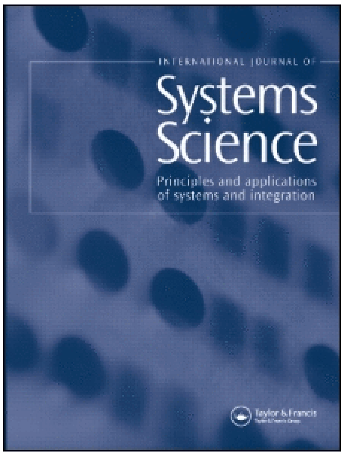


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# Robust exponential stability criteria of uncertain stochastic systems with time-varying delays

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# Robust exponential stability criteria of uncertain stochastic systems with time-varying delays

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This article investigates the problem of delay-dependent exponential stability in mean square for continuous-time linear stochastic systems with structured uncertainties and time-varying delays. By applying descriptor model transformation of the systems, a new type of Lyapunov–Krasovskii functional is constructed, and by introducing some free weighting matrices, some new delay-dependent and delay-independent stability criteria are derived respectively in terms of an LMI algorithm. The new stability criteria are less conservative than existing results. Numerical examples demonstrate that the new criteria are effective and are an improvement over existing results.

**Keywords:** stochastic systems; exponential stability in mean square; time-varying delays; linear matrix inequality



## 1. Introduction

During the last few decades, many authors have investigated the robust stability analysis of dynamic systems with delays. In general, delay-dependent stability criteria make use of the information on the size of delays, they are less conservative than delay-independent ones, especially for small delays. Therefore, much interest has been focused on the delay-dependent stability analysis of delay systems (see, e.g. Haussman 1978; Ichikawa 1982; Verriest and Florchinger 1995; Li and De Souza 1997; Mao, Koroleva, and Rodkina 1998; Moon, Park, Kwon, and Lee 2001; Yue and Won 2001; Lu and Tsai 2003; He, Wu, She, and Liu 2004; Chen, Guan, and Lu 2005; Liu and Zhang 2005; Lu, Su, and Tsai 2005). However, most of these results have focused on the deterministic systems with delays (see, e.g. Li and De Souza 1997; Moon et al. 2001; He et al. 2004; Liu and Zhang 2005), few authors (Haussman 1978; Ichikawa 1982; Verriest and Florchinger 1995; Mao et al. 1998; Yue and Won 2001; Lu and Tsai 2003; Lu et al. 2005) have discussed the delay-dependent stability for stochastic systems with delays. Verriest and Florchinger (1995) presented the stability of a linear stochastic differential equation via Riccati equations. Based on the LMI approach, Yue and Won (2001), Lu and Tsai (2003), Chen et al. (2005) and Lu et al. (2005) gave the delay-dependent robust stability criteria of uncertain stochastic systems. However, the criteria given in Yue and Won (2001) involved the parameterised model transformation to determine the stability of the systems. Lu and Tsai (2003) and Lu et al. (2005) used some inequality constraint. Chen et al. (2005) used a descriptor integral inequality constraint and the criteria given in Chen et al. (2005) and Lu et al. (2005) with matrix constraint $P \leq \alpha I$, ($\alpha > 0$ is a scalar, $P$ is the Lyapunov matrix). These results show considerable conservativeness.

This article presents some new delay-dependent stability criteria for continuous-time linear stochastic systems with structured uncertainties and time-varying delays. First, by extending the descriptor system, an approach was introduced in Fridman and Shaked (2002) for deterministic systems with delays to the stochastic systems case. We present the stochastic systems in the equivalent descriptor stochastic systems. Based on the equivalent descriptor form representation, we construct a new type of Lyapunov–Krasovskii functional. Second, we extend the idea proposed in He et al. (2004) to the stochastic systems case. In the derivative of the Lyapunov–Krasovskii functional, the terms $q(t)$ and $g(t)$ as state vectors are retained, and in order to exclude those constraint conditions in Yue and Won (2001), Lu and Tsai (2003), Chen et al. (2005) and Lu et al. (2005), some free weighting matrices are introduced. In consequence, the stability criteria of systems do not concern any product terms of the Lyapunov–Krasovskii functional matrices with the systems matrices; moreover, the influences of the terms $x(t), x(t-h(t)), \int_{t-h(t)}^{t} q(s)\mathrm{d}s$ and $\int_{t-h(t)}^{t} g(s)\mathrm{d}s$ in

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Newton–Leibniz formula are expressed in terms of free weighting matrices. Finally, based on an LMI algorithm, we obtain some less conservative delay-dependent stability criteria for uncertain linear stochastic systems. The results include the delay-independent/rate-dependent and delay-dependent/rate-independent stability criteria. Furthermore, the methods can be easily extended to the uncertain nonlinear stochastic systems with delays case and the stochastic systems with polytopic-type uncertainties and delays case. Numerical simulation examples show that the results are effective and are an improvement over existing criteria.

