$H_{\infty}$ filtering for stochastic time-delay systems

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Abstract—This paper presents the robust $H_{\infty}$ filter design for stochastic systems with time-varying delay. The aim is to design a stable linear filter ensuring exponential stability in mean-square and a prescribed $H_{\infty}$ performance level for the filtering error system. Based on the application of the descriptor model transformation and free weighting matrices, delay-dependent sufficient conditions are proposed in terms of linear matrix inequalities (LMIs). Numerical examples demonstrate the proposed approaches are effective and are an improvement over existing ones.

Index Terms—exponential stability in mean-square, robust $H_{\infty}$ filter, time-varying delay, linear stochastic system, linear matrix inequalities (LMIs).

I. INTRODUCTION

In recent years, stochastic $H_{\infty}$ filtering and control problems with system models expressed by Itô-type stochastic differential equations have become a interesting research topic and has gained extensive attention; (see [1]-[2]). There have been a lot of studies on $H_{\infty}$ control or state estimation in deterministic systems, (see [3]-[4]). Robust $H_{\infty}$ estimation problems for linear and nonlinear stochastic systems were discussed by [1] and [5], respectively.

This paper discusses robust $H_{\infty}$ filtering for stochastic systems with time-varying delay. Here, we focus on the design of a linear state estimator such that the dynamics of the estimation error is stochastically exponentially stable in mean square, whose $L_2$-induced gain with respect to uncertain disturbance signal is less than a prescribed level $\gamma$.

First, by extending the descriptor system approach introduced in [6] for deterministic systems with delay to the stochastic systems case. By using some zero equations, we introduce some free weighting matrices. To guarantee the existence of desired robust $H_{\infty}$ filters, Delay-dependent sufficient conditions are proposed based on LMI algorithm and the minimum $\gamma$ obtained are less conservative than corresponding results in the literature. Numerical simulation example shows the results are effective and are an improvement over existing methods.

For convenience, we adopt the following notations:

\[ Tr(A)(A^T) \] Trace (transpose) of the matrix $A$.
\[ A \geq 0 (A > 0) \] Positive semi-definite (positive definite) matrix $A$.
\[ L_2^2([0, \infty); R^n) \] space of nonanticipative stochastic processes $\phi(t)$ with respect to filtration $\mathcal{F}_t$ satisfying
\[ \|\phi(t)\|_2^2 = E \int_0^\infty \|\phi(t)\|^2 dt < \infty \]
\[ L_2^2([\tau, 0]; R^n) \] the family of $R^n-$valued stochastic processes $\eta(s), -\tau \leq s \leq 0$ such that $\eta(s)$ is $\mathcal{F}_s-$ measurable for every second and
\[ \int_0^\tau E\|\eta(s)\|^2 ds < \infty \]
\[ E\{\cdot\} \] mathematical expectation operator with respect to the given probability measure $P$.

II. MAIN RESULTS

Consider the following stochastic linear time-delay systems

\[ dx(t) = [A_0x(t) + A_0d(x(t - h(t)) + B_0v(t))]dt + [C_0x(t) + C_0d(x(t - h(t))]d\beta(t) \]

\[ x(t) = \varphi(t), t \in [-\tau, 0] \]

\[ dy(t) = [A_1x(t) + A_1d(x(t - h(t)) + B_1v(t))]dt + [C_1x(t) + C_1d(x(t - h(t))]d\beta(t) \]

\[ z(t) = Lx(t) \quad (3) \]

where $x(t) \in R^n$ is the state vector, the time delay $h(t)$ is a time-varying continuous function.
that satisfies
\[ 0 \leq h(t) \leq \tau \quad (4) \]
and
\[ h'(t) \leq \mu \leq 1 \quad (5) \]
y(t) \in R^m is the measurement output, \( \varphi(t) \) is a continuous vector-valued initial function and \( \varphi := \{ \varphi(s) : -\tau \leq s \leq 0 \} \in L^2_t((-\tau,0], R^n). \) \( z(t) \in R^r \) is the state combination to be estimated, \( v(t) \in L^2_t([0, \infty); R^n) \) stands for the exogenous disturbance signal. \( A_i, A_{id}, B_i, C_i, C_{id}, i = 1,2, \) and \( L \) are known constant matrices with appropriate dimensions. Where the variables \( \beta(t) \) is 1-D Brownian motion satisfying \( E\{d\beta(t)\} = 0, E\{d\beta(t)^2\} = dt. \)

