UNIVERSITÉ DU LUXEMBOURG

Ph.D.-FSTC-8-2009
The Faculty of Sciences, Technology and Communication

# DISSERTATION

Defense held on 14/5/2009 in Luxembourg

in candidature for the degree of

## DOCTOR OF THE UNIVERSITY OF LUXEMBOURG

## IN MATHEMATICS

by

Aron GOHR
Born on 9th of June 1979 in Munich

# ON NONCOMMUTATIVE DEFORMATIONS, COHOMOLOGY OF COLOR-COMMUTATIVE ALGEBRAS AND FORMAL SMOOTHNESS

## Dissertation defense committee

Dr. Saïd Benayadi,
*Maître de conferences, Université de Metz*

Dr. Martin Bordemann, Chairman,
*Professor, Université de Haute Alsace*

Dr. Martin Olbrich, Deputy Chairman,
*Professor, Université du Luxembourg*

Dr. Norbert Poncin,
*Professor, Université du Luxembourg*

Dr. Martin Schlichenmaier, dissertation supervisor,
*Professor, Université du Luxembourg*

# Contents

# Introduction

## 0.1 Notations

First let us fix some notations and conventions: $k$ will by default denote a commutative ring, $K$ a field. Rings, algebras and modules are always supposed unital. For a module $M$ over a ring $R$ with unit 1 we mean by this that $1.m = m$ for all $m \in M$ and in the bimodule case also $m.1 = m$ for all $m \in M$.
Tensor products of algebras are to be understood as tensor products over the base ring.
An algebra $A$ is if nothing else is specified an associative algebra over a commutative base ring $k$.
With $k$ a commutative ring and $X$ a set of variables, we denote by $k < X >$ the ring of non-commutative polynomials in the variables contained in $X$.
$G$ will be a group, in most cases abelian. The homogeneous component of degree $g \in G$ of a $G$-graded object $M$ will be denoted by $M_g$ and will include the zero element.
We will indicate it if at some point we choose not to abide by the conventions laid down here.

## 0.2 Background and motivation

The main subject of this thesis are homological questions related to mildly noncommutative algebras, with applications to the deformation theory and algebraic properties of these objects. To put what we will do on the following pages in perspective, we will explain first our motivations for studying these topics.
The first part of our motivation can be explained by recalling some facts from basic algebraic geometry. Our exposition will be very brief and very incomplete and we refer to the first chapter of (Hartshorne [32]) for a more appropriate treatment of the matters we are going to recall.
The basic objects of study in algebraic geometry were originally *algebraic sets* in affine space, hence solution sets of algebraic equations. In other words, given an algebraically closed field $K$ and the multivariate polynomial ring $K[X] := K[X_1, X_2, X_3, \dots, X_n]$ of $K$ in some finite number of variables and a set $S \subseteq K[X]$, one associates a zero set

$$Z(S) := \{x \in K^n : f(x) = 0 \text{ for all } f \in S\}.$$

On the other hand, one can also take a subset $Y \subset K^n$ and consider the vanishing ideal

$$I(Y) := \{f \in K[X] : f(x) = 0 \text{ for all } x \in Y\}.$$

It is not hard to verify that $I(Y)$ is always an ideal and it turns out that the two maps so induced, one going from sets of points to ideals of polynomials and one going from sets of polynomials to algebraic sets, are in a sense near inverses to each other. Indeed, one can show that

$$I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$$

for any ideal $\mathfrak{a} \subseteq K[X]$. Recall that for $A$ a commutative ring and $\mathfrak{a} \subseteq A$ an ideal in $A$, the *radical* of $\mathfrak{a}$ is defined by

$$\sqrt{\mathfrak{a}} := \{a \in A : \text{ there is } n \in \mathbb{N} \text{ such that } a^n \in \mathfrak{a}\},$$

which means that $\sqrt{\mathfrak{a}}$ is just the preimage of the ideal of nilpotent elements in $A/\mathfrak{a}$ under the projection $A \to A/\mathfrak{a}$.
In the other direction, $Z(I(S))$ becomes the smallest algebraic set containing $S$. Hence, we get a one-to-one correspondence between algebraic subsets of $n$-dimensional affine space over an algebraically closed field $K$ and radical ideals in $K[X]$. This sort of correspondence can be rediscovered in many other guises, for instance algebraic varieties, i.e. algebraic sets that cannot written as a union of strict algebraic subsets of themselves, correspond under this notion to prime ideals in $K[X]$, the set of polynomial functions on an algebraic set $Y$ corresponds to the factor ring $K[X]/I(Y)$, which is called the *coordinate ring* of the set $Y$. Under this scheme, subvarieties correspond to prime ideals, points correspond to maximal ideals of the coordinate ring of an algebraic variety and so on.
Using more sophisticated algebraic technology - namely, sheaves of commutative rings - similar correspondences can also be constructed for different kinds of geometric objects, such as differentiable manifolds. Thus, in a massive leap of oversimplification, one might arrive at the following philosophy:

*To study a space, study the commutative algebra of 'functions' compatible with that space.*

And indeed one finds that many interesting properties of a geometric object can be recovered from their associated function algebras. One such property which especially caught our attention is that of *smoothness*. For $K$ an algebraically closed field and $Y \subseteq K^n$ an algebraic variety and $A = K[X]/I(Y)$ its coordinate ring, one can set the following definition:

**Definition of smoothness for algebraic varieties** *$Y$ is called smooth if for every point $x \in Y$ localization of $A$ at the maximal ideal $\mathfrak{m}_x$ associated to the point $x$ yields a regular local ring, i.e. $A_{\mathfrak{m}_x}$ should be regular local for any $x \in Y$.*

It can be shown (Hartshorne [32], Ch. 1, Theorem 5.1) that this criterion corresponds well to a more intuitive definition through the rank of the jacobian matrix in $x$ given by formal derivatives of some system of generators of $I(Y)$. Note that this definition is completely algebraic. Also, it makes sense in principle for any commutative ring but not for any ring in general, because it uses localization.
A series of generalizations together with the dropping of finiteness conditions leads in (Cuntz, Quillen [17]) to the following

**Definition of formal smoothness** Let $K$ be a field and let $A$ be a $K$-algebra. Then $A$ is called *formally smooth* if for any $K$-algebra $B$ and any two-sided nilpotent ideal $I \subseteq B$ and for any $K$-algebra homomorphism $\varphi : A \to B/I$ a lifting exists to a morphism of $K$-algebras $\tilde{\varphi} : A \to B$.

This definition recovers the previous one if we restrict everything to commutative algebras of finite type over $A$, compare (Appendix E to Loday [38] by M. Ronco). Due to this massive extension of the class of 'test-algebras' $B$ compared to the commutative situation, the class of formally smooth algebras is very restricted, an equivalent criterion to the above definition being given by the vanishing of second Hochschild cohomology with coefficients in any $A$-bimodule $M$, see (Cuntz, Quillen [17], [18]), (Kontsevich, Rosenberg [35]). For commutative algebras, smoothness can analogously be characterized by vanishing of appropriate second commutative algebra cohomologies (see Appendix E of [38]) or by finiteness of Hochschild cohomology dimension as in (Avramov, Vigue-Poirrier [4]), or (Avramov, Iyengar [3]) for a stronger statement. One of our original motivations for developing a cohomology theory for color-commutative algebras was therefore to study the concept of smoothness for a category of algebras much more commutative than arbitrary associative algebras, but not quite as well-behaved as commutative algebras. We believe that the cohomology theory we develop is suitable for a cohomological characterization of formal smoothness in the color-commutative category if one limits oneself to the case of color-commutative algebras over fields.
Another motivation came from formal deformation theory. Suppose here that $k$ is a commutative ring and that $A$ is a $k$-algebra. Then formal deformation theory studies the possibilities to endow the power series ring $A[[X]]$ in one variable over $A$ with a nonstandard associative product $\star : A[[X]] \times A[[X]] \to A[[X]]$ which is derived from $k[[X]]$-linear continuation of a suitable formal infinite sum

$$a \star b := \sum_{i \geq 0} X^i \mu_i(a, b)$$

where $\mu_0$ is the original multiplication and where the $\mu_i$ are $k$-bilinear maps from $A \times A$ to $A$. The subject of formal deformation theory is the classification and construction of such deformed products. In the classification of formal deformations of associative algebras of this kind, the second and third Hochschild cohomologies of $A$ with coefficients in $A$ play a central role, see for instance (Cattaneo, Indelicato [13]) and the textbook (Waldmann [55]) for expositions of the topic. The deformation theory of associative algebras was initiated by Gerstenhaber in [27], and since then much more research has been devoted to it than we can hope to summarize. Interest in this area is due to a large extent to connections to the problem of quantizing classical physical theories, see (Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer [6], [7]). See also the textbook (Waldmann [55]) for a pedagogical introduction.
In classical formal deformation theory, the deformed version of the algebra $A$ will unlike the ordinary power series ring $A[[X]]$ also for $A$ a commutative algebra in general be noncommutative. However, the formal deformation parameter $X$ will still always commute with all elements of the original algebra, at least when the deformation keeps the unit of $A$. We got interested in the question what happens if this condition is relaxed. Again, color-commutative algebras provided a framework where an exploration of this topic was possible. We essentially do this

in chapter three of this thesis.
It is well-known that the theory of *commutative* formal deformations of commutative algebras is controlled by Harrison cohomology, see e.g. (Fronsdal [23], [24]) and (Fronsdal, Kontsevich [25]) for some recent work on commutative deformations of (singular) commutative algebras. As an application of color-commutative Harrison cohomology, we prove also that it plays the same role for color-commutative algebras.

## 0.3 Structure of our work, setting and main results

### 0.3.1 General setup

The point of departure from the pre-existing literature of this thesis seems to be our definition of the category of $(G,\chi)$-graded associative algebras. We will explain here how our situation differs from objects of previous exploration.
We explain shortly our setup. More detailed explanations will follow in section one of chapter one. Let $G$ be an abelian group, $k$ a commutative ring, and $\chi : G \times G \to k^*$ a map such that $\chi$ induces morphisms of abelian groups in both components and such that $\chi(g,h)\chi(h,g) = 1$ for all $g,h \in G$. Then we will call $\chi$ a *bicharacter* (see also Def. 1.1.5). We use the bicharacter on the one hand to change the category of $G$-graded algebras by twisting the definition of the tensor product of $G$-graded $k$-algebras as in Def. 1.1.8 and to construct enriched hom-sets $\mathbf{Hom}_A(M,N)$ between bimodules $M,N$ over a $(G,\chi)$-graded algebra $A$ as in Def. 1.1.6. The homomorphisms differ from ordinary (i.e. degree zero) graded module morphisms in that they are not necessarily degree zero but finite sums of morphisms of fixed degree and that they satisfy a twisted linearity condition. More precisely, left linearity over $A$ is for homogeneous $\varphi \in \mathbf{Hom}_A(M,N)$ replaced by the condition

$$\varphi(rm) = \chi(|\varphi|,|r|)r\varphi(m)$$

for all homogeneous $r \in A$. The set of degree zero $A$-module homomorphisms will in our work be denoted by $Hom_A(M,N)$. Members of $\mathbf{Hom}_A(M,N)$ are referred to as quasihomomorphisms in our work.
On the other hand, we use the bicharacter in the same way as previously existing literature, to define color-commutative algebras as $G$-graded associative algebras which for homogeneous elements $a$ and $b$ satisfy a commutation condition of the type $ab = \chi(|a|,|b|)ba$. These two functions of the bicharacter in the setup of the categories we will be working with can be well understood by thinking of the case of $G = \mathbb{Z}_2$ and $\chi(a,b) = (-1)^{ab}$, in which case the definitions of the category of $(G,\chi)$-graded algebras coincide with the definitions of the category of superalgebras, while the color-commutative specimens inside correspond in this case to supercommutative algebras.
We set these definitions up in the way we do on the one hand to obtain an ambient category with good properties for color-commutative algebras, and on the other hand because of the needs of the deformation and color-commutative algebra cohomology theories we develop in

chapters two to four.
The pre-existing work closest to this general setup that we know of is a very recent paper by (Bergh, Oppermann [9]), where the *cohomology of twisted tensor products* is studied. Essentially, they consider $G$-graded associative algebras over a field $K$ and a bicharacter on $G$ with target $K^*$. They use this bicharacter to induce a twisting of the tensor product of two $K$-algebras in essentially the way we do and study the $Ext$-groups of tensor products of two such algebras in relation to the $Ext$-groups of each component. However, their $Ext$ functor differs from the one we introduce in chapter two, because they do not use the bicharacter $\chi$ to twist also the extended $Hom$-sets between modules over their algebras. Instead, they define graded Hom-sets by using the definition

$$\underline{Hom}_A(M,N) := \oplus_{g\in G} grHom_A(M, N[g])$$

where $grHom_A(M,N)$ corresponds to our $Hom_A(M,N)$, $\underline{Hom}_A(M,N)$ is their graded Hom-set, and $N[g]$ denotes a version of $N$ shifted by $g$. They develop an $Ext$ functor from this, but it is not quite the one we need.
Similar remarks apply to (Chen, van Oystaeyen, Petit [14]).
Of course, definitions similar in spirit to ours are known in the super case, see e.g. the textbook (Varadarajan, [54]).