For convenience, we adopt the following notations: $Tr(A)(A^T)$ denotes the trace (transpose) of the matrix $A$ and $A \geq 0 (A > 0)$ means that $A$ is a positive semi-definite (positive definite) matrix. $L^2_{\mathcal{F}_0}([-\tau, 0]; R^n)$ represents the family of $R^n$-valued stochastic processes $\eta(s)$, $-\tau \leq s \leq 0$, such that $\eta(s)$ is $\mathcal{F}_0$-measurable for every second and $\int_{-\tau}^{0} E\|\eta(s)\|^2 \mathrm{d}s < \infty$. $E\{\cdot\}$ denotes the mathematical expectation operator with respect to the given probability measure $P$.

## 2. Preliminaries

Consider the following linear stochastic systems with time-varying structured uncertainties and time-varying state delays:

$$\begin{aligned} \mathrm{d}x(t) &= [(A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t - h(t))]\mathrm{d}t \\ &\quad + [(C + \Delta C(t))x(t) + (C_d + \Delta C_d(t)) \\ &\quad \times x(t - h(t))]\mathrm{d}\beta(t) \\ x(t) &= \varphi(t) \quad \forall t \in [-\tau, 0] \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, and the time delay $h(t)$ is a time-varying continuous function that satisfies

$$0 \leq h(t) \leq \tau \tag{2}$$

and

$$\dot{h}(t) \leq \mu \tag{3}$$

where $\tau$ and $\mu$ are constants, $\varphi(t)$ is a continuous vector-valued initial function and $\varphi := \{\varphi(s) : -\tau \leq s \leq 0\} \in L^2_{\mathcal{F}_0}([-\tau, 0], R^n)$. It is well known that (1) has a unique solution, denoted by $x(t, \varphi)$, which is square integrable, so (1) admits a trivial solution $x(t, 0) \equiv 0$. $A$, $A_d$, $C$, $C_d$ are known constant matrices with appropriate dimensions, $\Delta A(t)$, $\Delta A_d(t)$, $\Delta C(t)$, $\Delta C_d(t)$ are unknown matrix representing time-varying parameter uncertainties and are assumed to be of the form

$$\begin{aligned} &\begin{bmatrix} \Delta A(t) & \Delta A_d(t) & \Delta C(t) & \Delta C_d(t) \end{bmatrix} \\ &\quad = MF(t)\begin{bmatrix} E_0 & E_1 & E_2 & E_3 \end{bmatrix} \end{aligned} \tag{4}$$

where $F(t) \in R^{i \times j}$ is a real uncertain matrix with Lebesgue measurable elements and meets

$$F^T(t)F(t) \leq I \tag{5}$$

where $M$, $E_0$, $E_1$, $E_2$, $E_3$ are known real constant matrices of appropriate dimensions. The variables $\beta(t)$ are an $r$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ (i.e. $\mathcal{F}_t = \sigma\{\varpi(s) : 0 \leq s \leq t\}$).

**Definition:** System (1) is said to be exponentially stable in mean square if there exists a positive constant $\alpha$ such that

$$\lim_{t \to \infty} \sup \frac{1}{t} \log E\|x(t)\|^2 \leq -\alpha \tag{6}$$

## 3. Main results and proofs

### 3.1. *Delay-dependent exponential stability*

In order to discuss the stability of system (1), first, we consider the following stochastic systems without uncertainties, namely

$$\begin{aligned} \mathrm{d}x(t) &= [Ax(t) + A_d x(t - h(t))]\mathrm{d}t \\ &\quad + [Cx(t) + C_d x(t - h(t))]\mathrm{d}\beta(t) \\ x(t) &= \varphi(t) \quad \forall t \in [-\tau, 0] \end{aligned} \tag{7}$$

For this case, the following theorem holds.

**Theorem 3.1:** *Given scalars $\tau > 0$ and $\mu < 1$, system (7) is exponentially stable in mean square, if there exist symmetric positive definite matrices $P > 0$, $Q \geq 0$, $Z > 0$, $R_{ij}$ $(i, j = 1, 2, 3)$ and appropriately dimensioned matrices $N_k$ and $T_k$ $(k = 1, 2, 3, 4)$ such that $R_{ij} = R_{ji}$ and the following LMIs hold*:

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & R_{13} \\ * & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & R_{23} \\ * & * & \Sigma_{33} & \Sigma_{34} & 0 \\ * & * & * & \Sigma_{44} & 0 \\ * & * & * & * & -Z \end{bmatrix} < 0 \tag{8}$$

*and*

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12}^T & R_{22} & R_{23} \\ R_{13}^T & R_{23}^T & R_{33} \end{bmatrix} > 0 \tag{9}$$

