Unbiased $H_\infty$ filtering for stochastic systems with data packet losses

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**Abstract.** This paper presents the unbiased $H_\infty$ filter design for stochastic systems with data packet losses. By constructing unbiased filter, the complexity and computational burden of the real-time filtering process are reduced greatly. Delay-dependent sufficient conditions for stochastic system with data packet losses are proposed in terms of linear matrix inequalities (LMIs). Numerical example demonstrates the proposed approaches are effective.

**Introduction**

Recently, the $H_\infty$ filtering and control problems of stochastic system have gained extensive attentions and achieved comprehensive applications in many fields [1,2,3,4,5,6,7]. But, the order of filter error system in [1,2,3,4,5,6,7] is the twice as great as the original system, in the real time filtering process, which directly result in the complexity and computational burden. In addition, it is well known that data packet dropouts are commonly encountered in control field and it usually cause instability and poor performance of signals. Unfortunately, there are very few corresponding works dealing with the filter design problems for systems with data packet losses. These motivate us to investigate the unbiased filter design problem for stochastic systems with data packet losses.

This paper discusses the $H_\infty$ filtering problem of stochastic system with data packet losses. By constructing an unbiased filter such that the order of filter error dynamic system is the same as the original system, in contrast with the normal $H_\infty$ filtering approaches, which result in the order of filter error dynamic system is the twice as great as the original system. Therefore, the unbiased filter reduce the complexity and computational burden of the real-time filtering process. Based on LMI algorithm, delay-dependent sufficient conditions for stochastic system with data packet dropouts are proposed and the minimum $H_\infty$ performance lever $\gamma$ is obtained. Numerical simulation example shows the results are effective.

**Unbiased $H_\infty$ Filtering for Stochastic Systems Based on Data Packet Losses**

consider the following stochastic system

$$
dx(t) = [A_x x(t) + B_x v(t)] dt + C_x x(t) d\beta(t)$$

$$x(t) = \phi(t), t \in [-\tau, 0]$$

$$z(t) = Lx(t)$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state vector, $\phi(t)$ is a continuous vector-valued initial function, $z(t) \in \mathbb{R}^p$ is the state combination to be estimated, $v(t) \in L^2_c([0, \infty); \mathbb{R}^r)$ stands for the exogenous disturbance signal. $A_x, B_x, C_x$ and $L$ are known constant matrices with appropriate dimensions. Where the stochastic variables $\beta(t)$ is zero-mean real scalar Wiener processes.

Due to the data packet dropouts lead to the measurement signal contains the uncertainty. So we construct a new measurement model in the section and the model with uncertainty can be described as
\[ y_{\alpha} = \theta(t)y = \theta(t)A_{\alpha}x(t) + \theta(t)A_\omega x(t-h(t)) + \theta(t)B_\nu v(t). \]  
(2)

where \( h(t) = t - t_\omega + \tau_\omega + d(k)h \) and represents there are \( d(k) \) packets dropout at time \( t_\omega \). The delay part \( h(t) \) may vary with time \( t \) and it is assumed that \( 0 \leq \tau_\omega \leq h(t) = t - t_\omega + \tau_\omega + d(k)h \leq h + \tau_\omega \leq \tau_\omega, \ t \in [t_\omega, t_{\omega+1}) \) and \( \dot{h}(t) \leq \mu \). Assuming \( \theta(t) \) is a stochastic matrix of the form

\[ \theta(t) = \text{diag}(\theta_1, \ldots, \theta_m). \]  
(3)

where \( \theta(t), (i = 1, \ldots, m) \) taking the values of 0 and 1 is independent stochastic variable one another and satisfies the following probability

\[ \text{Prob}\{\theta_i(t) = 0\} = E[\theta_i(t)] = \rho_i, i = 1, \ldots, m, \text{Prob}\{\theta_i(t) = 1\} = E[1 - \theta_i(t)] = 1 - \rho_i. \]  
(4)