We take the following linear filter for the estimation of \( z(t) \)
\[
\begin{align*}
dx_f(t) &= A_f x_f(t)dt + B_f dy(t) \\
x_f(0) &= 0 \\
z_f(t) &= C_f x_f(t)
\end{align*}
\]
where \( x_f(t) \in R^n, \tilde{z}(t) \in R^r, \) the constant matrices \( A_f, B_f, C_f \) are filter parameters to be designed. Denoting \( \xi^T(t) = [x^T(t), x_f^T(t)] \) and \( \tilde{z}(t) = z(t) - z_f(t), \) then, we obtain the following augmented systems:
\[
\begin{align*}
d\xi(t) &= \tilde{A} \xi(t) + \tilde{A}_d \xi(t-h(t)) + \tilde{B} v(t) dt + [\tilde{C}_d \xi(t) + \tilde{C}_d \xi(t-h(t))] d\beta(t) \quad (7) \\
\tilde{z} &= \tilde{L} \xi(t)
\end{align*}
\]
where
\[
\begin{align*}
\tilde{A} &= \begin{bmatrix} A_0 & 0 \\ B_f A_1 & A_f \end{bmatrix}, \quad \tilde{A}_d &= \begin{bmatrix} A_{0d} & 0 \\ B_f A_{id} & A_f \end{bmatrix} \\
\tilde{B} &= \begin{bmatrix} B_0 \\ B_f B_1 \end{bmatrix}, \quad \tilde{L} = \begin{bmatrix} L & -C_f \end{bmatrix} \\
\tilde{C} &= \begin{bmatrix} C_0 & 0 \\ C_f C_1 & 0 \end{bmatrix}, \quad \tilde{C}_d &= \begin{bmatrix} C_{0d} & 0 \\ B_f C_{id} & 0 \end{bmatrix}
\end{align*}
\]

Definition 1: System (7) with \( v(t) \equiv 0 \) is said to be exponentially stable in mean square if there exists a positive constant \( \alpha \) such that
\[
\lim_{t \to \infty} \sup \frac{1}{t} \log E \|x(t)\|^2 \leq -\alpha
\]
The objective of this paper is to seek the filter parameters \( A_f, B_f \) and \( C_f \) such that the augmented system (7) with \( v(t) \equiv 0 \) is exponentially stable in mean square. More specifically, for a prescribed disturbance attenuation level \( \gamma > 0, \) such that the performance index
\[
\begin{align*}
\bar{J} &= \int_0^\infty (\tilde{z}^T \tilde{z} - \gamma^2 v^T v) dt \\
n &= \int_{-h(t)}^0 (\tilde{x}^T \tilde{x} + \gamma^2 v^T v) dt \quad (9)
\end{align*}
is negative \( \forall 0 \neq v(t) \in L^2_t((0, \infty), R^n). \)