## 0.3.2 Summary of our work

We will provide here a chapter-by-chapter summary of what we do. A list with what we view as the most important contributions of this thesis follows later.

### Overview of chapter one

Section 1.1 of this thesis is entirely devoted to setting up and explaining fundamental properties of the $(G,\chi)$-graded category. To this end, we recall a number of well-known facts about graded rings and modules. We define some concepts related to rings and algebras in this category which we consider useful. We refer to the super situation for the purpose of illustrating our constructions. We do not think that apart from the general setup as discussed above, anything here is new.
In Section 1.2, we concern ourselves in particular with color-commutative rings. We start by recalling the definition and some examples of supercommutative superalgebras for motivation. Then, we define color-commutative algebras as indicated above and give a number of examples for illustration. These include supercommutative and commutative superalgebras (Ex. 1.2.7), $(G,\chi)$-analogs of the symmetric algebra over a commutative base ring $k$ (Ex. 1.2.10), the quaternions and para-quaternions with appropriate graded structures (Ex. 1.2.11), full matrix algebras over rings with enough primitive roots of unity (Ex. 1.2.12), finite dimensional Clifford algebras (Ex. 1.2.13), quasicommutative algebras (Ex. 1.2.15) and group algebras over groups satisfying a very restrictive technical property (Ex. 1.2.17). The novelty status of these examples is as follows: the example last mentioned may be new. In the special case of $G=\mathbb{Z}^n$, "even" $(G,\chi)$-commuting polynomial rings are introduced in (Avramov, Gasharov, Peeva [2]).

Our example is more general than their definition, but the generalization is straightforward. The example of the quaternions as a color-commutative algebra may have been new until it was discovered independently from our work by (Morier-Genoud, Ovsienko in [42]). If so, *a forteriori* our treatment of Clifford algebras as commutative should be new. The beautiful example 1.2.12 comes essentially from (Caenepeel, Dascalescu, Nastasescu [12]).
We then show in Lemma 1.2.20, which form a number of useful properties of commutative algebras take when transplanted into the color-commutative context. This lemma will quite frequently be useful in particular for the constructions in chapter four. Afterwards, we show in Rem. 1.2.21 - 1.2.23 that certain attempts of strengthening Lemma 1.2.20 fail.
We finish the discussion of commutative algebra for color-commutative rings by a result that offers some insight into the properties of what we call *pseudofields*, i.e. color-commutative algebras in which all homogeneous elements are invertible. It says that their degree zero component is always a field, that their odd components are all zero, and that their even components are always one-dimensional as vector spaces over the degree zero component.
The section closes with some constructions of mostly technical importance, all of which are analogs of well-known commutative algebra constructions.
Section (1.3) treats direct and inverse limits in categories of graded objects, with some applications to the $(G, \chi)$-graded case. We comment specifically on difficulties that arise compared to the ungraded situation when one tries to apply certain constructions involving limiting processes to graded objects. Some of these difficulties are similar in nature to difficulties we will encounter later when we build graded power series rings. The results proven in Section (1.3) will be of some technical importance later on.

**Overview of chapter two**

The first task of chapter two is to adapt Hochschild cohomology and certain computational tools connected to it to the $(G, \chi)$-graded context. We start this task by recalling, in Subsection 2.1.1, some notions about *graded projective modules* and more specifically *graded projective resolutions*. These are standard material from graded ring theory and we include them only for self-containedness. The main point of Subsection 2.1.1 is Prop. 2.1.6, which is basically a technical tool which will be needed to prove well-definedness of derived functor constructions using projective resolutions in the $(G, \chi)$-graded category.
Subsection 2.1.2 sees the development of an *Ext* functor adapted to the category of $(G, \chi)$-graded associative algebras. Conceptually, the point in which our construction here differs the most from the classical case is that in a sense, *two* different categories instead of just one play a role in the construction of our *Ext* functor. Both categories share the same objects - graded modules over graded $k$-algebras - but they differ with respect to the hom-sets used. For the construction of resolutions, we admit only degree zero morphisms. But the hom-sets subsequently used in the *Ext* construction are sets of quasihomomorphisms. We prove that like in the classical situation, the definition of this *Ext* does not depend upon choice of projective resolution.
In Section 2.2 we introduce Hochschild cohomology for $(G, \chi)$-graded algebras in Prop./Def.

2.2.1. This involves slightly changing the differential. We have to set

$$\beta(\varphi)(a_0 \otimes a_2 \otimes a_n) = \chi(|\varphi|, |a_0|)a_0\varphi(a_1 \otimes \ldots \otimes a_n)$$
$$- \sum_{i=0}^{n-1}(-1)^i\varphi(a_0 \otimes \ldots \otimes a_ia_{i+1} \otimes \ldots \otimes a_n) - (-1)^n\varphi(a_0 \otimes a_2 \otimes \ldots \otimes a_{n-1})a_n$$

for $\varphi \in \mathbf{Hom}_k(A^{\otimes n}, N) =: C^n(A, M)$ a homogeneous cochain and homogeneous $a_i \in A$. Verification that this is indeed a differential works just like the usual case. If $A$ is graded projective over $k$, one can as in the classical case use the bar complex as a projective resolution of $A$ as $A \otimes A^{op}$-left module. This construction is influenced by the additional structure compared to the ungraded case only insofar as the $k$-algebra structure of $A \otimes A^{op}$ depends on the bicharacter and the grading on $A$. We use this to show that for $k$-projective $A$, Hochschild cohomology coincides with *Ext* as in the classical case.

Towards the end of Section 2.2, we illustrate calculation of Hochschild cohomology via *Ext* by first classifying all color-commutative structures on the dual numbers $K[\epsilon]$ over a field $K$ and then calculating Hochschild cohomology for a representative sample of these cases.

In Section 2.3, we recover the usual interpretations of $H^1(A, M)$ as the module of derivations modulo inner derivations from $A$ into $M$ in the colored case. In the case of color-commutative algebras, we also define *universal derivations* and prove existence and unicity as in the classical case. The main purpose, in the context of our work, of introducing the module of derivations is that composition of derivations induces a color Lie algebra structure on $HH^1(A)$ which coincides there with the *color Gerstenhaber bracket* which we define later.

To bring into place the algebraic tools necessary to understand this color Lie algebra is the goal of Section 2.4. We start by recalling the definition of color Lie algebras, as they were introduced in (Rittenberg, Wyler [49]), (Scheunert [51]). Essentially, the ordinary Jacobi identity and skew symmetry conditions are replaced by the conditions

$$[a, b] = -\chi(|a|, |b|)[b, a] \text{ (graded skew symmetry)}$$
$$0 = \chi(|c|, |a|)[a, [b, c]] + \chi(|a|, |b|)[b, [c, a]] + \chi(|b|, |c|)[c, [a, b]] \text{ (Jacobi identity)}$$

for homogeneous elements of our color Lie algebra. Since we work over an arbitrary commutative ring $k$, we add two constraints which in characteristic zero are automatic in order to prevent pathologies in case the underlying module of our color Lie algebra has torsion.

Thereafter, we introduce (Def. 2.4.4), (Def. 2.4.6) colored analogs of the pre-Lie-algebras and pre-Lie-systems originally invented in (Gerstenhaber [27]). We show that a $(G, \chi)$-pre-Lie-algebra induces by the color commutator bracket a $(G, \chi)$-Lie algebra (Prop. 2.4.5). We also show that our notion of $(G, \chi)$-pre-Lie-system induces on its underlying module a structure of pre-Lie-algebra and therefore ultimately of color Lie algebra similarly as in the classical case (Prop. 2.4.8).

Finally, in Section 2.5 we conclude chapter two by showing that a colored version of partial composition induces a color pre-Lie-system structure on the Hochschild complex of a $(G, \chi)$-graded asssociative $k$-algebra $A$, and that the color Lie algebra structure so induced on the modified Hochschild complex induces a color Lie algebra also on the level of cohomology. We call the bracket giving the color Lie structure on the Hochschild complex the *color Gerstenhaber bracket*.

**Overview of chapter three**

In chapter three, we turn our attention to the formal deformation theory of $(G, \chi)$-graded algebras. The first problem that presents itself in this context is to obtain for a $(G, \chi)$-graded $k$-algebra $A$ a notion of *trivial* deformation, i.e. a colored replacement for the concept of a univariate power series ring. There are three main difficulties with finding a suitable replacement for the ungraded power series ring construction. First, the ring to be constructed should naturally support a grading extending the grading of $A$ and such that the formal parameter $X$ is of the desired degree while still allowing as much freedom in building formal series as possible. Second, as we want $X$ to still color-commute with elements of the base algebra, the Cauchy-product has to be changed accordingly. Third, if we want $A[[X]]$ to be a color-commutative algebra whenever $A$ was, we get additionally relations of the form $X^2 = \chi(|X|, |X|)X^2$. In the super case, this means that for $X$ an odd variable we have $2X^2 = 0$, as e.g. for super polynomial rings.
One must in this case check that any such relations do not interfere with the well-definedness of the modified Cauchy product.
In the course of this discussion we find two apparently reasonable notions of a $(G, \chi)$-graded power series ring in one formal variable $X$ of degree $g \in G$. We refer to the power series ring over $A$ that arises from the first option as $A[[X]]$ and call it the (univariate) *power series ring in degree* $g$ over $A$, whereas we denote the other by $A[[\tilde{X}]]$ and call it the *alternative power series ring* in degree $g$ over $A$.
The primary difference between the two options is that $A[[X]]$ is designed to preserve color-commutativity of $A$, while $A[[\tilde{X}]]$ is not. We add, in Ex. 3.1.3 - 3.1.6 some examples to illustrate the behaviour of this construction. In the case $G = 0$, both of our notions of a graded power series of course recover exactly the usual notion of power series ring over a commutative algebra. The section closes with some technical remarks about $A[[X]]$, most importantly one saying that the usual criterion for invertibility of an element in a power series ring - invertibility of the degree zero component - holds in the context of graded power series rings only for homogeneous elements.
We did not find much literature on graded power series rings, and we are not aware of any previous discussion of graded power series rings with coloring. In the case of a $G$-graded algebra $A$ with trivial bicharacter $\chi$ and a formal variable $X$ of degree $g \in G$, our $A[[X]]$ is the subring generated by homogeneous power series of the ordinary univariate power series ring over $A$. For some discussion of graded power series rings over commutative $\mathbb{Z}$-graded rings, see e.g. (Fossum [21]), (Landweber [36]).
Section (3.2) is devoted to the theory of graded formal deformations. As in the classical situation, these are nonstandard products defined in a particular manner on the power series ring, here primarily $A[[X]]$. The deformation theory arising from the alternative power series ring $A[[\tilde{X}]]$ is given separate treatment only in remarks along the way as it is very similar.
The section starts by recalling a special case of the fact that a system of orthogonal idempotents in a commutative ring induces a splitting of the ring into a direct sum of smaller rings.
After these introductory remarks, we begin to build the infrastructure needed to do formal deformation theory in the $(G, \chi)$-graded context. In Prop. 3.2.8 - 3.2.12 we prove various

statements of technical importance about the process of extending, in the $(G,\chi)$-graded setting, a $k$-linear map $\mu : A \otimes A \to A[[X]]$ to a $k[[X]]$-linear map $A[[X]] \otimes A[[X]] \to A[[X]]$. After Prop. 3.2.12, we are in Def. 3.2.13 able to define a concept of formal deformation analogous to the classical case discussed in (Gerstenhaber [27]). We define also formal equivalences of formal deformations. These definitions work precisely as in the classical case.
In Prop. 3.2.18, we derive a deformation equation for $(G,\chi)$-graded algebras and do the same for an equivalence equation in Prop. 3.2.19. Compared to the classical case, the main difference is that these have to account precisely for color commutation relations and degrees of the appearing functions.
Finally, we apply the theory to obtain a cohomological classification of infinitesimal deformations of a $(G,\chi)$-graded algebra in Prop. 3.2.20 and of obstructions to extensions of formal deformations to higher order in Prop. 3.2.22. We see here that the definitions and propositions obtained in chapter two on Hochschild cohomology and related structures work together reasonably well in the framework of our deformation problem. The results are:

**Infinitesimal deformations (3.2.20)** The equivalence classes of infinitesimal deformations in degree $g \in G$ correspond one to one with elements of $HH^2(A)$ of
degree $-g$.

