Associative and preassociative aggregation functions

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Let $X$ be a nonempty set

$G : X^2 \to X$ is associative if

$$G(G(a, b), c) = G(a, G(b, c))$$

**Examples:** $G(a, b) = a + b$ on $X = \mathbb{R}$  
$G(a, b) = a \land b$ on $X = L$ (lattice)
Associative functions

\[ G(G(a, b), c) = G(a, G(b, c)) \]

Extension to \( n \)-ary functions

\[
\begin{align*}
G(a, b, c) &= G(G(a, b), c) = G(a, G(b, c)) \\
G(a, b, c, d) &= G(G(a, b, c), d) = G(a, G(b, c), d) = \cdots \\
\text{etc.}
\end{align*}
\]
Associative functions with indefinite arity

Let

\[ X^* = \bigcup_{n \in \mathbb{N}} X^n \]

\[ F : X^* \to X \] is associative if

\[ F(x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_r) = F(x_1, \ldots, x_p, F(y_1, \ldots, y_q), z_1, \ldots, z_r) \]

Example: \( F(x_1, \ldots, x_n) = x_1 + \cdots + x_n \) on \( X = \mathbb{R} \)
\( F(x_1, \ldots, x_n) = x_1 \land \cdots \land x_n \) on \( X = L \) (lattice)
We regard \( n \)-tuples \( x \) in \( X^n \) as \textit{n-strings} over \( X \)

0-string: \( \varepsilon \)

1-strings: \( x, y, z, \ldots \)

\( n \)-strings: \( x, y, z, \ldots \)

\( X^* \) is endowed with concatenation

**Example:** \( x \in X^n, y \in X, z \in X^m \Rightarrow xyz \in X^{n+1+m} \)

\(|x| = \text{length of } x\)

\[ F(x) = \varepsilon \iff x = \varepsilon \]
Associative functions with indefinite arity

$F : X^* \rightarrow X$ is *associative* if

\[ F(xyz) = F(xF(y)z) \quad \forall \ xyz \in X^* \]

Equivalent definitions

\[ F(F(xy)z) = F(xF(yz)) \quad \forall \ xyz \in X^* \]

\[ F(xy) = F(F(x)F(y)) \quad \forall \ xy \in X^* \]
Associative functions with indefinite arity

\[ F : X^* \to X \] is associative if

\[ F(xyz) = F(xF(y)z) \quad \forall \ xyz \in X^* \]

**Theorem**

We can assume that \(|xz| \leq 1\) in the definition above

That is, \( F : X^* \to X \) is associative if and only if

\[
\begin{align*}
F(y) &= F(F(y)) \\
F(xy) &= F(xF(y)) \\
F(yz) &= F(F(y)z)
\end{align*}
\]
Associative functions with indefinite arity

\[ F(yz) = F(F(y)z) \]

\[ F_n = F|_{X^n} \]

\[ F_n(x_1 \cdots x_n) = F_2(F_{n-1}(x_1 \cdots x_{n-1})x_n) \quad n \geq 3 \]

Associative functions are completely determined by their unary and binary parts.

**Proposition**

Let \( F : X^* \to X \) and \( G : X^* \to X \) be two associative functions such that \( F_1 = G_1 \) and \( F_2 = G_2 \). Then \( F = G \).
Associative functions with indefinite arity

Link with binary associative functions?

**Proposition**

A binary function $G : X^2 \to X$ is associative if and only if there exists an associative function $F : X^* \to X$ such that $F_2 = G$.

Does $F_1$ really play a role?

\[
F_1(F(x)) = F(x) \\
F(xyz) = F(xF_1(y)z)
\]
Associative functions with indefinite arity

\[ F_1(F(x)) = F(x) \]
\[ F(xyz) = F(xF_1(y)z) \]

**Theorem**

\( F : X^* \rightarrow X \) is associative if and only if

(i) \( F_1(F_1(x)) = F_1(x), \quad F_1(F_2(xy)) = F_2(xy) \)

(ii) \( F_2(xy) = F_2(F_1(x)y) = F_2(xF_1(y)) \)

(iii) \( F_2(F_2(xy)z) = F_2(xF_2(yz)) \)

(iv) \( F_n(x_1 \cdots x_n) = F_2(F_{n-1}(x_1 \cdots x_{n-1})x_n) \quad n \geq 3 \)
Associative functions with indefinite arity