*where an asterisk * denotes a block induced easily by symmetry and*

$$\Sigma_{11} = N_1A + A^TN_1^T + T_1C + C^TT_1^T + Q + \tau R_{11} + R_{13} + R_{13}^T$$
$$\Sigma_{12} = N_1A_d + A^TN_2^T + T_1C_d + C^TT_2^T + \tau R_{12} - R_{13} + R_{23}^T$$
$$\Sigma_{13} = P - N_1 + A^TN_3^T + C^TT_3^T$$
$$\Sigma_{14} = A^TN_4^T - T_1 + C^TT_4^T$$
$$\Sigma_{22} = N_2A_d + A_d^TN_2^T + T_2C_d - (1-\mu)Q + C_d^TT_2^T + \tau R_{22} - R_{23} - R_{23}^T$$
$$\Sigma_{23} = -N_2 + A_d^TN_3^T + C_d^TT_3^T$$
$$\Sigma_{24} = A_d^TN_4^T - T_2 + C_d^TT_4^T$$
$$\Sigma_{33} = -N_3 - N_3^T + \tau R_{33}$$
$$\Sigma_{34} = -N_4^T - T_3$$
$$\Sigma_{44} = P - T_4^T - T_4 + \tau Z$$

**Proof:** For convenience, set

$$q(t) = Ax(t) + A_dx(t-h(t)) \tag{10}$$

$$g(t) = Cx(t) + C_dx(t-h(t)) \tag{11}$$

then system (7) becomes the following descriptor stochastic systems:

$$\mathrm{d}x(t) = q(t)\mathrm{d}t + g(t)\mathrm{d}\beta(t) \tag{12}$$

Choose a Lyapunov–Krasovskii functional for system (7) to be

$$V(t) = \sum_{i=1}^{5} V_i(t)$$

in which

$$V_1(t) = x(t)^TPx(t), \quad V_2(t) = \int_{t-h(t)}^{t} x^T(s)Qx(s)\mathrm{d}s$$

$$V_3(t) = \int_{-\tau}^{0}\int_{t+\theta}^{t} q^T(s)R_{33}q(s)\mathrm{d}s\,\mathrm{d}\theta,$$

$$V_4(t) = \int_{-\tau}^{0}\int_{t+\theta}^{t} \mathrm{trace}[g^T(s)Zg(s)]\mathrm{d}s\,\mathrm{d}\theta$$

$$V_5(t) = \int_0^t\int_{\alpha-h(\alpha)}^{\alpha} \begin{bmatrix} x(\alpha) \\ x(\alpha-h(a)) \\ q(s) \end{bmatrix}^T R \begin{bmatrix} x(\alpha) \\ x(\alpha-h(a)) \\ q(s) \end{bmatrix} \mathrm{d}s\,\mathrm{d}\alpha$$

where

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12}^T & R_{22} & R_{23} \\ R_{13}^T & R_{23}^T & R_{33} \end{bmatrix}$$

$P$, $Q$, $Z$, $R_{ij}$ $(i, j = 1, 2, 3)$ are positive definite matrices with appropriate dimensions. Let $L$ be the weak infinitesimal operator of (12), then, by the Itô differential formula

$$L_{v=0}V_1 = 2x^T(t)Pq(t) + \mathrm{trace}\big[g^T(t)Pg(t)\big] \tag{13}$$

$$L_{v=0}V_2 \le x^T(t)Qx(t) - (1-\mu)x^T(t-h(t))Qx(t-h(t)) \tag{14}$$

$$L_{v=0}V_3 \le \tau q^T(t)R_{33}q(t) - \int_{t-h(t)}^{t} q^T(s)R_{33}q(s)\mathrm{d}s \tag{15}$$

$$L_{v=0}V_4 \le \tau\,\mathrm{trace}\big[g^T(t)Zg(t)\big] - \int_{t-h(t)}^{t}\mathrm{trace}\big[g^T(s)Zg(s)\big]\mathrm{d}s \tag{16}$$

$$\begin{aligned} L_{v=0}V_5 = {} & h(t)\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} \\ & + 2\int_{t-h(t)}^{t} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix} q(s)\mathrm{d}s \\ & + \int_{t-h(t)}^{t} q^T(s)R_{33}q(s)\mathrm{d}s \end{aligned} \tag{17}$$

The Newton–Leibniz formula provides

$$x(t) - x(t-h(t)) = \int_{t-h(t)}^{t}\dot{x}(s)\mathrm{d}s = \int_{t-h(t)}^{t} q(s)\mathrm{d}s + \int_{t-h(t)}^{t} g(s)\mathrm{d}\beta(s) \tag{18}$$

By (18), we get

$$\int_{t-h(t)}^{t} q(s)\mathrm{d}s = x(t) - x(t-h(t)) - \int_{t-h(t)}^{t} g(s)\mathrm{d}\beta(s) \tag{19}$$

