$, we have (without assumption)
\[
\bar{S}_{n-k} = \sum_{a \in T_k} c_0(a) \hat{\psi}(\bar{S}_{n_1-a_1}, \ldots, \bar{S}_{n_r-a_r}), \quad 0 \leq k \leq n. \quad (3)
\]
Here $\hat{\psi}$ is the multilinear extension of $\psi$. 

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Decomposability of $q$

The hypotheses for the decomposition:

**Definition**

Denote by $q^{C_j}$ the relative quality function associated with module $C_j$:

$$q^{C_j}(A) = \Pr\left( \max_{i \in C_j \setminus A} T_i < \min_{i \in A} T_i \right), \quad A \subseteq C_j.$$ 

**Definition**

Given a partition $C = \{C_1, \ldots, C_r\}$ of $C$, the relative quality function $q$ is $C$-decomposable if there exists a function $c : \prod_{i=1}^r \{0, \ldots, n_i\} \rightarrow \mathbb{R}$ such that

$$q(A) = c(|A \cap C_1|, \ldots, |A \cap C_r|) \prod_{j=1}^r q^{C_j}(A \cap C_j), \quad A \subseteq C. \quad (4)$$

**Remark:** This is a condition on the component lifetimes.

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Modular decomposition of $p$

Recall the tail signature $\overline{P}_k = \sum_{i=k+1}^{n} p_i = \Pr(T_S > T_{k:n})$ for $0 \leq k \leq n - 1$, $\overline{P}_0 = 1$ and $\overline{P}_n = 0$.

Theorem (Marichal, M., Spizzichino (submitted))

If the relative quality function $q$ is $C$-decomposable for some partition $C = \{C_1, \ldots, C_r\}$ of $C$, then for every semi-coherent system $(C, \phi, F)$ with a modular decomposition into $(C_j, \chi_j, G_j)$, $j = 1, \ldots, r$, connected according to $\psi: \{0,1\}^r \rightarrow \{0,1\}$, we have

$$\overline{P}_{n-k} = \sum_{a \in T_k} c(a) \hat{\psi}(\overline{P}_{n_1-a_1}^{1}, \ldots, \overline{P}_{n_r-a_r}^{r}), \quad 0 \leq k \leq n.$$  (5)
Theorem (Marichal, M., Spizzichino (submitted))

Consider a partition $C = \{C_1, \ldots, C_r\}$ of $C$ and a distribution $F$ of the component lifetimes. Assume that there exists a function $\gamma: \prod_{j=1}^{r} \{0, \ldots, n_j\} \to \mathbb{R}$ such that, for every semi-coherent system $(C, \phi, F)$ with a modular decomposition into $r$ disjoint modules $(C_j, \chi_j, G_j)$, $j = 1, \ldots, r$, connected according to a semi-coherent structure $\psi: \{0, 1\}^r \to \{0, 1\}$, we have

$$\overline{P}_{n-k} = \sum_{a \in T_k} \gamma(a) \widehat{\psi}(\overline{P}_{n_1-a_1}^1, \ldots, \overline{P}_{n_r-a_r}^r).$$

Then the relative quality function $q$ associated with $F$ is $C$-decomposable.
Some cases where $q$ is $C$-decomposable

- If $q$ and $q^{C_j}$ are symmetric, then it is $q$ $C$-decomposable for every $C$, but then $p = s$.
- This happens when the values $p_{\sigma} = \Pr(T_{\sigma(1)} < \cdots < T_{\sigma(n)})$ are equal to $\frac{1}{n!}$.

**Definition**

The function $q : 2^C \to \mathbb{R}$ is $C$-symmetric for a partition $C = \{C_1, \ldots, C_r\}$ if $q(A) = q(B)$ for every $A, B \subseteq C$ such that $|A \cap C_j| = |B \cap C_j|$ for every $j \in [r]$.

- If the function $q$ is $C$-symmetric for some partition $C = \{C_1, \ldots, C_r\}$ and the functions $q^{C_j}$ are symmetric for $j = 1, \ldots, r$, then $q$ is $C$-decomposable.
Decomposable systems

Modular decomposition of $\phi$ and corresponding decomposability of $q$ both play a role, so we propose a new definition:

**Definition**

We say that a semicoherent system $(C, \phi, F)$ is *decomposable* if, for some partition $C = \{C_1, \ldots, C_r\}$ of $C$,

(i) the structure $\phi: \{0, 1\}^n \to \{0, 1\}$ has a modular decomposition into $r$ disjoint semicoherent modules $(C_j, \chi_j, G_j)$, $j = 1, \ldots, r$, and

(ii) the function $q$ is $C$-decomposable.