**Obstruction theory (3.2.22)** Let $A$ be a $(G,\chi)$-graded associative algebra over a commutative ring $k$ with $2 \in k$ invertible and let $(A[[X]]/(X^N), \mu)$ be an associative deformation in degree $g \in G$ given up to order $N$. Then the obstructions to extending $\mu$ associatively to order $N+1$ can be identified with the elements of $HH^3(e_\lambda A)_{-(N+1)g}$.

Here $e_\lambda$ is an idempotent of $k$ determined by $\chi$ and $|X| =: g$, the subscripts indicate the degree of the responsible component of Hochschild cohomology in the $G$-grading. We close the section with a remark on how the color Gerstenhaber bracket can be used to rewrite the deformation equation of the power series ring $A[[X]]$ but not of the alternative power series ring $A[[\tilde{X}]]$.

### Overview of chapter four

In chapter four, we develop in imitation of the methods used in (Harrison [31]) a cohomology theory specifically for $(G,\chi)$-*commutative* algebras over a commutative ring $k$. We then proceed to show that color Harrison cohomology behaves similarly with respect to color polynomial rings as Harrison cohomology does with respect to polynomial rings in *ibid.* Finally, we show that color Harrison cohomology is suitable for obtaining a cohomological approach to the color-commutative deformation problem of color-commutative algebras. We discuss an additional problem related to the deformation theory of color-commutative algebras, pointing out that given an algebra that is endowable with a color-commutative structure, there are of course in general many color-commutative structures compatible to it and therefore many options to choose from for the definition of a color-commutative power series ring extending it.

Literature about graded-commutative Harrison cohomology seems to be scarce. Graded Harrison cohomology over strictly commutative, $\mathbb{Z}$-graded algebras is among the subjects treated in (Kadeishvili [34]). The possibility of developing Harrison cohomology for supercommutative algebras seems to be folklore, see (Fronsdal, Kontsevich [25]) in the differential graded algebra setting, but we could not find a detailed published account. In any case, we are not aware of any account of Harrison cohomology in the setting of arbitrary color-commutative algebras. In light of the fact that the general color-commutative setting includes some algebras the ungraded versions of which seem fairly non-commutative, such as full matrix algebras over suitable coefficient domains or arbitrary color polynomial rings, we are of the opinion that the possibility of obtaining a well-behaved extension of commutative algebra cohomology into these settings is interesting.

It should be mentioned in this context that other approaches than Harrison cohomology to commutative algebra cohomology exist, most notably André-Quillen cohomology (Quillen [48]) and an approach using Koszul duality between the commutative and the Lie operad, see e.g. the book (Markl, Shnider, Stasheff [41]). If $k$ is a ring containing $\mathbb{Q}$ and if $A$ is a flat $k$-algebra, the cohomologies induced by all these approaches coincide, but in general, they do not (see Loday [38], Quillen [48]). We do not try to adapt these to the $(G,\chi)$-commutative situation.

We begin the chapter by defining a $(G,\chi)$-commutative analog of the *shuffle product* in Def./Prop. 4.1.2. In the commutative case, this is a certain graded commutative product on the tensor algebra $T(V)$ of a module over a commutative ring $k$. The original definition would make perfect sense also in our context, but it would not be compatible with Hochschild cohomology. For $A$ a $(G,\chi)$-commutative $k$-algebra,

$$a := a_1 \otimes a_2 \otimes \ldots \otimes a_p, a' := a_{p+1} \otimes \ldots \otimes a_{p+q}$$

tensor products of homogeneous elements in $A$, we therefore introduce the changed product

$$a \diamond a' := \sum_{\sigma \in Sh(p,q)} sign(\sigma) F(\sigma, aa') \sigma.aa'$$

where $aa'$ is tensor concatenation of $a$ and $a'$, $\sigma$ operates on $aa'$ by permutation of elements, $F$ is a generalized permutation sign, and everything is linearly extended to cover inhomogeneous elements. We prove that this gives a color-commutative product on the tensor algebra $T(A)$ of $A$. One important point proven in this context is that $F$ satisfies the composition rule $F(\rho\sigma, c) = F(\sigma, c)F(\rho, c_{\sigma^{-1}})$, in analogy to the composition rule for permutations. This observation is used many times in the following proofs.

We acknowledge the fact that more general variations on the idea of a shuffle product have been discussed before, for instance in (Baez [5]), (Rosso [50]) or (Ospelt [45]). The shuffle product in the latter two papers is closely related to ours and (Ospelt [45]) gives some compatibility results to Hochschild homology in his context. However, he does not attempt to use his findings to generalize Harrison homology or cohomology to his setting and for our case the modification we propose is sufficient. We therefore believe that discussion of our case has independent merit. We learnt of this work after completion of chapter four.

We prove (Prop. 4.1.5) that the changed shuffle product is still compatible with the bar complex

differential in the same way as the original. After this, we have all the infrastructure we need in Def./Prop. 4.2.1 to define our extension of Harrison cohomology to the color-commutative context. We check that the modifications to the Hochschild differential and the modifications to the shuffle product work together well in terms of inducing a cohomology on the subcomplex vanishing on color-commutative shuffles. Other than that, the construction we used is very similar to the commutative case.
With $A$ a $(G, \chi)$-commutative algebra, we have by definition the commutation relation

$$ab = \chi(|a|, |b|)ba$$

for all homogeneous $a, b \in A$. Even if we changed the definition of a bicharacter to no longer force the axiom $\chi(g, h)\chi(h, g) = 1$, color-commutation itself would induce some relations similar to this one. We discuss along the way to what extent the Harrison construction would still work if we considered algebras with commutation relations that are not strictly speaking skew-symmetric.
Section (4.3) is devoted to developing a characterization by color Harrison cohomology of color polynomial rings in analogy to the characterization of polynomial rings by classical Harrison cohomology in (Harrison [31]). Problems compared to the classical case appear mostly in relation to limiting constructions as in Corollary 4.3.3, where they are solved by Section 1.3. Additionally, one has to be more careful with proving that all definitions are well-posed, which is most apparent in Lemma 4.3.6. Also the introduction of homomorphisms of nonzero degree again requires more care than in the classical case. Apart from these difficulties, Section 4.3 is not very different from the commutative case treated by Harrison.
In Section 4.4 we present another application of color Harrison cohomology, namely to the problem of obtaining color-commutative graded formal deformations of color-commutative rings.

**Chapter five and Appendix**

In chapter five, we discuss some possible directions of further research which are at varying stages of exploration. The appendix is devoted to the classification of color-commutative structures on a more complicated example algebra than the one we looked at in chapter two.

## 0.4 List of main contributions

The following is a synopsis of the main contributions of this thesis. These are ordered by chapter again:

1. We give some examples of color-commutative algebras, some of which may be new. We introduce a class of field-like color-commutative rings which we call *pseudofields*, and prove a statement characterizing their structure to some extent. As an application, we derive a necessary criterion for a full matrix ring over a field to admit a certain special kind of color-commutative structure. In Rem. 1.2.16 we discuss a construction originally intended to yield a generalization of color-commutativity which gives instead new examples of color-commutative algebras.

2. We define an *Ext*-functor and Hochschild cohomology adapted to the $(G, \chi)$-graded setting and show that the modified *Ext* functor can be used to compute our adapted Hochschild cohomology. We define colored analogs of some constructions related to deformation theory and show that they relate to each other in a good way. In particular, we introduce *color pre-Lie algebras* and *color pre-Lie-systems*. We use these to introduce on our version of the Hochschild complex a structure of color Lie multiplication, the colored analog of the Gerstenhaber bracket.

3. We discuss the development of a notion of power series ring in the $(G, \chi)$-graded setting and propose in the end two notions of power series ring slightly differing from each other which seem appropriate to our setting. We use the first of these notions to develop a deformation theory for $(G, \chi)$-graded algebras in analogy to the one developed by Gerstenhaber for associative algebras. We supply cohomological tools for classification of order one formal deformations respectively obstructions to extension to higher order using second and third color Hochschild cohomology. We discuss how the theory depends upon the choices inherent in the definition of the graded power series ring that was used as trivial deformation of our algebra.

4. We develop a colored analog of the shuffle product and show that the modified shuffle product is color-commutative. We show that it behaves with respect to the bar differential in the same way as its classical counterpart. We then verify that the $(G, \chi)$-graded Hochschild cohomology differential is compatible with this shuffle product and use this to obtain a cohomology for color-commutative algebras analogous to Harrison cohomology for commutative algebras. We translate a result due to Harrison on characterization of polynomial rings over fields into the color-commutative context. We close with some remarks on the deformation theory of color-commutative algebras. We also discuss, along the way, the extent to which the properties of the bicharacter were really needed to carry out the construction.

5. In the appendix, we classify color-commutative structures on the algebras

$$A_q := K < X, Y > /(X^2, Y^2, XY + qYX)$$

where $K$ is a field and $q \in K^*$ is some invertible element. This is the first step of a larger project to determine the $(G, \chi)$-graded Hochschild cohomology of these algebras when they are viewed as color-commutative algebras.

**Added in proof** After completion of this work, we learned of the existence of some papers, previously unknown to us, which are related to parts of or deserve mentioning as context to this thesis, namely (Lychagin [39]) and (Scheunert [52]) on color algebras as well as (Nadaud [43], [44]), (Pinczon [47]) and (Ginzburg [29]) on noncommutative deformations. We quickly explain how these articles relate to the present work.
Lychagin considers a somewhat more general type of color-commutative ring than the one

presented in this thesis. His definitions allow rings graded over an arbitrary, not necessarily commutative group $G$. Like our definitions of a bicharacter over a $G$-graded ring $R$ with $G$ not necessarily abelian in Section 1.2 , he requires of the bicharacter that $\chi(g,h) \in R_{[g,h]}$ and $\chi(g,h)\chi(h,g) = 1$ for all $g,h \in G$. However, he replaces the condition $\chi(g_1g_2,h) = \chi(g_1,h)\chi(g_2,h)$, which in our situation forces all elements of $R$ of the form $\chi(g,h)$ to be homogeneous elements of the center of $R$, by the conditions

$$\begin{aligned} \chi(g_1g_2,h) &= \chi(g_1,g_2hg_2^{-1})\chi(g_2,h) \text{ (hexagon identity)} \\ \chi(g_1g_2,h) &= \chi(g_1,g_2)\chi(g_2,g_1h)\chi(g_1,h) \text{ (compatibility condition).} \end{aligned}$$

For explanations concerning the naming of these identities, we refer to (Lychagin [39]). In our context, both of these properties are automatically true of $\chi$. Lychagin supplies examples very similar to ours, including the quaternion algebra, Clifford algebras, paraquaternions and the full matrix algebra over $\mathbb{C}$. In the case of the quaternions and Clifford algebras, he uses a different graded structure than we do. He defines a notion of module homomorphisms similar to our quasihomomorphisms and obtains the same notion of colored derivation and universal derivations over color-commutative algebras as we do in Section 2.3 . He finds the same structure of color Lie algebra on the module of derivations over a $(G,\chi)$-graded algebra $A$ as we do in Section 2.4 . Other than that, we do not see significant overlap between his work and ours.
In (Scheunert [52]), the same adaption of Hochschild cohomology to the setting of color algebras as discussed in this thesis is introduced. However, his subsequent discussion of deformation theory of graded associative algebras covers only the case of deformations with a formal variable of degree zero. Even in this special case, obstruction theory is not treated. Also, his article does not attempt to derive a notion of color *Ext* functor adapted to the problem of calculating color Hochschild cohomology.
(Nadaud [43], [44]) and (Pinczon [47]) treat a different approach to noncommutative deformations than the one explored in this thesis. Essentially, their concept of formal deformations is based on twisting the ordinary commutative action of the undeformed algebra on the formal deformation parameter by an algebra automorphism, i.e. with $t$ the deformation parameter and $a$ an element of the undeformed algebra $A$, their framework prescribes that $at = t\sigma(a)$, with $\sigma : A \to A$ a $k$-algebra automorphism. For fixed $g \in G$, our commutation relations for a formal parameter of degree $g$ can be described in this way, and insofar their deformations are potentially 'less commutative' than ours. However, our construction uses additional information in the form of a graded structure, making our approach more general in situations that are amenable to description by both. Closest to their version of noncommutative deformation theory in our work is the deformation theory arising from what we call the alternative power series ring on a color algebra $A$.
Finally, (Ginzburg [29]) considers a very different framework for noncommutative deformations not relying on any commutation relations between elements of the undeformed algebra and the deformation parameter. His setting only allows definition of infinitesimal deformations.