**Theorem**

\(F : X^* \rightarrow X\) is associative if and only if

(i) \(F_1(F_1(x)) = F_1(x), \quad F_1(F_2(xy)) = F_2(xy)\)

(ii) \(F_2(xy) = F_2(F_1(x) y) = F_2(x F_1(y))\)

(iii) \(F_2(F_2(xy) z) = F_2(x F_2(yz))\)

(iv) \(F_n(x_1 \cdots x_n) = F_2(F_{n-1}(x_1 \cdots x_{n-1}) x_n) \quad n \geq 3\)

Suppose \(F_2\) satisfying (iii) is given. What could be \(F_1\)?

**Example:** \(F_2(xy) = x + y\)

\[
\Rightarrow F_1(x + y) = F_1(F_2(xy)) \overset{(i)}{=} F_2(xy) = x + y
\]

\[
\Rightarrow F_1(x) = x
\]
Associative functions with indefinite arity

Theorem

\( F : X^* \rightarrow X \) is associative if and only if

(i) \( F_1(F_1(x)) = F_1(x), \ F_1(F_2(xy)) = F_2(xy) \)

(ii) \( F_2(xy) = F_2(F_1(x) y) = F_2(x F_1(y)) \)

(iii) \( F_2(F_2(xy) z) = F_2(x F_2(yz)) \)

(iv) \( F_n(x_1 \cdots x_n) = F_2(F_{n-1}(x_1 \cdots x_{n-1}) x_n) \quad n \geq 3 \)

Example: \( F_n(x_1 \cdots x_n) = \sqrt{|x_1|^2 + \cdots + |x_n|^2} \)

\[ F_1(x) = x \]

\[ F_1(x) = |x| \]
Let \( Y \) be a nonempty set

**Definition.** We say that \( F : X^* \rightarrow Y \) is *preassociative* if

\[
F(y) = F(y') \implies F(xyz) = F(xy'z)
\]

**Examples:**
- \( F_n(x) = x_1^2 + \cdots + x_n^2 \) \( (X = Y = \mathbb{R}) \)
- \( F_n(x) = |x| \) \( (X \text{ arbitrary, } Y = \mathbb{N}) \)
Preassociative functions

\[ F(y) = F(y') \implies F(xyz) = F(xy'z) \]

Equivalent definition

\[ F(x) = F(x') \quad \text{and} \quad F(y) = F(y') \]

\[ \Downarrow \]

\[ F(xy) = F(x'y') \]
Preassociative functions

\[ F(y) = F(y') \implies F(xyz) = F(xy'z) \]

**Fact.** If \( F : X^* \to X \) is associative, then it is preassociative.

**Proof.** Suppose \( F(y) = F(y') \)

Then \( F(xyz) = F(xF(y)z) = F(xF(y')z) = F(xy'z) \)
Preassociative functions

\[ F(y) = F(y') \implies F(xyz) = F(xy'z) \]

**Proposition**

\( F : X^* \rightarrow X \) is associative if and only if it is preassociative and \( F_1(F(x)) = F(x) \)

**Proof.** (Necessity) OK.

(Sufficiency) We have \( F(y) = F(F(y)) \)

Hence, by preassociativity, \( F(xyz) = F(xF(y)z) \)
Preassociative functions

**Proposition**

If \( F : X^* \rightarrow Y \) is preassociative, then so is the function

\[
x_1 \cdots x_n \mapsto F_n(g(x_1) \cdots g(x_n))
\]

for every function \( g : X \rightarrow X \)

**Example:** \( F_n(x) = x_1^2 + \cdots + x_n^2 \) \((X = Y = \mathbb{R})\)
Preassociative functions

**Proposition**

If $F : X^* \to Y$ is preassociative, then so is $g \circ F : x \mapsto g(F(x))$ for every function $g : Y \to Y$ such that $g|_{\text{ran}(F)}$ is one-to-one.

**Example:** $F_n(x) = \exp(x_1^2 + \cdots + x_n^2)$ \quad ($X = Y = \mathbb{R}$)
Preassociative functions

Proposition

Assume $F : X^* \to Y$ is preassociative.
If $F_n$ is constant, then so is $F_{n+1}$.