A. Delay-dependent \( H_\infty \) filtering
Before presenting the main results, we first give the following lemmas.

Lemma 1: Given scalars \( \tau > 0 \) and \( \mu < 1, \) the system (7) with \( v(t) \equiv 0 \) is exponentially stable in mean square if there exist symmetric positive definite matrices \( P > 0, Q \geq 0, Z > 0, R > 0, \) and appropriately dimensioned matrices \( N_i, T_j \) and \( Y_i, (i = 1, 2, 3, 4; j = 1, 2, 3, 4, 5) \) such that the following LMI holds.
\[
\begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \tau Y_1 & Y_1 \\
* & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \tau Y_2 & Y_2 \\
* & * & \Gamma_{33} & \Gamma_{34} & \tau Y_3 & Y_3 \\
* & * & * & \Gamma_{44} & \tau Y_4 & Y_4 \\
* & * & * & * & \tau R & 0 \\
* & * & * & * & * & -Z
\end{bmatrix} < 0
\quad (10)
\]
where an ellipsis * denotes a block induced essentially by symmetry, and
\[
\begin{align*}
\Gamma_{11} &= N_1 \tilde{A} + \tilde{A}^T N_1^T + T_1 \tilde{C} + \tilde{C}^T T_1^T + Q + Y_1 + Y_1^T \\
\Gamma_{12} &= N_1 \tilde{A}_d + \tilde{A}^T N_1^T + T_1 \tilde{C}_d + \tilde{C}_d^T T_1^T + Y_1 + Y_1^T \\
\Gamma_{13} &= P - N_1 \tilde{N}_1^T + \tilde{N}_1^T T_1 + \tilde{C}_d^T T_1^T + Y_1^T \\
\Gamma_{14} &= \tilde{A}_1 T_1^T - T_1 + \tilde{C}_d^T T_1^T + Y_1^T \\
\Gamma_{22} &= N_2 \tilde{A}_d + \tilde{A}_d^T N_2^T + T_2 \tilde{C}_d + \tilde{C}_d^T T_2 + (1 - \mu)Q \\
&- Y_2 - Y_2^T \\
\Gamma_{23} &= -N_2 + \tilde{A}_d^T N_2^T + \tilde{C}_d^T T_3 + Y_3^T \\
\Gamma_{24} &= \tilde{A}_1^T N_1^T - T_2 + \tilde{C}_d^T T_4 - Y_4^T \\
\Gamma_{33} &= -N_3 - \tilde{N}_3^T + \tau R, \quad \Gamma_{34} = -N_4^T - T_3 \\
\Gamma_{44} &= P - T_4^T - T_4 + \tau Z \\
\end{align*}
\]
Proof: For convenience, set
\[
\begin{align*}
q(t) &= \tilde{A} \xi(t) + \tilde{A}_d \xi(t-h(t)) + \tilde{B} v(t) \\
g(t) &= C \xi(t) + C_d \xi(t-h(t)) \\
\end{align*}
\]
then system (7) becomes the following descriptor stochastic systems
\[
\begin{align*}
d\xi(t) &= q(t) dt + g(t) d\beta(t) \\
\tilde{z} &= \tilde{L} \xi(t)
\end{align*}
\]
Choose a Lyapunov-Krasovskii functional for system (13) to be
\[
V(t) = \sum_{i=1}^{4} V_i(t)
\]
in which
\[
\begin{align*}
V_1(t) &= (\xi(t))^T P \xi(t) \\
V_2(t) &= \int_{t-h(t)}^t (\xi(s))^T Q \xi(s) ds
\end{align*}
\]
where $P, Q, Z, R$ are symmetric positive definite matrices with appropriate dimensions. $L$ be the weak infinitesimal operator of (13), the Newton-Leibniz formula provides

$$
\xi(t)-\xi(t-h(t)) = \int_{t-h(t)}^{t} q(s)ds + \int_{t-h(t)}^{t} g(s)d\beta(s)
$$

(14)

For appropriately dimensioned matrices $N_i, Y_i$ and $T_i$ ($i = 1, 2, 3, 4$), equations in (11) - (12) ensure that

$$
2\left[\xi^T(t)N_1 + \xi^T(t-h(t))N_2 + q^T(t)N_3 + g^T(t)N_4\right]
\equiv 0
$$

(15)

$$
2\left[\xi^T(t)T_1 + \xi^T(t-h(t))T_2 + q^T(t)T_3 + g^T(t)T_4\right]
\equiv 0
$$

(16)

$$
2\left[\xi^T(t)Y_1 + \xi^T(t-h(t))Y_2 + q^T(t)Y_3 + g^T(t)Y_4\right]
\equiv 0
$$