## 0.5 Acknowledgements

First of all, I want to thank my advisor *Martin Schlichenmaier* for introducing me to deformation theory and for his supervision of my work on this thesis these last four years. His guidance and support - both scientific and administrative - were very valuable for this thesis. Especially noted are the improvements he suggested upon reading earlier drafts of this document.

My thanks go out to *Norbert Poncin* and *Alexei Kotov* for a very instructive lecture series of theirs in 2007 which introduced me to noncommutative geometry and superalgebras.

I wish to also thank *Yael Fregier* and *Boris Shoikhet* for useful discussions. Especially the former's particularly inquisitive style of listening to any exposition of my work was very useful to me. Boris Shoikhet brought to my attention the topic of formal smoothness.

I am grateful to *Michel Van den Bergh* and *David Eisenbud* for answering questions of mine about complete intersections.

I thank the *University of Luxembourg* for providing excellent working conditions throughout the preparation of this thesis. The *Hausdorff Institute* in Bonn earned my gratitude by providing a stimulating working atmosphere at the workshop on *Algebraic and geometric deformation spaces* in 2008. It was there that I learned of Example 1.2.12 from a talk by *Claude Cibils*.

I also thank the *University of Sevilla*, *Moscow State University*, the *Steklov Institute Moscow* and the *village of Glanon* for their hospitality during other schools and conferences.

I used on some occasions the first order logic finite countermodel searcher *Mace4* to obtain examples of finite algebras - or monoids from which an algebra can be constructed - with certain properties. While none of these examples made it into this thesis, this certainly saved me working time which could then be spent on other problems. During my work on hom-algebras, some of which was carried out concurrently with work on this dissertation, I used the accompanying equational logic prover *Prover9* as well to verify numerous simple conjectures definitively faster than I could have by hand. I therefore thank *William McCune* for developing these programs.

In the investigations leading up to Example 1.2.17, the computational discrete algebra tool *GAP* [26] and in particular the *small groups library* were quite helpful by giving me easy access to examples of finite nonabelian groups satisfying the requirements of this example.

My *friends* both in Luxembourg and abroad have contributed tremendously to keeping me sane throughout these years by cheering me up when progress was slow, by sharing my joy when work was going well, and by simply being there when mathematics was not the topic under discussion. You know who you are and I thank you all.

Finally, also the constant support of my *parents* is warmly appreciated.

# Chapter 1

# Color commutative algebras and related structures

In this chapter, we will recall some basic algebraic concepts and some notations which will be useful for the rest of this thesis. In particular, we will introduce the types of algebras which we will be interested in later. The structure of this chapter is as follows: in the first section, we recall the notion graded modules, rings and algebras with gradings over an abelian group $G$, explain some constructions for graded modules, and introduce the category of graded modules equipped with a *bicharacter* $\chi$. We explain how the bicharacter influences various constructions in this category. In particular, we explain the definition of the tensor product in the $(G,\chi)$-graded category and the notion of a *quasihomomorphism* of modules over a $(G,\chi)$-graded associative algebra $A$. We illustrate these things by frequently referring to the special case of the superalgebra category, which arises from considering gradings over $\mathbb{Z}_2$ and a bicharacter given by $\chi(a,b) = (-1)^{ab}$ for $a, b \in \mathbb{Z}_2$.
In the second section, we turn our attention to the special case of *color-commutative* algebras, which are $(G,\chi)$-graded algebras which obey a commutation condition given by the bicharacter $\chi$. In the super category, the corresponding concept to this is the concept of a supercommutative algebra. We provide many examples and show some generalizations of constructions from commutative algebra which will be useful later on.
In the third section, we study the behaviour of direct and inverse limits in our category. We explain some problems that appear in this context in comparison to the ungraded situation and how to overcome them. The section provides a number of technical lemmas which will be useful towards the end of this thesis.

## 1.1 Graded rings and modules

We would also like to recall some easy notions about graded rings and modules. First, we define the notion of a graded ring:

**Definition 1.1.1** *Let $G$ be an abelian group and $R$ be a ring such that $(R,+)$ as an abelian* *group admits a decomposition of the form*

$$R = \oplus_{g \in G} R_g$$

*which shall be supposed to satisfy the condition*

$$ab \in R_{g+g'}$$

*for any* $a \in R_g, b \in R_{g'}$ *and* $g, g' \in G$. *With obvious notations, these latter conditions we will also write simply as* $R_g R_{g'} \subseteq R_{g+g'}$. *Then, R together with the data supplied by the decomposition indicated above is called a G-graded ring. An element* $a \in R_g$ *is called a* homogeneous *element of* degree $g$. *In this case, we write* $deg(a) = g$. *We will sometimes also use the notation* $|a|$ *to denote the degree of* $a$.
*A homomorphism of graded rings R and S, where R and S are supposed graded over the same group G, is a ring homomorphism* $\varphi : R \to S$ *which respects the grading, i.e. with*

$$\varphi(A_g) \subseteq S_g$$

*for all* $g \in G$.
*An ideal in a graded ring is called homogeneous if it is compatible with the graded structure, i.e. if there exists a system of homogeneous generators.*
*By default, when we speak of an ideal of a graded ring, we will mean a homogeneous ideal as above. If the need will arise to talk about ideals which are not homogeneous, we will explicitly view them as ideals in the underlying ungraded ring.*
*The same notions are introduced for algebras over a commutative ring k in the natural manner. The main additional restriction imposed in this case is that all structures are required to be compatible with the structure as k-module in this case.*

We remind the reader of some observations about graded ideals of a graded ring:

**Remark 1.1.2** *Let R be a ring graded over an abelian group G and* $I \subseteq R$ *be a left(right, two-sided) ideal of the ungraded underlying ring. Then, the following are equivalent:*

1. *I is homogeneous, i.e. generated as an R-module (left, right, twosided) by a set* $S \subseteq R$ *of homogeneous generators.*

2. *For each* $x \in I$ *we have in the decomposition* $x = \sum_{d \in G} x_d$ *into homogeneous elements that* $x_d \in I$ *for each* $d \in G$.

3. $I = \oplus_{g \in G} I \cap R_g$

*Let further* $\mathfrak{a}, \mathfrak{b} \subseteq R$ *be two-sided homogeneous ideals of R. Then, the ideals* $\mathfrak{ab}$, $\mathfrak{a} \cap \mathfrak{b}$ *and* $\mathfrak{a} + \mathfrak{b}$ *are homogeneous too.*

**Proof** We prove the equivalence for the two-sided case and leave the cases of left and right ideals to the reader.
($1 \Rightarrow 2$) Suppose first that there is a set $S$ consisting of homogeneous elements such that the

two-sided submodule of $R$ generated by $S$ is $I$ and let $x \in I$. Then, by definition, $x = \sum a_i s_i b_i$, with $a_i, b_i \in R$ and $s_i \in S$, the $s_i$ not necessarily pairwise distinct. Each of the $a_i, b_i$ can be decomposed into a sum of homogeneous elements, and we can hence using the distributive law assume without loss of generality that they are all homogeneous. Since then the products $a_i s_i b_i$ are also homogeneous and due to $I$ being a two-sided ideal members of $I$, we have found a decomposition of $x$ into homogeneous members of $I$. Unicity of the decomposition into homogeneous terms yields the desired conclusion.
($2 \Rightarrow 3$) Suppose that every $x \in I$ has a homogeneous decomposition with all summands in $I$. Then by definition, $x \in \oplus_{g \in G} I \cap R_g$, so indeed $I \subseteq \oplus_{g \in G} I \cap R_g$. Since the other inclusion is obvious, the equality given is obtained.
($3 \Rightarrow 1$) This is clear: in this case, a system of homogeneous generators is given for instance by $\cup_{d \in G}(R_g \cap I)$.
We will now prove the rest of the remark. Let $\mathfrak{a}, \mathfrak{b} \subseteq R$ be two ideals in $R$, as usual homogeneous. That $\mathfrak{a} + \mathfrak{b}$ is homogeneous is then clear. To see $\mathfrak{ab}$ homogeneous, suppose $x \in \mathfrak{ab}$. Then, by definition, $x = \sum a_i b_i$ with $a_i \in \mathfrak{a}, b_i \in \mathfrak{b}$ for all $i$. Again we may assume without loss of generality, using the distributive law and homogeneity of $\mathfrak{a}, \mathfrak{b}$, that all the $a_i, b_i$ are already homogeneous here. The conclusion is then immediate. Finally, consider $x \in \mathfrak{a} \cap \mathfrak{b}$, then we have as usual a unique $x = \sum_{d \in G} x_g$ and all the $x_g$ must, by the homogeneity of $\mathfrak{a}$ and $\mathfrak{b}$ be members of both ideals. This concludes the proof.

**Remark 1.1.3** *Note that the definitions force the unit element of the ring to be homogeneous of degree zero, but it is quite possible for other idempotents to exist which are not in the zero degree part of the ring because they are inhomogeneous.*

Note that any ring $R$ can be given a grading over any group by setting $R_0 := R$ and $R_g := 0$ for all $g \neq 0$. In the case where only one degree of a graded object of any kind is nonzero, we say that this object is *concentrated* in that degree. For any abelian group $G$, this construction can be used to obtain an embedding of the category of associative rings into the category of associative $G$-*graded* rings. In the language of category theory, the embedding functor so obtained is full and faithful.
In generalization of the definitions for graded algebras, one can define graded *modules* over a ring $R$ or a $k$-algebra $A$. We will for this assume from the start that our algebra is graded over some abelian group $G$. We will also only treat explicitly the case of modules over algebras, since this case will later concern us most. Also, the corresponding definitions for rings not endowed with an algebra structure are virtually identical. In the case of graded left modules over a $G$-graded algebra $A$, the complete definition is as follows:

**Definition 1.1.4** *A $G$-graded (left-) module $M$ over the $G$-graded algebra $A$ is an $A$-(left-)module together with a decomposition as a direct sum of $k$-modules*

$$M = \oplus_{g \in G} M_g$$

*such that for any $a \in A_g$ homogeneous and $m \in M_{g'}$ we have $am \in M_{g+g'}$. An element of $M_g$ is called* homogeneous of degree $g$. *A map $\varphi$ between graded $A$-modules $M$ and $N$ is said to* *be of degree $h \in G$ if for every $g \in G$ we have $\varphi(M_g) \subseteq M_{g+h}$. A homomorphism of graded $A$-modules is an $A$-module homomorphism which in addition is a degree zero map. The set of all such maps is denoted by $Hom_A(M,N)$, i.e. we have*

$$Hom_A(M,N) := \{f : M \to N | f \text{ is } A\text{-linear and degree zero}\}$$

*It is understood that linearity here means linearity of the appropriate kind (left, right, both) depending on the module category under consideration.*

The definitions for graded modules over graded rings are exactly the same. When we talk about graded modules over rings or algebras which themselves carry no graded structure, the latter will be viewed as being concentrated in degree zero.