Substituting (19) into $L_{v=0}V_5$, we get

$$\begin{aligned} L_{v=0}V_5 = {} & h(t)\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} \\ & + 2\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} R_{13} & -R_{13} \\ R_{23} & -R_{23} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} \\ & + \int_{t-h(t)}^{t} q^T(s)R_{33}q(s)\mathrm{d}s - 2\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \\ & \times \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix} \int_{t-h(t)}^{t} g(s)\mathrm{d}\beta(s) \end{aligned}$$

By the lemma in Wang, Xie, and De Souza (1992), for any matrix $Z > 0$

$$-2\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix} \int_{t-h(t)}^{t} g(s)\mathrm{d}\beta(s)$$
$$\leq \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix} Z^{-1} \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix}^T \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}$$
$$+ \left[\int_{t-h(t)}^{t} g(s)\mathrm{d}\beta(s)\right]^T Z \left[\int_{t-h(t)}^{t} g(s)\mathrm{d}\beta(s)\right]$$

Obviously,

$$L_{v=0}V_5 \leq \tau \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}$$
$$+2\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} R_{13} & -R_{13} \\ R_{23} & -R_{23} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}$$
$$+\int_{t-h(t)}^{t} q^T(s)R_{33}q(s)\mathrm{d}s + \left[\int_{t-h(t)}^{t} g(s)\mathrm{d}\beta(s)\right]^T$$
$$\times Z\left[\int_{t-h(t)}^{t} g(s)\mathrm{d}\beta(s)\right] + \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix}$$
$$\times Z^{-1} \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix}^T \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}$$

For appropriately dimensioned matrices $N_k$ and $T_k$ $(k = 1, 2, 3, 4)$, equations in (10) and (11) ensure that

$$2\left[x^T(t)N_1 + x^T(t-h(t))N_2 + q^T(t)N_3 + g^T(t)N_4\right] * [A_0x(t) + A_dx(t-h(t)) - q(t)] \equiv 0 \tag{20}$$

$$2\left[x^T(t)T_1 + x^T(t-h(t))T_2 + q^T(t)T_3 + g^T(t)T_4\right] * [Cx(t) + C_dx(t-h(t)) - g(t)] \equiv 0 \tag{21}$$

Combining $L_{v=0}V_i(i = 1, 2, 3, 4, 5)$ and adding the terms on the left-hand side of (20) and (21) to $L_{v=0}V$ allow us to express $L_{v=0}V$ as

$$L_{v=0}V \leq \eta^T(t)\Sigma\eta(t) - \int_{t-h(t)}^{t} \mathrm{trace}\left[g^T(s)Zg(s)\right]\mathrm{d}s$$
$$+ \left[\int_{t-h(t)}^{t} g(s)\mathrm{d}\beta(s)\right]^T Z \left[\int_{t-h(t)}^{t} g(s)\mathrm{d}\beta(s)\right]$$

where

$$\eta^T(t) = \left[x^T(t), x^T(t-h(t)), q^T(t), g^T(t)\right]$$
$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\ * & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} \\ * & * & \Sigma_{33} & \Sigma_{34} \\ * & * & * & \Sigma_{44} \end{bmatrix} + \begin{bmatrix} R_{13} \\ R_{23} \\ 0 \\ 0 \end{bmatrix} Z^{-1} \begin{bmatrix} R_{13} \\ R_{23} \\ 0 \\ 0 \end{bmatrix}^T$$

Since

$$E\left(\left[\int_{t-h(t)}^{t} g(s)\mathrm{d}\beta(s)\right]^T Z \left[\int_{t-h(t)}^{t} g(s)\mathrm{d}\beta(s)\right]\right)$$
$$= E\int_{t-h(t)}^{t} \mathrm{trace}\left[g^T(s)Zg(s)\right]\mathrm{d}s$$

it follows that

$$EL_{v=0}V(t) \leq E\eta^T(t)\Sigma\eta(t) \tag{22}$$

By Schur's complement, $\Sigma < 0$ is equivalent to LMI (8). From the proof of Theorem 1 Li, Guan, Peng, and Luo (2008), there exists a scalar $\alpha$ such that

$$\lim_{t\to\infty} \sup \frac{1}{t} \log E\|x(t)\|^2 \leq -\alpha$$

which implies that system (7) is exponentially stable in mean square. The proof of Theorem 3.1 is completed. □

**Remark 1:** Theorem 1 presents a new delay-dependent exponential stability criterion for stochastic systems by retaining $q(t)$ and $g(t)$ as state vectors to construct the Lyapunov–Krasovskii functional. The new criterion is less conservative than the results in existing the literature. However, by means of the idea of delay fractionising (Mou, Gao, Qiang, and Chen 2008), we could construct a new Lyapunov–Krasovskii functional and obtain some new results, which might help to improve the existing ones. Moreover, the conservatism reduction might increase as the delay fractionising becomes thinner, it deserves further research.