\( \theta(t) = 0 \) represents the case that data packet losses occur, while \( \theta(t) = 1 \) represents the case that no data packets are dropped, and \( \theta_i(t) = 0 \) means the \( i \)-th output signal dropout, \( \theta_i(t) = 1 \) means the \( i \)-th output signal is transmitted normally. \( \rho_i \in [0, 1] \) reflects the occurrence probability of the \( i \)-th output signal losses. Assuming \( \theta_i(t), i = 1, \ldots, m \) are independent stochastic variable one another and satisfy

\[ E(\theta_i(t)\theta_j(t)) = E(\theta_i(t))E(\theta_j(t)), i \neq j, j = 1, \ldots, m. \]  
(5)

We set up the following unbiased filtering system

\[ d\hat{x}(t) = [A_\nu \hat{x}(t) - \theta(t)B_\nu A_\omega \hat{x}(t-h(t))]dt + B_\nu d\gamma(t) + C_\nu \hat{x}(t)dB(t) \]
\[ \hat{x}(0) = 0 \]
\[ \hat{z}(t) = C_\nu \hat{x}(t). \]  
(6)

where \( \hat{x}(t) \in R^r \) is the filter state, \( \hat{z}(t) \in R^r \), the constant matrices \( A_\nu, B_\nu, C_\nu \) are filter parameters to be designed. Denote \( x_i(t) = x(t) - \hat{x}(t) \) and \( z_i(t) = z(t) - \hat{z}(t) \) then, we obtain the following filtering error system:

\[ dx_i(t) = [A_\omega x_i(t) + (A_\nu - \theta(t)B_\nu A_\omega)x_i(t) - \theta(t)B_\nu A_\omega x_i(t-h(t)) + (B_\nu - \theta(t)B_\nu B_\nu)v(t)]dt + C_\nu x_i(t)dB(t) \]
\[ z_i(t) = C_\nu x_i(t) + (L - C_\nu)x_i(t). \]  
(7)

Unbiasedness of the filter requires that the estimation error system be independent of the system state \( x \), that is the following conditions are satisfied:

\[ A_\nu = A_\nu - \theta(t)B_\nu A_\nu, C_\nu = L. \]  
(8)

then we obtain an unbiased filter and the filtering error system can be described as

\[ dx_i(t) = [A_\nu x_i(t) + A_\nu x_i(t-h(t)) + B_\nu v(t)]dt + C_\nu x_i(t)dB(t) \]
\[ z_i(t) = C_\nu x_i(t). \]  
(9)

where

\[ A_\nu = A_\nu - \theta(t)B_\nu A_\nu, A_\omega = -\theta(t)B_\nu A_\nu, B_\nu = B_\nu - \theta(t)B_\nu B_\nu. \]  
(10)

Based on the above discussion, the problem to be addressed in this paper is stated as follows. Design an unbiased filter of the form (6), such that the filtering error system (9) with \( v(t) \equiv 0 \) is exponentially stable in mean square, and the \( H_\infty \) performance

\[ \int_0^\infty \| z_i(t) \|^2 dt \leq \gamma \int_0^\infty \| v(t) \|^2 dt \]  

is satisfied for zero initial conditions and all nonzero \( v(t) \in L_2((0, \infty), R^r) \) with a prescribed \( \gamma > 0 \).
Theorem 1: Consider the system (1) and (2). Given scalars $\tau, \tau > 0$ and $\mu$ for a prescribed $\gamma > 0$, the system (9) with $v(t) = 0$ is exponentially stable in mean square, and the $H_\infty$ performance $\hat{J} < 0$ holds for all nonzero $v(t), v(t) \in L_2((0, \infty), R^r)$, if there exist symmetric positive definite matrix $P > 0, Q > 0, R > 0$, scalars $\gamma > 0$, and appropriately dimensioned matrices $S, X, T$ and $Y(i = 1, 2, 3, 4, 5, 6)$ satisfying the following LMI

$$
\begin{bmatrix}
\Omega & \tilde{V} & K & \tilde{F}(\rho) \\
* & -(\frac{1}{\gamma^2} + Z) & 0 & 0 \\
* & * & -I & 0 \\
* & * & * & -\gamma^2 I \\
\end{bmatrix} < 0.
$$

If a solution of the LMI exists then the filter that guarantees the estimation error level of $\hat{e}$ is given by (6) with $A_e = A_0 - S^+X\rho A_1, B_e = S^+X$ and $C_e = L$.