**Definition 1.1.5** *Let $R$ be a $G$-graded ring. Denote by $Z(R)$ the center of $R$ viewed as an ungraded algebra, i.e. the set of all elements in $R$ which commute with everybody else. Denote further by $R^*$ the set of invertible elements of $R$. Let $\chi : G \times G \to R^* \cap Z(R) \cap R_0$ be a map taking elements of $G$ to invertible elements of degree zero of the center of $R$. Suppose that $\chi$ satisfies the following two properties:*

1. *For every $x \in G$, the map $\varphi : G \to R^*$ given by $\varphi(y) := \chi(x,y)$ is a homomorphism of abelian groups.*

2. *$\chi$ is skew-symmetric, meaning $\chi(g,h)\chi(h,g) = 1$ for all $g,h \in G$.*

*Then we call $\chi$ a* bicharacter *and the triple $(R,G,\chi)$ a* color associative ring.
*Suppose now that $k$ is a commutative ring and that $A$ is a $G$-graded $k$-algebra. We introduce then basically the same concepts but in this case instead of demanding that the bicharacter $\chi$ have as target a multiplicative subgroup of $A_0^*$, we ask that it be a map with target $k^*$.*

To illustrate how the presence of a bicharacter will influence further definitions and constructions, it is maybe good to recall the case of superrings. By definition, a superring is simply a $\mathbb{Z}_2$-graded ring $R$. If we posit that in addition a bicharacter is defined on $\mathbb{Z}_2$ through

$$\chi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to R, \chi(x,y) = (-1)^{xy}$$

then the following well-known superalgebraic constructions can conveniently be described using the bicharacter:

1. The grade involution automorphism $\varphi : R \to R$ may be written as $\varphi(x) := \chi(|x|,|x|)x$ for homogeneous elements.

2. Likewise, the supercommutator is for homogeneous elements defined by

   $$[x,y] := xy - \chi(|x|,|y|)yx$$

   and the supercenter can be defined by

   $$Z_s(R) := \{x \in R : [x,y] = 0 \text{ for all } y \in R\}$$

   $R$ is called *supercommutative* if $Z_s(R) = R$.

3. Suppose that $A$ and $B$ are superalgebras over the commutative base ring $k$. Then, the tensor product of $A$ and $B$ as graded $k$-modules is simply their ordinary tensor product as $k$-modules equipped with the grading given on homogeneous generators by $deg(a \otimes b) = deg(a) + deg(b)$. The product on $A \otimes B$ in the super category is in terms of the bicharacter given by
$$a_1 \otimes b_1 \cdot a_2 \otimes b_2 := \chi(|b_1|, |a_2|) a_1 a_2 \otimes b_1 b_2$$
and one of the primary motivations for setting this definition, different from the corresponding definition for the algebra structure on tensor products of associative algebras in general, is that with this definition the tensor product of two supercommutative algebras becomes supercommutative again.

These and similar observations in the super case motivate the following definition:

**Definition 1.1.6** *Suppose now that $M$ is a graded $R$-bimodule, where $R$ is supposed a $(G,\chi)$-colored ring. We call $M$ $(G,\chi)$-*symmetric *if for all homogeneous elements $a \in A$ and $m \in M$ the condition*
$$am = \chi(|a|, |m|) ma$$
*is satisfied.*
*For $M, N$ two $G$-graded $R$-bimodules, we define two notions of a homomorphism module from $M$ to $N$. The first one is given by the homomorphisms of two-sided graded modules over a graded ring, i.e. maps which are supposed linear on both sides and of degree zero. We denote these by $Hom_R(M,N)$. The second kind of homomorphism-like maps of type $M \to N$ that we want to consider we will call* quasihomomorphisms*. They are defined to be finite sums of maps $\varphi : M \to N$ of fixed degree $|\varphi|$ satisfying the $R$-linearity conditions*
$$\begin{array}{lrcl} (A) & \varphi(m+n) & = & \varphi(m) + \varphi(n) \\ (B) & \varphi(mr) & = & \varphi(m) r \\ (C) & \varphi(rm) & = & \chi(|\varphi|, |r|) r \varphi(m) \end{array}$$
*for any homogeneous $m, n \in M$ and $r \in R$. The set of those maps naturally forms a $G$-graded bimodule over $Z(R) \cap R_0$. We denote this bimodule by*
$$\boldsymbol{Hom}_R(M,N) := \{f \in App(M,N) : f \text{ is a quasihomomorphism}\}$$
*where $App(M,N)$ denotes the set of all maps from $M$ to $N$. The definitions for one-sided $A$-modules are set accordingly.*
*We easily $\boldsymbol{Hom}_R(M,N)_0 = Hom_R(M,N)$. The corresponding concepts for modules over $k$-algebras are defined in the natural way.*
*In the case of a $(G,\chi)$-symmetric bimodule, it is easily checked that condition $(C)$ on a quasi-homomorphism follows from condition $(B)$.*
*For $g \in G$ an arbitrary group element we define $\xi_g := \chi(g,g)$. The map so given is called the* signature *of $\chi$.*

**Remark 1.1.7** *Let $R$ be a $(G, \chi)$-colored ring. Then the category of $R$-bimodules together with the sets of quasihomomorphisms $\boldsymbol{Hom}_R(M, N)$ as morphism sets and function composition as composition law for morphisms forms a category. We will call this category $\boldsymbol{Bimod}(R)$.*

**Proof** We need only prove that for $G$-graded $R$-bimodules $M_1, M_2, M_3$ and quasihomomorphisms $\varphi_1 : M_1 \to M_2$, $\varphi_2 : M_2 \to M_3$ we have $\varphi_2 \circ \varphi_1 \in \mathbf{Hom}_R(M_1, M_3)$. We need to show this only for homogeneous morphisms, as the general case follows easily by distributing arbitrary sums of homogeneous morphisms through the composition. Also, all defining properties except $(C)$ are obvious. Assuming $\varphi_1, \varphi_2$ homogeneous and taking homogeneous $m \in M_1$ and $r \in R$, one then computes

$$\varphi_2(\varphi_1(rm)) = \chi(|\varphi_2|, |r|)\chi(|\varphi_1|, |r|)r\varphi_2(\varphi_1(m)) = \chi(|\varphi_1| + |\varphi_2|, |r|)\varphi_2(\varphi_1(m))$$

as desired.

Let in the rest of this section $A$ always denote a $G$-graded $k$-algebra. We now define the tensor product of graded modules over $A$:

**Definition 1.1.8** *The tensor product of the graded modules $M$ and $N$ over the base ring $k$ is given by*

$$(M \otimes N)_g := \oplus_{g'+g''=g}(M_{g'} \otimes M_{g''})$$

*If $A$ and $B$ are $(G, \chi)$-colored associative algebras, we define a $(G, \chi)$-colored tensor product in analogy to the tensor product of superalgebras by defining on the graded tensor product of the two algebras viewed as $k$-modules the product*

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := \chi(|b_1|, |a_2|)a_1a_2 \otimes b_1b_2.$$

*In this definition, the $a_i$ and $b_i$ are supposed homogeneous and the product is then extended linearly to non-homogeneous arguments[1]. The opposite algebra $A^{op}$ of a color associative algebra will be defined by creating a new multiplication on $A$, namely*

$$a * b := \chi(|a|, |b|)ba$$

*Both of these constructions give again a $(G, \chi)$-associative algebra. We regard this as clear for the opposite algebra. As for the tensor product algebra, the multiplication so defined is distributive by definition and compatibility with the gradings is clear. We check associativity. It suffices to do this for the case where $a_i, b_i$ appearing are homogeneous:*

$$\begin{aligned}(a_1 \otimes b_1)((a_2 \otimes b_2)(a_3 \otimes b_3)) &= \chi(|b_1|, |a_2|)\chi(|b_1|, |a_3|)\chi(|b_2|, |a_3|)a_1a_2a_3 \otimes b_1b_2b_3 = \\ &= \chi(|b_1|, |a_3|)\chi(|b_2|, |a_3|)\chi(|b_1|, |a_2|)a_1a_2a_3 \otimes b_1b_2b_3 \\ &= ((a_1 \otimes b_1)(a_2 \otimes b_2))(a_3 \otimes b_3)\end{aligned}$$

[1] To be more precise, we define for each second argument $a_2 \otimes b_2$ a *$k$-bilinear* map $A \times B \to A \otimes B$ on homogeneous arguments and then extend bilinearly to nonhomogeneous arguments. After that, we use the universal property of the $k$-module tensor product to produce from this a linear map of type $A \otimes B \to A \otimes B$. Finally, we check that this linear map depends linearly on $a_2 \otimes b_2$. Of course this last step uses essentially the same line of reasoning *again*. This type of argument will come up more often and we will in the future usually keep it implicit.

With this definition in mind, we are able to introduce the tensor algebra of a $G$-graded $k$-module as follows:

**Definition 1.1.9** *Let $V$ be a $G$-graded $k$-module. Consider then the $k$-module*

$$T(V) := \sum_{i=0}^{\infty} V^{\otimes i}$$

*where the tensor product is defined as above. Denote the submodule $V^{\otimes n}$ also by $T^n(V)$. Then, $T(V)$ naturally becomes a $G$-graded module, with the degree of a product of the form*

$$a_1 \otimes a_2 \otimes \ldots \otimes a_n$$

*being given by the sum of the degrees of all factors when all the factors $a_i$ are homogeneous. With this gradation and concatenation of tensors as multiplication operation, $T(V)$ becomes an associative graded $k$-algebra. We call the degree of a homogeneous element of $T(V)$ so given the* internal *degree of that element. The tensor algebra also becomes a graded algebra if every element of $T^n(V)$ is considered to be of degree $n$. This grading on $T(V)$ we call the* tensor *grading on $T(V)$.*

Finally, we want to introduce the definitions of certain kinds of rings which in the sequel we shall meet more often:

**Definition 1.1.10** *Suppose $R$ is a graded ring with a grading over an abelian group $G$. Then we set the following definitions:*

1. *If $G = \mathbb{Z}_2^n$, we call $R$ an $n$-ring.*
2. *If all nonzero homogeneous elements of $R$ are invertible, $R$ is called a* pseudo division ring.
3. *If $R$ has no proper homogeneous two-sided ideals, $R$ will be called* graded simple *or* simple *for short if the context is sufficiently clear to avoid confusion.*
4. *Further, we recall that a two-sided ideal $\mathfrak{p}$ of the underlying ungraded ring is called prime (completely prime) if for $a, b \in R$ such that $arb \in \mathfrak{p}$ for all $r \in R$ ($ab \in \mathfrak{p}$) implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. In the graded category, we correspondingly set the following definitions: if $R$ is a $G$-graded ring, the* prime ideals *respectively* completely prime *ideals of $R$ are by definition precisely those prime (completely prime) ideals of the underlying ring which are homogeneous. An ideal is called* pseudoprime *if for* homogeneous *$a, b \in R$ the implication $\forall r \in R.arb \in \mathfrak{p} \Rightarrow a \in \mathfrak{p} \vee b \in \mathfrak{p}$ holds (respectively completely pseudoprime if $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p} \vee b \in \mathfrak{p}$ already).*

*The corresponding concepts for $k$-algebras and the concept of graded simple modules are defined in the natural way.*

We believe that in the course of this thesis it will become reasonably apparent that setting these definitions makes some sense. For now, we will only remark that some of the terminology was inspired by our needs as well as by the existence of similar terminology for the super case. Literature in this direction may be found e.g. in (Deligne: *Notes on spinors* in [20]).

## 1.2 Color-commutative rings and algebras: first definitions

### Motivation: the super case

Of particular interest to us will be algebras which satisfy a weakened commutativity condition. We will call these algebras *color-commutative*. An in our view particularly intuitive - even though as we will see in some ways anomalous - special case of these is given by $\mathbb{Z}_2^n$-graded algebras satisfying a sign condition which is a very natural generalization of supercommutativity, namely

$$ab = (-1)^{|a|^t|b|}ba$$

for $a, b \in A$ homogeneous elements. We will call these $n$-commutative algebras. For purposes of motivation, we quickly remind the reader of some examples of supercommutative superalgebras. We will assume for these examples that 2 is invertible in the base ring $k$. Alternatively, we could also ask for the slightly weaker condition that all modules appearing be 2-torsionfree.