**Remark 2:** The matrices $N_2$, $T_2$, $R_{12}$, $R_{22}$ and $R_{23}$ in $\Sigma$ provide some extra freedom in the selection of the weighting matrices, which have the potential to yield less conservative results. when they are all zero, we obtain the following corollary.

**Corollary 3.2:** *Given scalars $\tau > 0$ and $\mu < 1$, system (7) is exponentially stable in mean square, if there exist symmetric positive definite matrices $P > 0$, $Q > 0$, $Z > 0$, $R = \begin{bmatrix} R_{11} & R_{13} \\ R_{13}^T & R_{33} \end{bmatrix} > 0$, and appropriately dimensioned matrices $N_k$ and $T_k(k = 1, 3, 4)$ such that the following LMI holds*:

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & R_{13} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} & 0 \\ * & * & \Pi_{33} & \Pi_{34} & 0 \\ * & * & * & \Pi_{44} & 0 \\ * & * & * & * & -Z \end{bmatrix} < 0 \tag{23}$$

*where*

$$\begin{aligned}
\Pi_{11} &= N_1A + A^TN_1^T + T_1C + C^TT_1^T \\
&\quad + Q + \tau R_{11} + R_{13} + R_{13}^T \\
\Pi_{12} &= N_1A_d + T_1C_d - R_{13} \\
\Pi_{13} &= P_1 - N_1 + A^TN_3^T + C^TT_3^T \\
\Pi_{14} &= A^TN_4^T - T_1 + C^TT_4^T \\
\Pi_{22} &= -(1-\mu)Q \\
\Pi_{23} &= A_d^TN_3^T + C_d^TT_3^T \\
\Pi_{24} &= A_d^TN_4^T + C_d^TT_4^T \\
\Pi_{33} &= -N_3 - N_3^T + \tau R_{33} \\
\Pi_{34} &= -N_4^T - T_3 \\
\Pi_{44} &= 2P - T_4^T - T_4 + \tau Z
\end{aligned}$$

### 3.2. *Delay-dependent robust exponential stability*

This method can easily be extended to provide an LMI-based delay-dependent robust exponentially mean-square stability condition for stochastic systems with structured uncertainties and time-varying delays.

**Theorem 3.3:** *Given scalar $\tau > 0$ and $\mu < 1$, and time-varying delay satisfying* (2) *and* (3), *system* (1) *is robustly exponentially stable in mean square if there exist symmetric positive definite matrices $P > 0$, $Q > 0$, $Z > 0$, $R_{ij}$ and appropriately dimensioned matrices $N_k$ and $T_k$ $(k = 1, 2, 3, 4)$ such that $R_{ij} = R_{ji}$ $(i, j = 1, 2, 3)$ and the following LMIs hold*:

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Sigma_{13} & \Sigma_{14} & R_{13} & N_1M & T_1M \\ * & \Omega_{22} & \Sigma_{23} & \Sigma_{24} & R_{23} & N_2M & T_2M \\ * & * & \Sigma_{33} & \Sigma_{34} & 0 & N_3M & T_3M \\ * & * & * & \Sigma_{44} & 0 & N_4M & T_4M \\ * & * & * & * & -Z & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{24}$$

*and*

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12}^T & R_{22} & R_{23} \\ R_{13}^T & R_{23}^T & R_{33} \end{bmatrix} > 0 \tag{25}$$

*where*

$$\begin{aligned}
\Omega_{11} &= \Sigma_{11} + E_0^TE_0 + E_2^TE_2 \\
\Omega_{12} &= \Sigma_{12} + E_0^TE_1 + E_2^TE_3 \\
\Omega_{22} &= \Sigma_{22} + E_1^TE_1 + E_3^TE_3
\end{aligned}$$

*and $\Sigma_{ij}(i, j = 1, 2, 3, 4)$ are defined in* (8).

**Proof:** Replacing $A$, $A_d$, $C$ and $C_d$ in (8) with $A + MF(t)E_0$, $A_d + MF(t)E_1$, $C + MF(t)E_2$ and $C_d + MF(t)E_3$, respectively, we find that (8) for system (1) is equivalent to the following condition:

$$\Sigma + \Phi F(t)\Psi + \Psi^TF^T(t)\Phi^T + \Gamma F(t)\Lambda + \Lambda^TF^T(t)\Gamma^T < 0$$

where

$$\begin{aligned}
\Phi^T &= \begin{bmatrix} M^TN_1^T & M^TN_2^T & M^TN_3^T & M^TN_4^T & 0 \end{bmatrix} \\
\Psi &= \begin{bmatrix} E_0 & E_1 & 0 & 0 & 0 \end{bmatrix} \\
\Gamma^T &= \begin{bmatrix} M^TT_1^T & M^TT_2^T & M^TT_3^T & M^TT_4^T & 0 \end{bmatrix} \\
\Lambda &= \begin{bmatrix} E_2 & E_3 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