Where

$$
\Omega = 
\begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16} \\
\Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & \Omega_{26} \\
\Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} & \Omega_{35} & \Omega_{36} \\
\Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} & \Omega_{45} & \Omega_{46} \\
\Omega_{51} & \Omega_{52} & \Omega_{53} & \Omega_{54} & \Omega_{55} & \Omega_{56} \\
\Omega_{61} & \Omega_{62} & \Omega_{63} & \Omega_{64} & \Omega_{65} & \Omega_{66} \\
\end{bmatrix}, \tilde{F}(\rho) = 
\begin{bmatrix}
h(SB_0 - X\rho B_1) \\
h(SB_0 - X\rho B_1) \\
h(SB_0 - X\rho B_1) \\
h(SB_0 - X\rho B_1) \\
h(SB_0 - X\rho B_1) \\
h(SB_0 - X\rho B_1) \\
\end{bmatrix}, K = 
\begin{bmatrix}
L^r \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}.
$$

$$
\rho = \text{diag}(\rho, \rho, \cdots, \rho_n)
$$

and

$$
\Omega_{11} = hSA + hA'_sS' - hX\rho A - hA'_s\rho'X' + T\text{diag}(\rho) + C_iT_i + Q + Q_s + Q, \\
\Omega_{12} = -hX\rho A_1 + hA'_sS' - hA'_s\rho'X' + C_iT_i, \Omega_{21} = hA'_sS' - hA'_s\rho'X' + C_iT_i + Y_i, \\
\Omega_{13} = hA'_sS' - hA'_s\rho'X' + C_iT_i - Y_i, \Omega_{22} = P - hS + hA'_sS' - hA'_s\rho'X' + C_iT_i, \\
\Omega_{31} = hA'_sS' - hA'_s\rho'X' - T_i + C_iT_i, \Omega_{32} = -hX\rho A_1 - hA'_s\rho'X' - (1 - \mu)Q_s, \\
\Omega_{33} = -hA'_s\rho'X' + Y_s, \Omega_{41} = -hA'_s\rho'X' - Y_s, \Omega_{42} = -hS - hA'_s\rho'X', \Omega_{51} = -T_i - hA'_s\rho'X', \\
\Omega_{52} = -Q_s + Y_s + Y_s', \Omega_{53} = -hS + Y_s', \Omega_{54} = -T_s + Y_s', \\
\Omega_{55} = -Q_s - Y_s - Y_s', \Omega_{61} = -hS - hS' + (\tau - \tau_i)R, \\
\Omega_{62} = -hS' - T_s, \Omega_{63} = P - T_s + (\tau - \tau_i)Z.
$$

Proof: For convenience, set

$$
q(t) = Ax(t) + A_1x(t - h(t)) + Bv(t) \\
g(t) = Cx(t)
$$

then system (9) becomes the following descriptor stochastic systems

$$
dx(t) = q(t)dt + g(t)d\beta(t) \\
z = Cx(t).
$$

Choose a Lyapunov-Krasovskii functional for system (14) to be

$$
V(t) = \sum_{i=1}^{r} V_i(t)
$$

in which
\[ V_i(t) = x_i(t)^T P x_i(t), V_z(t) = \sum_{i=1}^{\infty} \int_{-T}^{0} x_i(s)^T Q x_i(s) ds, V_j(t) = \int_{-T}^{0} x_j(s)^T Q x_j(s) ds \]

\[ V_j(t) = \int_{-T}^{0} g^j(s) R q_j(s) ds d\theta, V_i(t) = \int_{-T}^{0} \text{trace}[g^i(s)Z(g(s))] ds d\theta. \]

where \( P, Q, Q_i, Z, R \) are symmetric positive definite matrices with appropriate dimensions. \( L \) be the weak infinitesimal operator of (15), the Newton-Leibniz formula provides

\[
x_i(t - \tau_i) - x_i(t - \tau_i) = \int_{-\tau_i}^{0} g(s) ds + \int_{-\tau_i}^{0} g(s) d\beta(s).
\]