(17)

$$
\tau\Lambda - \int_{t-h(t)}^{t} \Lambda ds \geq 0
$$

(18)

where

$$
x^T(t) = \int_{t-h(t)}^{t} q^T(s)ds, \quad x(t) = \int_{t-h(t)}^{t} g^T(s)d\beta(s), \quad \Lambda = \eta^T(t)\tilde{Y}R^{-1}\tilde{Y}^T\eta(t).
$$

Adding the terms on the left of (15)-(18) to $L_{v=0} V$. Moreover, by Lemma[8], for any matrix $Z > 0$

$$
-2\eta^T(t)\tilde{Y}Z\xi \leq \eta^T(t)\tilde{Y}Z^{-1}\tilde{Y}^T\eta(t) + \xi^T Z\xi
$$

then $L_{v=0} V$ can be expressed as

$$
L_{v=0} V \leq \eta^T(t)\Xi\eta(t) - \int_{t-h(t)}^{t} \Theta R^{-1}\Theta^T ds
$$

$$
- \int_{t-h(t)}^{t} trace\left[g^T(t)Zg(t)\right] ds + \xi^T Z\xi
$$

where

$$\eta^T(t) = \left[\xi^T(t), \xi^T(t-h(t)), q^T(t), g^T(t)\right]
$$

$$\Xi = \Gamma + \tau\tilde{Y}R^{-1}\tilde{Y}^T + \tilde{Y}Z^{-1}\tilde{Y}^T
$$

$$\Theta = \eta^T(t)\tilde{Y} + q^T(t)R
$$

$$\tilde{Y}^T = \left[Y^T_1 Y^T_2 Y^T_3 Y^T_4\right]
$$

$$\Gamma = \left[
\begin{array}{cccc}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\
* & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\
* & * & \Gamma_{33} & \Gamma_{34} \\
* & * & * & \Gamma_{44}
\end{array}
\right]
$$

(19)

(20)

Since

$$
E(\xi^T Z\xi) = E\int_{t-h(t)}^{t} trace\left[g^T(t)Zg(t)\right] ds
$$

It follows that

$$
EL_{v=0} V(t) \leq E\eta^T(t)\Xi\eta(t)
$$

(21)

By Schur’s complement, $\Xi < 0$ is equivalent to LMI (10).

Set $\lambda_0 = \lambda_{\min}(-\Xi), \lambda_1 = \lambda_{\min}(P), by (21)

$$
EL_{v=0} V(t) \leq -\lambda_0 E\eta^T(t)\eta(t) \leq -\lambda_0 E\tilde{X}^T(t)x(t)
$$

(22)

From the definitions of $V(t), q(t)$ and $g(t)$, there exist positive scalars $\alpha_1, \alpha_2$ such that

$$
\lambda_1 \|x(t)\|^2 \leq V(t) \leq \alpha_1 \|x(t)\|^2 + \alpha_2 \int_{t-2\tau}^{t} \|x(s)\|^2 ds
$$

(23)

Choose a scalar $\alpha_0 > 0$ such that

$$
\alpha_0 (\alpha_1 + 2\alpha_2 \tau e^{2\alpha_0 \tau}) \leq \lambda_0
$$

(24)

*then, by Itô differential formula, for $t_0 \geq 2\tau$, it has

$$
Ee^{\alpha_0 t}V(t) - Ee^{\alpha_0 t_0}V(t_0) = E\int_{t_0}^{t} L_{v=0}(e^{\alpha_0 s}V(s)) ds
$$

$$
\leq E\int_{t_0}^{t} e^{\alpha_0 s}[\alpha_0 (\alpha_1 \|x(s)\|^2 + \alpha_2 \int_{t-s-2\tau}^{t} \|x(u)\|^2 du) - \lambda_0 \|x(s)\|^2] ds
$$

(25)

By (23) and (24), getting

$$
\lim_{t \to \infty} \sup \frac{1}{t} \log E \|x(t)\|^2 \leq -\alpha_0
$$

which implies system (13) is exponentially stable in mean square. The proof of Lemma 1 is completed.