**Example 1.2.1** *Let $V$ be a module over $k$ and $A = \bigwedge^\bullet V$ be the exterior algebra over $V$. Setting $|v| = 1$ for all $v \in V$ and extending this to a grading on the exterior algebra, we obtain a superalgebra which clearly is supercommutative.*

**Example 1.2.2** *Taking any commutative algebra $A$, we can give it a superalgebra structure making it into a supercommutative superalgebra simply by setting $deg(a) = 0$ for any $a \in A$.*

**Remark 1.2.3** *While the above construction gives a very natural embedding of the category of commutative algebras into the category of supercommutative superalgebras, one should keep in mind that there exist superalgebras which are commutative after forgetting the graded structure but which are nonetheless not supercommutative. An easy example of this is given by the algebra $K[X]$ with grading given by $deg(X) = 1$. On the other hand, it is quite possible for a commutative algebra to allow other gradings than the trivial one which make it into a supercommutative superalgebra. For instance, the algebra of dual numbers $K[\epsilon] \cong K[X]/(X^2)$ can be viewed as a superalgebra coming from a commutative algebra in the usual way, but it can also be viewed e.g. as the exterior algebra of a one-dimensional $K$-vector space. It is for these reasons that we will in this thesis not adopt a common abuse of language and instead always distinguish clearly between superalgebras which happen to be commutative after forgetting the grading and supercommutative superalgebras.*

**Example 1.2.4** *Let $V$ and $W$ be finite-dimensional vector spaces of dimensions $m$ and $n$ respectively over our field $K$. It is well known that then, the symmetric algebra over $V$ is isomorphic to the commutative polynomial ring in $m$ variables over $k$. We set*

$$k[X_1, X_2, \ldots, X_m|Y_1, Y_2, \ldots, Y_n] := S(V) \otimes \wedge^\bullet W$$

*and $deg(X_i) := 0, deg(Y_i) := 1$. This gives an important example of a supercommutative superalgebra, the polynomial algebra in $m$ even and $n$ odd variables.*

**Color commutative rings: Definition and examples**

Supercommutative superalgebras are a special case of a wider class of algebras which behave similarly to commutative algebras in many respects. The development and study of a cohomology theory for these algebras corresponding to Harrison cohomology for commutative algebras will be among the main topics of this thesis, so we introduce them here:

**Definition 1.2.5** *Suppose that* $(R, G, \chi)$ *is a* $G$*-colored ring or a* $G$*-colored* $k$*-algebra. Suppose that for homogeneous elements* $a, b \in R$ *we have the commutation rule*

$$ab = \chi(|a|, |b|)ba$$

*Then* $R$ *is called* color-commutative*. Alternatively, we will in this case also refer to* $R$ *as* $(G, \chi)$*-commutative or simply as* $G$*-commutative if the context is sufficiently clear.*
*If* $R$ *is not viewed as carrying a grading, but if a suitable grading can be given to it,* $R$ *shall be called* ungraded $G$*-commutative.* $R$ *is called* even *if the graded components of* $R$ *with indices outside the subset of* $G$ *given by the equation* $\chi(g, g) = 1$ *are all zero.*
*If* $G = \mathbb{Z}_2^n$ *and* $\chi(v, w) := (-1)^{v^t w}$*, we call* $R$ $n$-commutative*. If* $G = \mathbb{Z}_2^n$ *and* $\chi(v, w) := (-1)^{v^t S w}$ *with* $S \in M_n(\mathbb{Z}_2)$ *a symmetric matrix, we call* $R$ generalized $n$*-commutative.*

Color-commutative rings have been previously investigated in the literature, mostly in relation to problems concerning color Lie algebras as in (Chen, van Oystaeyen, Petit [14]), (Passman [46]), (Su, Zhao, Zu [53]). Also, (Avramov, Gasharov, Peeva [2]) use what in our terminology would be even color-commutative algebras over $G = \mathbb{Z}^n$ in work on complete intersections.
Sometimes, we will encounter a situation where $\chi$ is not supposed skew-symmetric. In this case, we will abuse the same terminology introduced here but under these circumstances we ask that $\chi$ induce group homomorphisms on both sides, which without skew-symmetry of course is no longer automatic.
We will now illustrate these definitions with some examples and remarks:

**Remark 1.2.6** *Suppose that* $A$ *is a* $(G, \chi)$*-commutative* $k$*-algebra, with* $k$ *an integral domain. Assume that as a* $k$*-module,* $A$ *is* 2*-torsionfree. Then, the even subgroup of* $G$*, i.e. the group of elements satisfying* $\chi(g, g) = 1$*, is a subgroup of index at most two. If* $a \in A_g$ *is a nonzero homogeneous element and* $\chi(g, g) \neq 1$*, then we find under these circumstances* $\chi(g, g) = -1$ *and* $a^2 = 0$*.*

**Proof** Set $\varphi(g) := \chi(g, g)$ for any $g \in G$, then $\varphi : G \to k^*$ is a group homomorphism, since

$$\varphi(x + y) = \chi(x, x)\chi(x, y)\chi(y, x)\chi(y, y) = \varphi(x)\varphi(y)$$

due to the antisymmetry of $\chi$. Clearly, for any $g \in G$ we have $\varphi(g)^2 = 1$ also due to antisymmetry of $\chi$, which because $k$ was an integral domain means $\varphi(g) \in \{-1, 1\}$. The even subset of $G$ is the kernel of $\varphi$, so a subgroup of index one or two. Since multiplication by two was supposed to be an injection on $A$, it follows also that $a^2 = 0$ whenever $a \in A_g$ with $\chi(g, g) = -1$.

**Example 1.2.7** *Both supercommutative and commutative superalgebras are easily seen to be examples of* 2*-commutative algebras.*

Adding to the previous example one can remark that indeed it is obvious that a $\mathbb{Z}_2$-graded algebra having a bicharacter with target a subset of $\{-1, 1\}$ must be either commutative or supercommutative. However, it is worth noting that if the commutative base ring contains zero-divisors, the definitions given may allow the construction of 'evil twins' of supercommutative superalgebras. For instance, taking as base the paracomplex numbers given by $k = \mathbb{R}[j]$ with $j^2 = 1$, we can define a 'supercommutative' algebra over $k$ by adding an odd variable $X$ and imposing for homogeneous elements the commutation relation $ab = j^{|a||b|}ba$. The result is the ring $k[X]/(X^2 - jX^2)$. In particular, in this case odd variables need no longer square to zero.

**Example 1.2.8** *Set*

$$A := k < X_1, X_2, Y_1, Y_2 > /(X_iY_j = Y_jX_i, X_1X_2 = -X_2X_1, Y_1Y_2 = -Y_2Y_1, X^2, Y^2)$$

*and set* $deg(X_i) = (1, 0)^t, deg(Y_i) = (0, 1)^t$. *Then* $A$ *becomes a* 2*-commutative algebra which is neither commutative nor supercommutative.*

**Example 1.2.9** *The algebra*

$$k < X, Y > /(XY + YX)$$

*can be made into a* 3*-commutative algebra by setting* $deg(X) := (1, 1, 0)^t, deg(Y) := (0, 1, 1)^t$. *It can also be made into a generalized* 2*-commutative algebra by setting* $deg(X) := (1, 0)^t, deg(Y) := (0, 1)^t$ *and using*

$$S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

*as signature matrix.*

**Example 1.2.10** *Let* $G$ *be an abelian group,* $k$ *a commutative base ring,* $M$ *a* $G$*-graded* $k$*-module and* $\chi : G \to k^*$ *be a bicharacter. Then, the tensor algebra* $T(M)$ *over* $M$ *naturally becomes a* $G \oplus \mathbb{Z}$*-graded algebra in which for homogeneous* $x, y \in V$ *elements of the form* $x \otimes y - \chi(|x|, |y|)y \otimes x$ *are homogeneous. Hence, the two-sided ideal* $I$ *generated by these elements is a homogeneous ideal in* $T(M)$ *and we can set*

$$\bigvee(M) := k[M] := T(M)/I$$

*which will by construction give a* $\mathbb{Z}$*-graded* $(G, \chi)$*-commutative algebra. We will call it the* graded symmetric algebra *over* $M$. *If* $M$ *happens to be free as a graded* $k$*-module (which is not equivalent to being free as a* $k$*-module) and* $B$ *is a basis of* $M$ *consisting of homogeneous elements, we will also refer to* $k[M]$ *as the algebra of* $G$*-commutative polynomials over* $k$ *with variables in* $B$ *and denote it by* $k[B]$. *When defining an* $G$*-commutative polynomial ring in this way, by giving a base ring and a set of variables, the degrees of the variables shall be indicated by suitable superscripts, e.g. a variable of degree* $(0, 1, 0)$ *may be written as* $X^{010}$.

**Example 1.2.11** *Consider the algebra* $A := \mathbb{R}[X^{110}, Y^{101}, Z^{011}]$ *and therein the (two-sided) homogeneous ideal* $(X^2 = -1, Y^2 = -1, Z^2 = -1, XYZ = -1)$. *Then, the quotient* $A' := A/I$ *gives a representation of the quaternions as a three-commutative, even* $\mathbb{R}$*-algebra.*

Of course, the same procedure will work e.g. also with the para-quaternions. These are well known to be representable as $M_2(\mathbb{R})$, which leads us naturally to the question which matrix algebras fit into our framework. It will follow from (Lemma 1.2.24) that finite-dimensional $n$-commutative graded simple algebras over a field have to be of vector space dimension a power of two, which means that $M_m(K)$ for $m \neq 2^i$ for all $i$ carries no $n$-commutative grading for any $n$; this together with the existence of $n$-commutative structures in general on finite-dimensional orthogonal Clifford algebras suggests viewing $n$-commutative algebras as in some sense generalized analogs of Clifford algebras. In the full $G$-commutative category, a wide range of matrix algebras can be made to fit in, as the following related example shows:

**Example 1.2.12** *Let $k$ be a commutative ring having a subfield $K \subseteq k$ containing a primitive $n$-th root of unity $q$. Then, the matrix algebra $M_n(K)$ can be given a grading making it color-commutative with $G := \mathbb{Z}_n^2$.*

**Proof** We first define $\chi : G \times G \to k^*$ by $\chi(v, w) := q^{v^t S w}$ where

$$S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_n)$$

We will show in a minute that the two matrices

$$A := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & q & 0 & \dots & 0 \\ 0 & 0 & q^2 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & q^{n-1} \end{pmatrix} \text{ and } B := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

form a system of generators of $M_n(k)$ as $k$-algebra. We immediately note that $A^n = E_n, B^n = E_n$, meaning that for $m \in \mathbb{Z}_n$ the expressions $A^m, B^m$ are well-defined. Given that, a grading with suitable properties will be given by setting

$$deg(A) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, deg(B) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and what remains to be shown is then that the family $\{A^i B^j : i, j \in \mathbb{Z}_n\}$ is a $k$-basis of $M_n(k)$ and that multiplication and commutation of these matrices behave as prescribed by the grading. Assume first that $k = K$. Consider now the set $S := \{A^i B^j : 0 \leq i \leq n-1, 0 \leq j \leq n-1\}$. It is clearly contained in the subalgebra $K < A, B > \subseteq M_n(K)$ generated by $A$ and $B$ over $K$ and we will show that it is a linearly independent family, which will imply that $dim_K(K < A, B >) = n^2$

and hence $K < A, B >= M_n(K)$.To see this, we calculate

$$A^kB^l = \begin{pmatrix} 0 & 0 & \dots & & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & & 0 & 0 & q^k & \dots & 0 \\ \vdots & 0 & \dots & & & & & \ddots & \vdots \\ 0 & 0 & \dots & & 0 & & & \dots & q^{k(n-l-1)} \\ q^{k(n-l)} & 0 & \dots & & & & & \dots & 0 \\ 0 & q^{k(n-l+1)} & 0 & \dots & & & & \dots & 0 \\ \vdots & 0 & \ddots & \dots & & 0 & & \dots & \vdots \\ 0 & \dots & & & q^{k(n-1)} & 0 & & \dots & 0 \end{pmatrix}$$

and see that any nontrivial linear combination

$$\sum_{i,j} \lambda_{ij} A^i B^j = 0$$

decomposes into subterms

$$\sum_{i} \lambda_{ij} A^i B^j = 0$$

which implies $\lambda_{ij} = 0$ for all $i, j$ since $B^j$ is an invertible matrix and the matrices $E_n, A, A^2, \dots, A^{n-1}$ form a linearly independent family due to the minimal polynomial of $A$ being $X^n - 1$ (note that $A$ has $n$ distinct eigenvalues). Further, it is easy to check that $B^n A^m = q^{nm} A^m B^n$ in line with the proposed grading and commutation rule and that the graded structure makes all graded components of the graded algebra generated over $k$ by $A$ and $B$ even. Hence, we have proven all statements in the case $K = k$.
In the general case, $A$ and $B$ also generate all of $M_n(k)$, since we can get any matrix by adding suitable multiples of elementary matrices. This concludes the proof.