By the Lemma in Xie (1996), a sufficient condition guaranteeing (8) for system (1) is that there exist positive scalars $\varepsilon_0$ and $\varepsilon_1$ such that

$$\Sigma + \varepsilon_0\Phi\Phi^T + \varepsilon_0^{-1}\Psi^T\Psi + \varepsilon_1\Gamma\Gamma^T + \varepsilon_1^{-1}\Lambda^T\Lambda < 0 \tag{26}$$

Setting $\varepsilon_0 = \varepsilon_1 = 1$, and applying Schur's complement shows that (26) is equivalent to (24). The proof of Theorem 3.3 is completed. □

Theorem 3.3 gives a delay- and rate-dependent robust criterion for a delay satisfying (2) and (3) . Note that a delay-dependent and rate-independent criterion for a delay satisfying (2) and (3) can be derived from Theorem 3.3 by choosing $Q = 0$ as follows.

**Corollary 3.4:** *Given scalar $\tau > 0$ and time-varying delay satisfying* (2), *system* (1) *is robustly exponentially stable in mean square if there exist symmetric positive definite matrices $P$, $Z$, $R_{ij}$ and appropriately dimensioned matrices $N_k$ and $T_k$ $(k = 1, 2, 3, 4)$ such that $R_{ij} = R_{ji}$ $(i, j = 1, 2, 3)$ and the following LMIs hold*:

$$\tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{12} & \Sigma_{13} & \Sigma_{14} & R_{13} & N_1M & T_1M \\ * & \tilde{\Omega}_{22} & \Sigma_{23} & \Sigma_{24} & R_{23} & N_2M & T_2M \\ * & * & \Sigma_{33} & \Sigma_{34} & 0 & N_3M & T_3M \\ * & * & * & \Sigma_{44} & 0 & N_4M & T_4M \\ * & * & * & * & -Z & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & *-I & \end{bmatrix} < 0 \tag{27}$$

*and*

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12}^T & R_{22} & R_{23} \\ R_{13}^T & R_{23}^T & R_{33} \end{bmatrix} > 0 \tag{28}$$

*where*

$$\begin{aligned}\tilde{\Omega}_{11} &= N_1A + A^TN_1^T + T_1C + C^TT_1^T + \tau R_{11} \\ &\quad + R_{13} + R_{13}^T + E_0^TE_0 + E_2^TE_2 \\ \tilde{\Omega}_{12} &= \Sigma_{12} + E_0^TE_1 + E_2^TE_3 \\ \tilde{\Omega}_{22} &= N_2A_d + A_d^TN_2^T + T_2C_d + C_d^TT_2^T + \tau R_{22} \\ &\quad - R_{23} - R_{23}^T + E_1^TE_1 + E_3^TE_3\end{aligned}$$

and $\Sigma_{ij}(i=1,2,3;\ j=3,4)$ *are defined in* (8).

In the case, when there is no stochastic uncertainty in the system (1), that is, $\beta(t)$ is assumed to be zero, Theorem 3.3 is specialised to the following corollary.

**Corollary 3.5:** *Given scalars* $\tau > 0$ *and* $\mu < 1$, *system* (1) *with* $\beta(t)=0$ *and time-varying delay satisfying* (2) *and* (3) *is asymptotically stable, if there exist symmetric positive definite matrices* $P > 0$, $Q \geq 0$, $R_{ij}$ $(i, j=1,2,3)$ *and appropriately dimensioned matrices* $N_k$ $(k=1,2,3,4)$ *such that* $R_{ij}=R_{ji}$ *and the following LMIs hold*:

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & N_1M \\ * & \Xi_{22} & \Xi_{23} & N_2M \\ * & * & \Xi_{33} & N_3M \\ * & * & * & -I \end{bmatrix} < 0 \qquad (29)$$

*and*

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12}^T & R_{22} & R_{23} \\ R_{13}^T & R_{23}^T & R_{33} \end{bmatrix} > 0 \qquad (30)$$

*where*

$$\begin{aligned}\Xi_{11} &= N_1A + A^TN_1^T + Q + \tau R_{11} + R_{13} + R_{13}^T + +E_0^TE_0 \\ \Xi_{12} &= N_1A_d + A^TN_2^T + \tau R_{12} - R_{13} + R_{23}^T + E_0^TE_1 \\ \Xi_{13} &= P - N_1 + A^TN_3^T \\ \Xi_{22} &= N_2A_d + A_d^TN_2^T - (1-\mu)Q + \tau R_{22} \\ &\quad - R_{23} - R_{23}^T + E_1^TE_1 \\ \Xi_{23} &= -N_2 + A_d^TN_3^T, \quad \Xi_{33} = -N_3 - N_3^T + \tau R_{33}\end{aligned}$$

### 3.3. *Delay-independent rate-dependent robust exponential stability*

If we set the matrices $Z$, $R_{ij}(i=1,2;\ j=1,2)$ and $R_{33}$ to zero, then we can obtain a delay-independent and rate-dependent criterion. In this case, Theorem 3.3 becomes the following corollary.