\textbf{Theorem 1:} Consider the system (1)-(3) with (4) and (5). Given scalars $\tau > 0$ and $\mu < 1$, for a prescribed $\gamma > 0$, the $H_\infty$ performance $J < 0$ holds for all nonzero $v(t) \in L^2_{\infty}((0, \infty), R^n)$, if there exist symmetric positive definite matrix $P > 0, Q \geq 0, Z > 0, R > 0$, scalars $h_i > 0, e_i > 0$ and appropriately dimensioned matrices $S_i, W_i$ and $Y_i$ ($i = 1, 2, 3, 4$) satisfying the following LMI

$$
\begin{bmatrix}
\Gamma & \tau\tilde{Y} & \tilde{Y}^T & \tilde{K} & \tilde{F} \\
* & -\tau R & 0 & 0 & 0 \\
* & * & -Z & 0 & 0 \\
* & * & * & -I & 0 \\
* & * & * & -\gamma^2 I
\end{bmatrix} < 0
$$

(25)
If a solution of the LMI exists then the filter that guarantees the estimation error level of $\gamma$ is given by (6) with

$$A_f = S^{-1}_1 X_4, \ B_f = S^{-1}_1 X_5 \ and \ C_f,$$

where $\Gamma$ and $\tilde{Y}$ are defined in (19) and (20), respectively,

$$\tilde{F} = \begin{bmatrix} \gamma_i & \gamma_{i+1} & \gamma_{i+2} \\ \gamma_{i+1} & \gamma_{i+2} & \gamma_{i+3} \\ \gamma_{i+2} & \gamma_{i+3} & \gamma_{i+4} \end{bmatrix}, \ \tilde{K} = \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix} \ (26)$$

and

$$\Gamma_{ii} = \begin{bmatrix} \Gamma^1_{ii} & \Gamma^2_{ij} & \Gamma^3_{ij} & \Gamma^4_{ij} \\ \Gamma^2_{ij} & \Gamma^3_{ij} & \Gamma^4_{ij} \end{bmatrix}, \ \Gamma_{ij} = \begin{bmatrix} \Gamma^1_{ij} & \Gamma^2_{ij} & \Gamma^3_{ij} & \Gamma^4_{ij} \end{bmatrix} \ (27)$$

$$\Gamma^1_{11} = h_1 S_1 A_0 + h_1 A^T_0 S^T_1 + h_1 X_2 A_1 + h_1 A^T_1 X^T_2 + e_1 W_1 C_0 + e_1 C^T_0 W^T_1 + e_1 X_3 C_1 + e_1 C^T_1 X^T_3 + Q_1 + Y_{11} + Y_{12}$$

$$\Gamma^2_{11} = h_1 X_1 + h_1 A^T_0 S^T_3 + h_1 A^T_1 X^T_2 \ (28)$$

$$\Gamma^3_{11} = h_1 X_1 + h_1 X^T_2 + Q_2 + Y_{12} + Y_{13}$$

$$\Gamma^4_{11} = h_1 X_1 + h_1 X^T_2 + Q_1 + Y_{11} + Y_{12} + Y_{13}$$

$$\Gamma^1_{12} = h_1 S_1 A_0 + h_1 X_1 A_1 + h_2 A^T_0 S^T_1 + h_2 A^T_1 X^T_2 + e_1 W_1 C_0 + e_1 X_1 C_1 + e_1 C^T_0 W^T_1 + e_1 C^T_1 X^T_3 + Q_1 + Y_{11} + Y_{12}$$

$$\Gamma^2_{12} = h_2 A^T_0 S^T_3 + h_2 A^T_1 X^T_2 + e_2 C^T_0 W^T_3 + e_2 C^T_1 X^T_3 - Y_{11} + Y_{12}$$

$$\Gamma^3_{12} = h_2 A^T_0 S^T_3 + h_2 A^T_1 X^T_2 + e_2 C^T_0 W^T_3 + e_2 C^T_1 X^T_3 - Y_{11} + Y_{12}$$

$$\Gamma^4_{12} = h_2 X^T_2 - Y_{12} + Y_{13}$$

$$\Gamma^1_{13} = P_2 - h_1 S_2 + h_3 A^T_1 S^T_3 + h_3 A^T_1 X^T_2 + e_2 W_1 + e_3 C^T_1 X^T_3 + Y_{13}$$