**Example 1.2.13** *Another example of even $n$-commutative algebras is given by the finite-dimensional Clifford algebras over fields of characteristic $\neq 2$. To see this, we will first recall shortly the definition of a Clifford algebra. Let $V$ be a vector space over $K$, where $K$ is a field with $char(K) \neq 2$, and let $Q : V \to K$ be a quadratic form on $V$. Then the Clifford algebra $C(V, Q)$ associated to $V$ and $Q$ is defined as the quotient of the tensor algebra $T(V)$ over $V$ by the two-sided ideal generated by the elements $v \otimes v - Q(v)$. Note that this is a homogeneous ideal when the tensor algebra is viewed as being $\mathbb{Z}_2$-graded, with the grading coming from the ordinary $\mathbb{Z}$-grading modulo two. Equivalently, one can define a bilinear form*

$$B(v, w) := \frac{Q(v + w) - Q(v) - Q(w)}{2}$$

*and impose on the tensor algebra relations of the form*

$$v \otimes w + w \otimes v = B(v, w)$$

*Suppose $V$ finite-dimensional. By e.g. (Crumeyrolle [16], Prop. 1.2.3) we can choose an orthogonal basis of $V$, say $\{e_1, e_2, \dots, e_n\}$, and then the relations given above simplify to $e_ie_j = -e_je_i$ and $e_i^2 = Q(e_i) \in K$. It is obvious that we can give these elements suitable degrees in $\mathbb{Z}_2^m$, with $m = n$ if $n$ odd and $m = n - 1$ if $n$ even.*

We would like to point out also one algebra which is color-commutative and looks a bit like a Weyl algebra:

**Example 1.2.14** *Let $K$ be a field and suppose that $k := K[[t]]$. Set $G := \mathbb{Z}^2$ and define on $G$ the bicharacter*

$$\chi(u, v) := (1 + t)^{u^t S v}$$

*where*

$$S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

*Then if*

$$A := k[X^{10}, Y^{01}]$$

*we find $XY = YX + tYX$. We cannot make this into a Weyl algebra proper as $tXY \in A_{11}$ and so $tXY = 1$ would be an inhomogeneous relation.*

**Example 1.2.15** *The next example are quasicommutative algebras in the sense of e.g. (Levandovskyy [37], Def. 3.2).*
*There, a* quasicommutative *algebra $A$ over a field $K$ is defined as a $K$-algebra of the form*

$$K < X_1, \ldots, X_n > /I$$

*where $I$ is the two-sided ideal generated by the terms $X_i X_j - \lambda_{ij} X_j X_i$ with $\lambda_{ii} = 1$ and otherwise $\lambda_{ij}\lambda_{ji} = 1$. Such an algebra has a natural $\mathbb{Z}^n$-grading given by setting $deg(X_i) = e_i$. Setting*

$$\chi(e_i, e_j) := \lambda_{ij}$$

*and extending linearly, we see that the quasicommutative algebra $A$ is $(\mathbb{Z}^n, \chi)$-commutative.*

**Remark 1.2.16** *Suppose that $G$ is a not a priori necessarily abelian group and that $R$ is a $G$-graded ring. Suppose further that $\chi : G \times G \to R^*$ is a map satisfying for $g_1, g_2, g_3 \in G$ the conditions*

$$\begin{aligned} \chi(g_1, g_2) &\in R_{[g_1, g_2]} \\ \chi(g_1, g_2)\chi(g_2, g_1) &= 1 \text{ and} \\ \chi(g_1 g_2, g_3) &= \chi(g_1, g_3)\chi(g_2, g_3) \end{aligned}$$

*where $[\cdot, \cdot]$ denotes the group-theoretic commutator bracket. Consider then the subring $S := R_{[G,G]}$ defined by taking the direct sum over the components of $R$ indexed by the elements of the commutator subgroup of $G$. Then $S$ is a commutative ring and contained in the center of $R$. $R$ supports a color-commutative structure as $S$-algebra with the same bicharacter as the original and a $G/[G, G]$-grading induced from the original one by the group homomorphism $G \to G/[G, G]$.*

**Proof** The proof proceeds by showing commutation for successively more general homogeneous elements of $R$. First, we note that for any $g \in G$ the element $\chi(g, g)$ is in the center of $R$ since it is of degree zero. Skew-symmetry of $\chi$ implies that $\chi(g, g)$ is self-inverse for all $g \in G$. As $\chi(g, h)$ is by definition of degree $ghg^{-1}h^{-1}$ for all $g, h \in G$, we find that also $\chi(g_1, h)$ and $\chi(g_2, h)$ commute for any $g_1, g_2, h \in G$:

$$\begin{aligned} \chi([g_1, h], [g_2, h]) &= \chi(g_1, g_2)\chi(g_1, h)\chi(g_1, g_2)^{-1}\chi(g_1, h)^{-1}\chi(h, g_2)\chi(h, h)\chi(h, g_2)^{-1}\chi(h, h)^{-1} \\ &\cdot \chi(g_1, g_2)^{-1}\chi(g_1, h)^{-1}\chi(g_1, g_2)\chi(g_1, h)\chi(h, g_2)^{-1}\chi(h, h)^{-1}\chi(h, g_2)\chi(h, h) \\ &= \chi(g_1, g_2)\chi(g_1, h)\chi(g_1, g_2^{-1})\chi(g_1, h^{-1})\chi(g_1, g_2^{-1})\chi(g_1, h^{-1})\chi(g_1, g_2)\chi(g_1, h) \\ &= \chi(g_1, [g_2, h])\chi(g_1, [g_2, h])^{-1} = 1. \end{aligned}$$

As a consequence, one sees that for any $g, h_1, h_2 \in G$ we have

$$\chi([h_1, h_2], g) = \chi(h_1, g)\chi(h_2, g)\chi(h_1, g)^{-1}\chi(h_2, g)^{-1} = 1$$

which means that any element of $R$ commutes with any element in $S$. This proves that $S$ is in the center of $R$. Since $\chi$ is constant equal to 1 on $[G, G]$, it induces also a bicharacter on $G/[G, G]$, thereby proving the statement completely.

**Example 1.2.17** *As an application of the preceding remark, we construct one last example of a $(G, \chi)$-commutative algebra. Take $k$ to be an arbitrary commutative ring and $G$ to be a group such that $[\cdot, g]$ induces an endomorphism of $G$ for every $g \in G$. Then the group algebra $k[G]$ with $\chi(g, h) := [g, h]$ satisfies all properties required by the previous remark. Consequently, $k[G]$ becomes a color-commutative algebra over the base ring $k[H]$ in this case, where $H := [G, G]$.*

While the property required of $G$ in the preceding remark is no doubt special, there exist quite a few nonabelian groups which satisfy it. For instance, one can check by direct calculation that for $p$ a prime number the two nonabelian groups of order $p^3$ satisfy this property. By taking the direct product with an abelian group of arbitrary order, one can then construct a group of size $n$ exhibiting this property for every $n$ which is a multiple of such a prime power. We have not tried to prove that there are no such finite groups of cubefree size.

### Properties, constructions, and further examples of color-commutative algebras

Apart from its objects, in order to define the category of color-commutative algebras, we need to specify the set of morphisms between any two of them. These are given simply as in the general case of graded algebras by ordinary algebra morphisms which in addition are degree zero maps. We define also $\mathbf{Hom}_A(M, N)$ as in the case without commutation constraints. Note that while we will need these quasihomomorphisms sometimes, they need not have very nice properties as they are right-linear but not left-linear over $A$. For instance, in general the image and kernel of such a map are not homogeneous submodules of the target and source module respectively. The graded structure is obviously much better preserved by degree zero maps.
The following (Lemma 1.2.20) is intended to establish some basic properties of $G$-commutative algebras which have well-known analogs in the commutative situation. The hope here is obviously that $G$-commutative algebras will be close enough to the commutative situation that at

least basic constructions in algebraic geometry or commutative algebra will carry over to them without too much change. First, we need yet another definition.

**Definition 1.2.18** *Let $R$ be a $(G,\chi)$-commutative ring. $R$ is called an $G$-*commutative integral domain *if it has no zero-divisors and a* pseudo integral domain *if it has no homogeneous zero-divisors. A $G$-commutative pseudo division ring is called a pseudofield.*

**Remark 1.2.19** *One could ask at this point whether pseudofields deserve their name, as some examples - e.g. the paracomplex numbers, which have zero-divisors - are as ungraded algebras not very field-like. However, it is indeed the case that many nice properties carry over from fields to pseudo-fields. The reason for this is that the restrictions imposed by the graded structure protect those properties. For instance, it is an easy exercise in pseudo linear algebra to see that $(G,\chi)$-symmetric modules over pseudofields always have bases, that two finite bases are of the same length, that bases of subspaces can always be extended to bases of the ambient pseudo vector space, that arbitrary degree-preserving functions on a basis can be extended linearly and so on.*

**Lemma 1.2.20** *Let $R$ be a $(G,\chi)$-commutative ring. Recall that we have adopted the convention that by default, an ideal in a graded ring means a homogeneous ideal. Then we have:*

1. *All ideals of $R$ are two-sided. Ideals $I \subseteq R$ which are generated by a single homogeneous element $x \in R$ are called* principal *and they are of the form $Rx$.*

2. *Pseudo-prime ideals in $R$ are completely pseudoprime.*

3. *$R$ is a pseudofield exactly if $R$ is simple (as a $G$-ring).*

4. *Every ideal in $R$ lies in a maximal ideal. Maximal ideals are pseudoprime.The quotient of a $G$-commutative ring by a maximal ideal is a pseudofield. The quotient of $R$ by a pseudoprime ideal is a pseudo integral domain.*

5. *Set $\mathfrak{n} := (U)$, with $U$ being the subset formed by all elements of 'odd degree', i.e.*

$$U := \bigcup_{\substack{g\in G \\ \chi(g,g)\neq 1}} R_g$$

*Suppose $\mathfrak{p} \subset R$ is a pseudoprime ideal and that for any $g \in G$ we have $\chi(g,g) = 1$ or $1-\chi(g,g)$ not a zero divisor. Then $\mathfrak{n} \subseteq \mathfrak{p}$.*

**Proof** 1. Assume that $R$ is a $(G,\chi)$-commutative ring. Assume further that $I \subset R$ is a proper (homogeneous) left ideal of $R$. We have to prove that for any $r \in I$ and any $s \in R$ also the right multiplication $r$ with $s$ is in $I$, i.e. $rs \in I$. To this end, we note that it suffices to check this on homogeneous elements, since by definition of a homogeneous ideal $r$ decomposes into a sum of elements that are homogeneous and still elements of $I$. But, for $r$ and $s$ homogeneous $rs = \chi(|r|,|s|)sr \in I$, so if $sr \in I$ then also $rs \in I$. To prove the second claim, it suffices to show that $Rx$ is a two-sided ideal, which is clear since we have just proven that (homogeneous) left ideals in $R$ are always two-sided.