**Corollary 3.6:** *Given scalar* $\mu < 1$ *and time-varying delay satisfying* (3), *system* (1) *is robustly exponentially stable in mean square, if there exist symmetric positive definite matrices* $P$, $Q$, $Z$, $R_{i3}(i=1,2)$ *and appropriately dimensioned matrices* $N_k$ *and* $T_k$ $(k=1,2,3,4)$ *such that* $R_{i3}=R_{3i}$ $(i=1,2)$ *and the following LMI holds*:

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & N_1M & T_1M \\ * & \Phi_{22} & \Phi_{23} & \Phi_{24} & N_2M & T_2M \\ * & * & \Phi_{33} & \Phi_{34} & N_3M & T_3M \\ * & * & * & \Phi_{44} & N_4M & T_4M \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \qquad (31)$$

*where*

$$\begin{aligned}\Phi_{11} &= N_1A + A^TN_1^T + T_1C + C^TT_1^T + Q + R_{13} \\ &\quad + R_{13}^T + E_0^TE_0 + E_2^TE_2 \\ \Phi_{12} &= N_1A_d + A^TN_2^T + T_1C_d + C^TT_2^T - R_{13} \\ &\quad + R_{23} + E_0^TE_1 + E_2^TE_3 \\ \Phi_{13} &= P - N_1 + A^TN_3^T + C^TT_3^T \\ \Phi_{14} &= A^TN_4^T - T_1 + C^TT_4^T \\ \Phi_{22} &= N_2A_d + A_d^TN_2^T + T_2C_d - (1-\mu)Q + C_d^TT_2^T \\ &\quad - R_{23} - R_{23}^T + E_1^TE_1 + E_3^TE_3 \\ \Phi_{23} &= -N_2 + A_d^TN_3^T + C_d^TT_3^T \\ \Phi_{24} &= A_d^TN_4^T - T_2 + C_d^TT_4^T \\ \Phi_{33} &= -N_3 - N_3^T \\ \Phi_{34} &= -N_4^T - T_3 \\ \Phi_{44} &= 2P - T_4^T - T_4\end{aligned}$$

**Remark 3:** When $\mu=0$, the delay is time-invariant, from Theorem 3.3 and Corollary 3.6, we can easily obtain the delay-dependent and delay-independent stability criteria for uncertain linear stochastic systems with time-invariant state delays, respectively.

## 4. Numerical simulation

In this section, some simulation examples are provided to illustrate the advantage of our results by comparing it with recently reported results on delay-dependent exponential stability of uncertain stochastic systems.

**Example 1:** Consider the uncertain system $\Sigma_0$ with the following parameters (Lu and Tsai 2003):

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1.4 & 0 \\ -0.8 & -1.5 \end{bmatrix}, \quad C = C_d = 0$$

$$M = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad E_2 = E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_0 = E_1 = 0$$

The method of Lu and Tsai (2003) cannot handle the case $0.45 \leq \mu < 1$, our method finds the upper bound

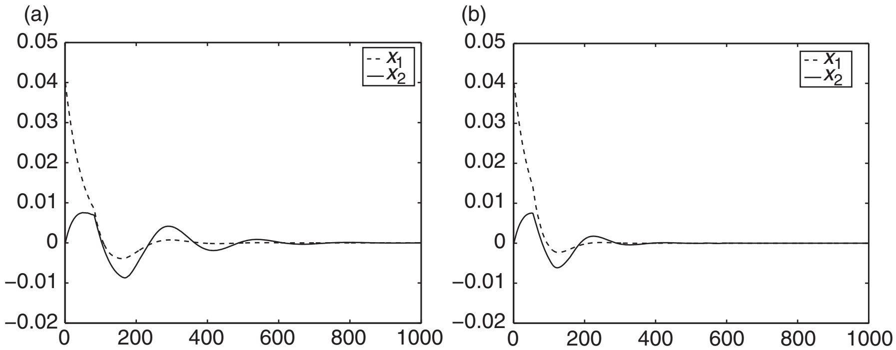


Figure 1. The response of state $x_1$ (dashed) and $x_2$ (solid). (a) $\mu = 0.5$, $\tau_{\max} = 0.8111$; (b) $\mu = 1$, $\tau_{\max} = 0.5360$.



on the time-delay in the case of $0.45 \le \mu < 1$, such as applying Theorem 2 of Lu and Tsai (2003), when $\mu = 0.5$ and $\mu = 0.9$, system $\Sigma_0$ is exponentially stable in mean square for any delay $\tau$ satisfying $0 \le \tau \le 2.5669$ and $0 \le \tau \le 0.6026$, respectively. It is clear that our method is better than the method of Lu and Tsai (2003).