$$\Gamma^2_{13} = P_2 - h_1 S_2 + h_3 A^T_1 S^T_3 + h_3 A^T_1 X^T_2 + e_2 W_1 + e_3 C^T_1 X^T_3 + Y_{13}$$

$$\Gamma^3_{13} = P_2 - h_1 S_2 + h_3 A^T_1 S^T_3 + h_3 A^T_1 X^T_2 + e_2 W_1 + e_3 C^T_1 X^T_3 + Y_{13}$$

$$\Gamma^1_{14} = h_4 X^T_1 - e_1 W_1 + Y_{14}$$

$$\Gamma^2_{14} = h_4 X^T_1 - e_1 W_1 + Y_{14}$$

$$\Gamma^3_{14} = h_4 X^T_1 - e_1 W_1 + Y_{14}$$

$$\Gamma^4_{14} = h_4 X^T_1 - e_1 W_1 + Y_{14}$$

$$\Gamma^1_{15} = \tau Y_{11}, \ \Gamma^2_{15} = \tau Y_{12}, \ \Gamma^3_{15} = \tau Y_{13}, \ \Gamma^4_{15} = \tau Y_{14} \ (29)$$

$$\mathbf{Y}_i = \begin{bmatrix} Y_{i1} & Y_{i2} & Y_{i3} & Y_{i4} \end{bmatrix}, i = 1, 2, 3, 4$$

$$\tilde{L} = \begin{bmatrix} L & -C_f \end{bmatrix} \ (29)$$

**Proof:** First by the proof of the Lemma 1, we can deduce that the augmented system (7) with $v(t) = 0$ to be robust exponential stable in mean square. Second, we prove $\tilde{J} < 0$ for all nonzero
that for any $T > 0$

$$\dot{J}(T) = E \int_0^T (\| \dot{z}(t) \|^2 - \gamma^2 \| v(t) \|^2) dt$$

$$\leq E \int_0^T (\| \dot{z}(t) \|^2 - \gamma^2 \| v(t) \|^2 + L_v V(\xi(t))) dt$$

$$= E \int_0^T \left[ \eta(t) \right]^T \Psi \left[ \eta(t) \right] dt$$

$$- E \int_0^T \Theta R \Theta^T ds dt$$

Where $\tilde{F}$ and $\tilde{K}$ are defined in (26), and

$$\Psi = \begin{bmatrix} \tilde{Z} & \tilde{F} \\ \ast & -\gamma^2 I \end{bmatrix}$$

$$\tilde{Z} = \Gamma + \tau \bar{Y} R^{-1} \bar{Y}^T + \bar{Y} \tilde{Z}^{-1} \tilde{Y}^T + \tilde{K} \tilde{K}^T$$

Therefore, if $\Psi < 0$, then

$$\dot{J}(T) \leq -\lambda_{\min}(\Psi) E \int_0^T (\| \eta(t) \|^2 + \| v(t) \|^2) dt$$

$$\leq -\lambda_{\min}(\Psi) E \int_0^T \| v(t) \|^2 dt < 0$$

for any nonzero $v(t) \in L^2_2([0, +\infty), R^n)$, which yield $\dot{J}(T) < -\lambda_{\min}(\Psi) E \int_0^T \| v(t) \|^2 dt < 0$. If we take

$$N_i = h_i \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}, Y_i = \begin{bmatrix} Y_{i1} & Y_{i2} \\ Y_{i3} & Y_{i4} \end{bmatrix}$$

$$T_i = e_i \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}, i = 1, 2, 3, 4$$

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix}, P = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix}$$

$$R = \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix}, Z = \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix}$$

Substituting (8) and (32) into (29), and letting $S_2 A_f X_1, S_2 B_f W_2 X_3, S_4 A_f X_4, S_4 B_f W_4 X_5$ yields $\dot{\Psi} < 0$ is equivalent to (25). From our assumption, $A_f = S_4^{-1} X_4, B_f = S_4^{-1} X_5$, and an $H_\infty$ filter is constructed as in the form of (6), and the proof of Theorem 1 is completed.