Proof. If $F_n(y) = F_n(y')$ for all $y, y' \in X^n$, then $F_{n+1}(xy) = F_{n+1}(xy')$ and hence $F_{n+1}$ depends only on its first argument...
Preassociative functions

We have seen that $F : X^* \rightarrow X$ is associative if and only if it is preassociative and $F_1(F(x)) = F(x)$.

Relaxation of $F_1(F(x)) = F(x)$:

\[
\text{ran}(F_1) = \text{ran}(F)
\]

\[
\text{ran}(F_1) = \{ F_1(x) : x \in X \}
\]

\[
\text{ran}(F) = \{ F(x) : x \in X^* \}\]
Preassociative functions

Preassociative functions
\[ \text{ran}(F_1) = \text{ran}(F) \]

Associative functions
We now focus on preassociative functions $F: X^* \to Y$ satisfying $\text{ran}(F_1) = \text{ran}(F)$

**Proposition**

These functions are completely determined by their unary and binary parts
Theorem

Let $F : X^* \to Y$. The following assertions are equivalent:

(i) $F$ is preassociative and satisfies $\text{ran}(F_1) = \text{ran}(F)$

(ii) $F$ can be factorized into $F = f \circ H$

where $H : X^* \to X$ is associative and $f : \text{ran}(H) \to Y$ is one-to-one.
Axiomatizations of function classes

**Theorem (Aczél 1949)**

$H : \mathbb{R}^2 \to \mathbb{R}$ is

- continuous
- one-to-one in each argument
- associative

if and only if

$$H(xy) = \varphi^{-1}(\varphi(x) + \varphi(y))$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and strictly monotone

$$H_n(x) = \varphi^{-1}(\varphi(x_1) + \cdots + \varphi(x_n))$$
Axiomatizations of function classes

**Theorem**

Let $F : \mathbb{R}^* \to \mathbb{R}$. The following assertions are equivalent:

(i) $F$ is preassociative and satisfies $\text{ran}(F_1) = \text{ran}(F)$, $F_1$ and $F_2$ are continuous and one-to-one in each argument

(ii) we have

$$F_n(x) = \psi(\varphi(x_1) + \cdots + \varphi(x_n))$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are continuous and strictly monotone
Axiomatizations of function classes

Recall that a \textit{triangular norm} is a function $T : [0, 1]^2 \rightarrow [0, 1]$ which is nondecreasing in each argument, symmetric, associative, and such that $T(1x) = x$

\begin{theorem}
Let $F : [0, 1]^* \rightarrow \mathbb{R}$ be such that $F_1$ is strictly increasing. The following assertions are equivalent:

(i) $F$ is preassociative and $\text{ran}(F_1) = \text{ran}(F)$,
    $F_2$ is symmetric, nondecreasing, and $F_2(1x) = F_1(x)$

(ii) we have
    \[ F = f \circ T \]
    where $f : [0, 1] \rightarrow \mathbb{R}$ is strictly increasing and $T : [0, 1]^* \rightarrow [0, 1]$ is a triangular norm
\end{theorem}
**Strongly preassociative functions**

**Definition.** We say that $F : X^* \to Y$ is strongly preassociative if

$$F(xz) = F(x'z') \Rightarrow F(xyz) = F(x'yz')$$

**Theorem**

$F : X^* \to Y$ is strongly preassociative if and only if $F$ is preassociative and $F_n$ is symmetric for every $n \in \mathbb{N}$.
Open problems

(1) Find new axiomatizations of classes of preassociative functions from existing axiomatizations of classes of associative functions

(2) Find interpretations of preassociativity in fuzzy logic, artificial intelligence,...
Back to the factorization theorem

**Theorem**

Let $F : X^* \to Y$. The following assertions are equivalent:

(i) $F$ is preassociative and $\text{ran}(F_1) = \text{ran}(F)$

(ii) $F$ can be factorized into

$$F = f \circ H$$

where $H : X^* \to X$ is associative

$f : \text{ran}(H) \to Y$ is one-to-one.
String functions

A *string function* if a function

\[ F : X^* \rightarrow X^* \]

\[ F : X^* \rightarrow X^* \] is *associative* (E. Lehtonen) if

\[ F(\text{xyz}) = F(\text{x}F(\text{y})\text{z}) \quad \forall \ \text{xyz} \in X^* \]