**Example 2:** Consider the uncertain stochastic delay system $\Sigma_1$ (Mao 1996; Example 5.1):

$$\begin{aligned} \mathrm{d}x(t) = {} & [(A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t-h)]\mathrm{d}t \\ & + g(t, x(t), x(t-h))\mathrm{d}\beta(t) \end{aligned}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -0.5 & -1 \end{bmatrix}$$

$$\|\Delta A(t)\| \le 0.1, \quad \|\Delta A_d(t)\| \le 0.1$$

$$\mathrm{trace}[g^T(t,x,y)\,g(t,x,y)] \le 0.1\|x\|^2 + 0.1\|y\|^2$$

The above system is the form of system (1) with

$$M = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad C = C_d = \begin{bmatrix} \sqrt{0.1} & 0 \\ 0 & \sqrt{0.1} \end{bmatrix},$$
$$E_0 = E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The upper bounds on the time delay obtained in Mao (1996), Yue and Won (2001) and Chen et al. (2005) and are 0.8635, 1.1997 and 0.175, respectively. However, by our method the system $\Sigma_1$ is exponentially stable in mean square for any delay $\tau$ satisfying $0 \le \tau \le 4.2940$, which is better than the values in Mao (1996), Yue and Won (2001) and Chen et al. (2005).

**Example 3:** Consider the robust stability of the uncertain stochastic delay system $\Sigma_2$ with the following parameters:

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -0.8 & -1.5 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.1 & -0.1 \\ 0 & 0.1 \end{bmatrix}, \quad C_d = \begin{bmatrix} -0.3 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$M = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad E_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i = 0, \ldots, 3$$

When $\mu = 0$, the delay is time-invariant, by Theorem 3.3, the upper bound on the time-delay is found to be 1.7517. The upper bound on the time-delay for any $\mu \ne 0$ can be obtained when $\mu = 0.1$, $\mu = 0.5$ and $\mu = 1$; based on Theorem 3.3, system $\Sigma_2$ is exponentially stable in mean square for any delay $\tau$ satisfying $0 \le \tau \le 1.4367$, $0 \le \tau \le 0.8111$ and $0 \le \tau \le 0.5360$, respectively. Set the initial conditions as $x(0) = [0.04;\ 0]^T$, Figures 1 and 2 show the state response of system, as $\mu = 0.5$, $\tau_{\max} = 0.8111$, $\mu = 1$, $\tau_{\max} = 0.5360$ and as $\mu$ is any value, $\tau_{\max} = 0.3541$, respectively. It shows that our method is effective and feasible.

In addition, Theorem 3.3 is also applicable to cases in which the derivative of the time delay is greater than 1. For example, if we let $h(t) = 2\cos^2 t$, then $\tau = \mu = 2$. The simulation results show that the system is exponentially stable in mean square, however, compared with the case in which $\mu \le 1$, the stability effect of system is poor. Figure 2(b) displays the state response of system, as $\mu = 2$, $\tau = 2$.

## 5. Conclusion

This article presents some new stability criteria for a class of uncertain stochastic systems with time-varying delays. New techniques are developed to make the criteria less conservative. First, based on the equivalent descriptor form of the original system, a new type of Lyapunov–Krasovskii functional is constructed.

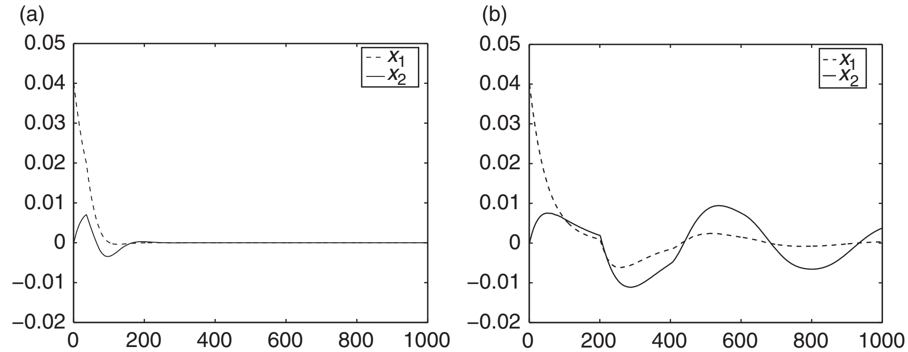


Figure 2. The response of state $x_1$ (dashed) and $x_2$ (solid). (a) any $\mu$, $\tau_{\max} = 0.3541$; (b) as $\mu = 2$, $\tau = 2$.

And then, in order to avoid some unnecessary constraints during the deducing process, some free weighing matrices are introduced. Finally, numerical examples demonstrate that the results obtained are very effective and are an improvement over existing results.


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