III. NUMERICAL SIMULATION

Consider the stochastic time-varying delay system\cite{10} with parameters as follows:

$$A_0 = \begin{bmatrix} 0 & 3 \\ -4 & -5 \end{bmatrix}, B_0 = \begin{bmatrix} -0.4545 \\ 0.9090 \end{bmatrix}$$

$$A_{od} = \begin{bmatrix} -0.1 & 0 \\ 0.2 & -0.2 \end{bmatrix}, A_{id} = \begin{bmatrix} 0.5 & 0.3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}, L = \begin{bmatrix} 3 & 4 \end{bmatrix}, B_1 = 1$$

$$C_0 = C_{od} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, C_1 = C_{id} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

For the simple choice of $h_1 = 0.7, h_2 = 0.41, h_3 = 0.2, h_4 = 0.2, e_1 = 0.5, e_2 = 0.3, e_3 = 0.1, e_4 = 0.5, e_5 = 0.01,$

The minimum achievable noise attenuation level obtained in \cite{10} is $\gamma = 0.6074$. However, applying the delay-dependent criterion of Theorem 1, we obtain that the minimum achievable bound on the noise attenuation level is $\gamma_{\min} = 8 \times 10^{-7}$ (for $\tau = 0.1, \mu = 0.3$), which is far smaller than the result obtained in \cite{10}. When we take $\gamma = 0.6074$, the corresponding filter matrices obtained by Theorem 1 and \cite{10} respectively are

$$A_f = \begin{bmatrix} -9.7029 & 1.1090 \\ 0.9684 & -9.4364 \end{bmatrix}, B_f = \begin{bmatrix} -0.1195 \\ -0.1562 \end{bmatrix}$$

$$C_f = \begin{bmatrix} 0.0399 \\ 0.0965 \end{bmatrix}$$

and

$$A_f = \begin{bmatrix} 0.3065 \\ -4.4618 \end{bmatrix}, B_f = \begin{bmatrix} 0.1797 \\ -0.5045 \end{bmatrix}$$

$$C_f = \begin{bmatrix} -2.8091 \\ -3.4756 \end{bmatrix}$$

Set the initial condition as $x(0) = [0.1, 0]^T, x_f = [0.04, 0]^T$ respectively. The exogenous disturbance input $v(t)$ is set as random noise and $v(0) = 0.1$. Fig. 1 gives the time responses of the system states and filter states for the filter (33) with $v(t) = 0$. It shows the filtering error system (7) is exponentially stable in mean square with $v(t) = 0$. Fig. 2 shows the time response of the function $\omega(t) = \int_0^\infty \| \tilde{z}(s) \|^2 ds / \int_0^\infty \| v(s) \|^2 ds$ with $v(t) \neq 0$. It is seen that the maximum value of this function is less than 0.25, which reveals that the $H_\infty$ performance level $\sqrt{0.25} = 0.5$ is much less than the prescribed level 0.6074. Therefore, we can see that the designed $H_\infty$ filter meets the specified requirements. When $v(t) \neq 0$, the time responses of the system states $x_1, x_2$ and filter state $x_f 1, x_f 2$ for the filter (33) and (34) are displayed in Fig. 3 and Fig. 4 respectively. The simulation results imply that the desired goal is well achieved. Moreover, our method is better than the method in \cite{10}.

IV. Conclusion

The problem of robust $H_\infty$ filtering for stochastic systems with time-varying delay has been addressed in this paper. LMI-based technique, exponentially stable in mean square linear filter are designed which guarantee $L_2$ gain to be less than a prescribed level $\gamma > 0$. Delay-dependent sufficient condition for stochastic system is presented, the
minimum $\gamma$ obtained are less conservative than corresponding results in the literature due to applying descriptor model transformation of the system and introducing some free weighting matrices. Numerical example has clearly indicated the less conservatism of our design.

REFERENCES