For decomposable systems, the signatures both admit a modular decomposition.
Final remarks

- We can make the same computations for the Barlow-Proschan index (2012);
- We have a simple analytic method to compute $s$ in terms of $\phi$ (i.i.d. case only) (2012);
- The structure signature can be defined in a purely geometric way (2011), (least squares).

Thank you.
A word about the proof

- Use the multilinear extension of $\psi$ to obtain

$$S_{n-k} = \sum_{|A|=k} q_0(A) \phi(A) = \sum_{|A|=k} q_0(A) \psi(\chi_1(A \cap C_1), \ldots, \chi_r(A \cap C_r))$$

$$= \sum_{B \subseteq [r]} \psi(B) \sum_{|A|=k} q_0(A) \prod_{j \in B} \chi_j(A \cap C_j) \prod_{j \in [r] \setminus B} (1 - \chi_j(A \cap C_j)).$$

where $q_0(A) = 1/\binom{n}{|A|}$.

- Decompose $q_0$ in terms of $c_0$ and $q_0^{C_j}$, $j \leq r$.
- Arrange the sums in a suitable way (easy since we have products).
- Use the fundamental property of $q_0$ and $q_0^{C_j}$:

$$\sum_{A \subseteq C, |A|=k} q_0(A) = 1 \quad \sum_{A_j \subseteq C_j, |A_j|=a_j} q_0^{C_j}(A_j) = 1.$$
Decomposition of reliability

How to extend Samaniego’s decomposition?

**Proposition (Samaniego (1985))**

If $F$ is absolutely continuous and i.i.d., then

$$R_S(t) = \sum_{k=1}^{n} s_k R_{k:n}(t),$$

for all $t > 0$, and every coherent system $S = (n, \phi, F)$.

In the i.i.d. case we have $p = s$, so the decomposition also writes

$$R_S(t) = \sum_{k=1}^{n} p_k R_{k:n}(t). \quad (7)$$
Some questions

Assume that $F$ has no ties.

1. Find necessary and sufficient conditions (on $F$) so that

$$R_S(t) = \sum_{k=1}^{n} s_k R_{k:n}(t),$$

holds for every system and all $t > 0$.

2. Find necessary and sufficient conditions so that

$$R_S(t) = \sum_{k=1}^{n} p_k R_{k:n}(t),$$

holds for every system and all $t > 0$.

3. Find necessary and sufficient conditions so that

$$s = p,$$

holds for every system.

Sufficient conditions (e.g. i.i.d. or exchangeability) are known.
Necessary and sufficient conditions I

For every $t > 0$, the state variables $X_k(t)$ are defined by

$$X_k(t) = \text{Ind}(T_k > t).$$

**Theorem (Marichal, M., Waldhauser (2011))**

For every $t > 0$, (Samaniego) decomposition

$$R_S(t) = \sum_{k=1}^{n} s_k R_{k:n}(t)$$

holds for every coherent structure $\phi$ if and only if the state variables $X_1(t), \ldots, X_n(t)$ are exchangeable.

The condition is weaker than exchangeability of component lifetimes.
Theorem (Marichal, M., Waldhauser (2011))

For every $t > 0$, decomposition

\[ R_S(t) = \sum_{k=1}^{n} p_k \cdot R_{k:n}(t) \]

holds for every coherent structure $\phi$ iff

\[ \Pr(X(t) = x) = q(x) \left( \sum_{|z|=|x|} \Pr(X(t) = z) \right), \]

where $X(t) = (X_1(t), \ldots, X_n(t))$. 
Equality of $s$ and $p$

**Proposition (Marichal, M., Waldhauser (2011))**

*We have $p = s$ for every coherent structure $\phi$ if and only if $q$ is symmetric.*
All the new results in this talk are in one of the following papers.

- J.-L. Marichal, P. M., T. Waldhauser, *On signature-based expressions of system reliability*, Journal of Multivariate analysis, 102 (10), 1410–1416 (2011);
- Everything is on the ArXiv.