(same equivalent definitions)
Associative string functions

\( F : X^* \rightarrow X^* \) is associative if

\[
F(xyz) = F(xF(y)z) \quad \forall \ xyz \in X^*
\]

Examples

- \( F = \text{id} \)
- \( F = \) sorting data in alphabetic order
- \( F = \) transforming a string of letters into upper case
- \( F = \) removing a given letter, say ‘a’
- \( F = \) removing all repeated occurrences of letters

\[
F(\text{mathematics}) = \text{matheics}
\]
Preassociative functions

**Theorem**

Let $F: X^* \to Y$. The following assertions are equivalent:

(i) $F$ is preassociative

(ii) $F$ can be factorized into

$$F = f \circ H$$

where $H: X^* \to X^*$ is associative

$f: \text{ran}(H) \to Y$ is one-to-one.

We can add:

(i) $\text{ran}(F) = \text{ran}(F_1) \cup \cdots \cup \text{ran}(F_m)$

(ii) $H: X^* \to X^1 \cup \cdots \cup X^m$
Preassociative functions

Open question:
Find characterizations of classes of associative string functions
Barycentrically associative functions

Notation

\[ x^n = x \cdots x \quad (n \text{ times}) \]

\[ |x| = \text{length of } x \]

\( F : X^* \to X \) is \textit{B-associative} if

\[
F(xyz) = F(xF(y)|y|z) \quad \forall \ xyz \in X^*
\]

Alternative names: decomposability, associativity of means.
Barycentrically associative functions

\[ F(xyz) = F(xF(y)|y|z) \quad \forall \ xyz \in X^* \]
**Theorem** (Kolomogoroff-Nagumo, 1930)

$F : \mathbb{R}^* \rightarrow \mathbb{R}$ is B-associative,
every $F_n$ is

- symmetric
- continuous
- idempotent (i.e., $F_n(x^n) = x$)
- str. increasing in each argument

if and only if

$$F_n(x) = \varphi^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} \varphi(x_i)\right)$$

where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and strictly monotone
B-preassociative functions

Let $Y$ be a nonempty set

**Definition.** We say that $F : X^* \rightarrow Y$ is **B-preassociative** if

$$|y| = |y'| \quad \text{and} \quad F(y) = F(y') \quad \Rightarrow \quad F(xyz) = F(xy'z)$$

**Examples:**

$F_n(x) = x_1^2 + \cdots + x_n^2 \quad (X = Y = \mathbb{R})$

$F_n(x) = |x| \quad (X \text{ arbitrary, } Y = \mathbb{N})$

**Fact.** Preassociative functions are B-preassociative
B-preassociative functions

**Proposition**

$F : X^* \rightarrow X$ is B-associative if and only if it is B-preassociative and $F(F(x)^{|x|}) = F(x)$

\[
F(F(x)^{|x|}) = F(x) \iff \delta_{F_n} \circ F_n = F_n \quad (n \in \mathbb{N})
\]

\[
\delta_{F_n}(x) = F_n(x^n)
\]

**Relaxation:**

\[
\text{ran}(\delta_{F_n}) = \text{ran}(F_n) \quad (n \in \mathbb{N})
\]
B-preassociative functions

ran(\delta_{F_n}) = \text{ran}(F_n)

B-associative functions
Theorem

Let $F : X^* \to Y$. The following assertions are equivalent:

(i) $F$ is B-preassociative and $\text{ran}(\delta_{F_n}) = \text{ran}(F_n)$ for all $n \in \mathbb{N}$

(ii) $F$ can be factorized into

$$F_n = f_n \circ H_n$$

where $H : X^* \to X$ is B-associative

$f_n : \text{ran}(H_n) \to Y$ is one-to-one.

Open question: Describe the class of B-preassociative functions
Theorem

$F : \mathbb{R}^* \to \mathbb{R}$ is B-preassociative,
every $F_n$ is

- symmetric
- continuous
- strictly increasing in each argument

if and only if

$$F_n(x) = \psi_n \left( \frac{1}{n} \sum_{i=1}^{n} \varphi(x_i) \right)$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi_n : \mathbb{R} \to \mathbb{R}$ ($n \in \mathbb{N}$) are continuous and strictly increasing.
Thank you for your attention!