For appropriately dimensioned matrices \( N_j, T \) and \( Y_j \) \( (j = 1, 2, 3, 4, 5, 6) \), equations in (13) ensure that

\[ 2[x_i(t) T_i + x_i(t - h(t)) T_i + x_i(t - \tau_i) T_i + q_i(t) T_i + g_i(t) T_i] = 0. \]

\[ 2[x_i(t) Y_i + x_i(t - h(t)) Y_i + x_i(t - \tau_i) Y_i + q_i(t) Y_i + g_i(t) Y_i] = 0. \]

\[ x_i(t - \tau_i) - x_i(t - \tau_i) - \chi - \varsigma = 0. \]

\[ (\tau_i - \tau_i) \Lambda - \int_{-\tau_i}^{0} \Lambda ds \geq 0. \]

where \( \chi = \int_{-\tau_i}^{0} g_i(s) ds, \varsigma = \int_{-\tau_i}^{0} g_i(s) d\beta(s), \Lambda = \eta_i(t) \tilde{R}^T \tilde{Y} \eta_i(t) \). Adding the terms on the left of (17)-(20) to \( L_{\text{ss}} V \), by Lemma [8], for any matrix \( Z > 0, -2\eta_i(t) \tilde{Y} \varsigma \leq \eta_i(t) \tilde{Y} Z \tilde{Y}^T \eta_i(t) + \varsigma^T Z \varsigma \). Then \( L_{\text{ss}} V \) can be expressed as

\[ L_{\text{ss}} V \leq \eta_i(t) \Xi \eta_i(t) - \int_{-\tau_i}^{0} \Theta R \Theta^T ds - \int_{-\tau_i}^{0} \text{trace}[g_i(s)Zg_i(s)] ds + \varsigma^T Z \varsigma. \]

where

\[ \Xi = \Omega + \tilde{Y} \left( \frac{R}{\tau_i - \tau_i} + Z \right)^{-1} \tilde{Y}^T, \Theta = \eta_i(t) \tilde{Y} + q_i(s) R. \]

\[ \eta_i(t) = [x_i(t) x_i(t - h(t)) x_i(t - \tau_i) q_i(t) g_i(t)]^T, \tilde{Y}^T = [Y_i Y_i Y_i Y_i Y_i Y_i]^T. \]

where \( \Omega \) is defined in (12).

Since \( E(\varsigma^T Z \varsigma) = E \int_{-\tau_i}^{0} \text{trace}[g_i(s)Zg_i(s)] ds \). It follows that \( EL_{\text{ss}} V(t) \leq E\eta_i(t) \Xi \eta_i(t) \). By [9] a scalar \( \alpha_0 \) exists, such that \( \limsup_{t \to \infty} \frac{1}{t} \log E\|x_i(t)\|^2 \leq -\alpha_0 \). Which implies system \( (9) \) is exponentially stable in mean square. Next, we prove \( \dot{J} < 0 \) for all nonzero \( v(t) \in L^r((0, +\infty), R^r) \) with \( x_i(t, 0) = 0 \).

Note that for any \( T > 0 \)

\[ \dot{J}(T) = E \int_{0}^{T} \|z_i(t)\|^{\gamma} \|v(t)\|^{\gamma} dt \leq E \int_{0}^{T} \|z_i(t)\|^{\gamma} \|v(t)\|^{\gamma} + L V(z_i(t), t) dt \]

\[ = E \int_{0}^{T} \|v(t)\|^T \Psi \|v(t)\|^T dt - E \int_{0}^{T} \Theta \Theta^T ds dt. \]

Where

\[ \Psi = \left[ \tilde{\Xi} \ F \right], \tilde{\Xi} = \Omega + \tilde{Y} \left( \frac{R}{\tau_i - \tau_i} + Z \right)^{-1} \tilde{Y}^T + K^T K. \]

Therefore, if \( \Psi < 0 \) then \( \dot{J}(T) < -\lambda_{\text{ss}} (-\Psi) E \|v(t)\|^T dt < 0 \). for any nonzero \( v(t) \in L^r((0, +\infty), R^r) \)}
which yield \( \dot{J}(t) < -\lambda_{\text{ss}}(-\Psi) E [\|v(t)\|^2] dt < 0 \). Substituting (10) and (22) into (25), and letting \( SB_r = X \) then \( \Psi < 0 \) is equivalent to (12). From our assumption, \( B_r = S^r X \) and a \( H_\infty \) filter is constructed as in the form of (6). The proof of Theorem 1 is completed.

**Numerical Simulation**

Consider a CH-47 double helix helicopter who makes standard 40 kt (1kt=1.85 km/h) for level flight and its linear model is as follows:

\[
\dot{x} = A_x x(t) + B_x v(t) + C_x x(t) \dot{\beta}(t) \\
z = Lx(t)
\]

\[
x(t) = \varphi(t), t \in [-\tau, 0].
\]

where

\[
A_x = \begin{bmatrix} -9 & 1 \\ -2 & -10 \end{bmatrix}, B_x = \begin{bmatrix} -0.3 \\ -0.42 \end{bmatrix}, C_x = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, L = \begin{bmatrix} 1 & -1 \end{bmatrix}.
\]

(27)

state \( x \) denotes flight altitude; \( x_1 \) means forward velocity; \( v(t) \) is an exogenous disturbance signal, \( \dot{\beta}(t) \) is a white noise. Assume the measurement output experiences the network transmission delay and results in the data packet losses. The output can be given as

\[
y = \theta(t)[A_x x(t) + A_{x_v} x(t - h(t)) + B_x v(t)].
\]

(28)

\[
A = \begin{bmatrix} -8 & 1 \\ 0.5 & -15 \end{bmatrix}, A_{x_v} = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & -0.4 \end{bmatrix}, B = \begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix}.
\]

(29)

where stochastic matrix \( \theta(t) \) and the probability of output signal losses are as follows, respectively

\[
\theta(t) = \begin{bmatrix} \theta_1(t) & 0 \\ 0 & \theta_2(t) \end{bmatrix}, \rho_1 = 0.3, \rho_2 = 0.
\]

(30)

Let \( h = 0.1, h_i = 0.3, h = 0.1, h_i = 0.5, h_i = 0.012, h_i = 0.0513 \). By Theorem 1, \( \gamma_{\text{ss}} = 1.5 \times 10^{-4} \) (for \( t_i = 0, t = 0.1, \mu = 0.3 \)). When we take \( \gamma = 0.5 \), the corresponding filter matrices are

\[
A_f = \begin{bmatrix} -7.1648 & 0.7706 \\ -2.7471 & -9.9066 \end{bmatrix}, B_f = \begin{bmatrix} 0.7647 & 0 \\ -0.3113 & 0 \end{bmatrix}, C_f = \begin{bmatrix} 1 & -1 \end{bmatrix}.
\]

(31)

Figure 1 gives the time responses of states \( x_1 \) and \( x_2 \) with \( v(t) = 0 \). The trajectories of real states \( x_1 \), \( x_2 \) and filter states \( \hat{x} \), \( \hat{x} \), for the filter (31) are displayed in Figure 2. The simulation results demonstrate that the prescribed performance requirements on the filtering process are guaranteed by the Theorem 1.

![Figure 1. The state responses of the filtering error system](image1.png)

(a)  

![Figure 2. The time responses of state x and x̂ for the filter (31)](image2.png)
Summary

LMI-based technique, delay-dependent sufficient conditions for stochastic system with data packet losses are presented. By constructing unbiased filter, the complexity and computational burden are reduced greatly in the process of seeking the filter parameters. Numerical example has clearly indicated our design is effective.

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