2. We first note that the condition for an ideal $\mathfrak{p} \subseteq R$ to be pseudoprime is clearly equivalent to asking that $a, b \in R$ homogeneous such that $arb \in \mathfrak{p}$ for all *homogeneous* $r \in R$ imply that $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Taking any pseudoprime ideal $\mathfrak{p} \subseteq R$ and any homogeneous $a, b \in R$ with $ab \in \mathfrak{p}$, we then see without difficulty that $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ since for any homogeneous $r \in R$ we have $arb = \chi(|a|, |r|)rab \in \mathfrak{p}$, as desired.

3. If $R$ is a pseudofield, by definition all homogeneous elements are invertible and hence every nonzero homogeneous element generates the whole ring. If on the other hand $R$ is graded simple, then every nonzero homogeneous element generates the whole ring, which due to $G$-commutation means $Rx = R$ for every homogeneous $0 \neq x \in R$, and so invertibility of $x$.

4. The standard Zornian argument shows that every homogeneous ideal is contained in a maximal homogeneous ideal; this part does not make any use of commutation conditions and would work just as well for two-sided ideals of an arbitrary $G$-graded ring for instance. That an (homogeneous) ideal $I \subseteq R$ is pseudoprime exactly if $R/I$ is a pseudo integral domain is evident from the definitions. Likewise, such an $I$ is maximal iff $R/I$ is simple, which in turn is the case iff $R/I$ is a pseudofield. Finally, since pseudofields are in particular pseudo integral domains, this implies that maximal ideals are necessarily pseudoprime.

5. By definition, $\mathfrak{n}$ has a system of homogeneous generators, with every element of $\mathfrak{n}$ being an $R$-linear combination of elements of

$$U = \bigcup_{\substack{g \in G \\ \chi(g,g) \neq 1}} R_g$$

Suppose $x \in U$ with degree $g \in G$ odd, then $(1 - \chi(g, g))x^2 = 0$. Since $1 - \chi(g, g)$ is not a zero divisor, this means $x^2 = 0$ and so $x \in \mathfrak{p}$, implying $\mathfrak{n} \subseteq \mathfrak{p}$ as desired.

Some of the statements in the preceding lemma invite natural attempts of strengthening. The following sequence of remarks is meant to show counterexamples to such attempts:

**Remark 1.2.21** *Prime ideals in $G$-commutative rings are not in general completely prime. A counterexample is given e.g. by the zero ideal in $M_2(\mathbb{R})$, when we equip that ring with a suitable graded structure.*

**Remark 1.2.22** *To see that pseudofields are not necessarily division rings and that maximal homogeneous ideals are not necessarily prime, consider the case of the paracomplex numbers $\mathbb{R}[j]$, where $j^2 = 1$ and $j$ is supposed to be of degree $(1, 1)^t$. With the grading so given, it is clear that the paracomplex numbers form a pseudofield and the zero ideal is maximal among proper homogeneous ideals, but they are certainly no division ring. Also, since they form a strictly commutative ring, the existence of zero divisors in $R[j]$ is sufficient to prove that the zero ideal is not prime.*

**Remark 1.2.23** *Define $\mathfrak{n}$ as in the lemma. Suppose that $R$ is a $(G,\chi)$-commutative ring without the constraint that $1-\chi(g,g)$ not be a zero divisor unless itself zero. In this case, $\mathfrak{n}$ is not necessarily contained in every pseudoprime. The simplest example of a $G$-commuting algebra where $\mathfrak{n}$ is not contained in every pseudoprime ideal may be $\mathbb{Q}[j,X]/(j^2=1, jX^2=X^2)$ with $G=\mathbb{Z}_2$, $\chi(v,w):=j^{vw}$, $|X|=1$, $\mathbb{Q}[j]/(j^2-1)$ as basefield and $\mathfrak{p}=(j-1)$. Then $A/\mathfrak{p}$ is the integral domain $\mathbb{Q}[X]$, so $\mathfrak{p}$ is certainly pseudoprime and as certainly $X$ is not inside the ideal generated by the element $j-1$. However, one can prove also in this situation that at least $(1-\chi(|x|,|x|))x$ is contained in every pseudoprime for every odd $x$.*

The significance of the last remark lies in the fact that it shows that a certain trick which is sometimes useful when dealing with questions about prime ideals of supercommutative superalgebras, namely reduction to similar questions about prime ideals of the commutative ring $A/\mathfrak{n}$, is not available in general even for $\mathbb{Z}_2$-graded color-commutative rings.
We would like to also mention the following observation about pseudofields:

**Proposition 1.2.24** *Let $R$ be a $(G,\chi)$-commuting pseudofield and $K:=R_0$. Then $K$ is a field, $R$ is a $K$-algebra, all odd components of $R$ are zero, and the dimension of any even component over $K$ is either zero or one. If $G$ admits a structure as a vector space over $\mathbb{F}_p$, then the dimension of $R$ as a $K$-vectorspace is a power of $p$ if $R$ is finite-dimensional.*

**Proof** It is clear that $K$ as defined is a field. It is likewise clear that all odd components must be zero, as they contain only zero-divisors (indeed, as $R$ is an algebra over a field, only nilpotents) and thus no units. The multiplication in $R$ makes all homogeneous components of $R$ into $K$-vectorspaces. Suppose that $g\in G$ with $R_g\neq 0$ and $0\neq x\in R_g$, then $x$ is invertible as $R$ is a pseudofield and $x^{-1}\in R_{-g}$. Multiplication by $x^{-1}$ gives an invertible $K$-linear map from $R_g$ to $R_0$, implying $dim_K(R_g)=1$. If $g,h\in G$ and $0\neq x\in R_g, 0\neq y\in R_h$, then $0\neq xy\in R_{g+h}$ as well as $0\neq x^{-1}\in R_{g^{-1}}$, so the indices $g\in G$ for which $R_g\neq 0$ form a subgroup of $G$ and in the case that $G$ admits a vector space structure over $F_p$ a subspace of $G$, which gives the statement about dimension.

**Corollary 1.2.25** *In particular, the matrix ring $M_r(K)$ over a field $K$ can be endowed with an $n$-commutative structure only if $r$ is a power of two.*

**Proof** $M_r(K)$ is simple, hence if endowed with a suitable $n$-commutative structure also graded-simple and therefore a pseudofield. With the notations of (Lemma 1.2.24) the $n$-commutative case corresponds to $G=\mathbb{F}_2^n$, so the vector space dimension of $M_r(K)$ over the subfield $K'=M_r(K)_0$ in degree zero of the grading must be a power of two. Since $K'$ is in the center of $M_r(K)$, we know that $K'=K$. Therefore $dim_K(M_r(K))=r^2$ and by extension $r$ itself is forced to be a power of two.

When we talk of a module over a $G$-graded algebra, we will by default mean a module in the graded sense. The following remark points out some easy properties of the graded tensor product and of the opposite algebra construction with regards to color-commutative algebras:

**Remark 1.2.26** *Let $A$ and $B$ be $(G,\chi)$-colored associative $k$-algebras. Then the graded $k$-module $A\otimes B$ becomes an $(G,\chi)$-colored associative algebra by setting*

$$a_1\otimes b_1\cdot a_2\otimes b_2 := \chi(|b_1|,|a_2|)a_1a_2\otimes b_1b_2$$

*for homogeneous elements $a_i, b_i$. Let $A$ be a $(G,\chi)$-colored associative $k$-algebra. Then $A$ is color-commutative if and only if $A^{op}$ is. If $B$ is another $(G,\chi)$-colored algebra and if both $A$ and $B$ are $(G,\chi)$-commutative, then so is $A\otimes B$.*

**Proof** That $A^{op}$ is $G$-commutative if and only if $A$ was is obvious.
What remains to be proven is $G$-commutativity of $A\otimes B$ in the case that $A$ and $B$ were $G$-commutative. Using again homogeneous generators, we get by simple calculation:

$$(a_2\otimes b_2)(a_1\otimes b_1) = \chi(|b_2|,|a_1|)\chi(|a_2|,|a_1|)\chi(|b_2|,|b_1|)a_1a_2\otimes b_1b_2$$

and using skew-symmetry of the bicharacter we hence see that this is $\lambda(a_1\otimes b_1)(a_2\otimes b_2)$ with

$$\lambda = \chi(|b_2|,|a_1|)\chi(|a_2|,|a_1|)\chi(|b_2|,|b_1|)\chi(|a_2|,|b_1|) = \chi(|a_2|+|b_2|,|a_1|+|b_1|).$$

We state here some basic properties of the $(G,\chi)$-colored tensor product:

**Proposition 1.2.27** *Let $A, B, C$ be $(G,\chi)$-colored associative algebras. Then, we have*

$$\begin{aligned} A\otimes B &\cong B\otimes A \\ A\otimes(B\otimes C) &\cong (A\otimes B)\otimes C \\ (A\oplus B)\otimes C &\cong (A\otimes C)\oplus(B\otimes C) \end{aligned}$$

*where on direct sums of $k$-algebras we install componentwise multiplication and the Hadamard grading (i.e., a $(a,b)$ is homogeneous of degree $g$ if and only if both $a$ and $b$) are individually). The claimed isomorphies are isomorphies of graded algebras.*

**Proof** The proof runs as in the ungraded case, by using the universal property of the tensor product to construct suitable $k$-linear maps between the tensor products appearing in each statement and then showing that the maps so obtained are indeed isomorphisms of graded rings. The only difference to the usual case is that one has to check compatibility of the isomorphisms used to the additional signs induced by the color structure and in the case of the first isomorphism this has the consequence that instead of the twist map $\tau$ given by

$$\tau(a\otimes b) := b\otimes a$$

one has to use

$$\tau_\chi(a\otimes b) := \chi(|a|,|b|)b\otimes a$$

for homogeneous $a$ and $b$.

**Remark 1.2.28** *Suppose now that $A$ and $A'$ are $G$-commutative $k$-algebras and that $M$ is a $G$-graded $k$-module together with multiplications $A \times M \to M$ and $A' \times M \to M$ (and the corresponding operations on the right) making it into an bimodule over both $A$ and $A'$. Suppose further that both module structures are compatible, i.e. that we have*

$$a(a'm) = \chi(|a|, |a'|)a'(am)$$

*then $M$ becomes a bimodule over the algebra $A \otimes A'$ by setting*[2]

$$\begin{aligned} (a \otimes a')m &:= aa'm \\ m(a \otimes a') &:= maa' \end{aligned}$$

*for homogeneous elements and extending linearly. If $M$ was a $G$-symmetric bimodule over $A$ and $A'$, then it is also $G$-symmetric over $A \otimes A'$.*

**Proof** The given operations are distributive and compatible with the $k$-module structure and the grading by construction. What remains to be proven are compatibility of scalar and ring multiplication and in the case that $M$ was a $G$-symmetric module also symmetry. It suffices to check these properties on homogeneous generators. Suppose now $a_1, a_2, b_1, b_2 \in A$ and $m \in M$ homogeneous. We then calculate

$$((a_1 \otimes b_1)(a_2 \otimes b_2))m = \chi(|b_1|, |a_2|)a_1 a_2 b_1 b_2 m$$

and

$$(a_1 \otimes b_1)((a_2 \otimes b_2)m) = a_1 b_1 a_2 b_2 m.$$

By the compatibility conditions on the multiplicative operations of $A$ and $A'$ on $M$, the expressions on the righthand side of both equations are equal, as desired. The same arguments apply as well for right multiplication.
Assume now that $M$ is a $(G, \chi)$-symmetric module over both $A$ and $A'$. Then for homogeneous $a, b \in A$ and $m \in M$ we calculate

$$(a \otimes b)m = abm = \chi(|a|, |m| + |b|)\chi(|b|, |m|)mba = \chi(|a| + |b|, |m|)mab$$

and $m(a \otimes b) = mab$ as desired.

**Example 1.2.29** *We want to illustrate the preceding construction by an example which will be useful more often. Suppose for this that $A$ is a $(G, \chi)$-graded algebra and that $A^{op}$ is its opposite. Let further $M$ be an $A$-bimodule. Then, $M$ is an $A^{op}$-left module with multiplication defined on homogeneous elements by $a.m := \chi(|a|, |m|)ma$ and an $A^{op}$-right module with multiplication $m.a := \chi(|m|, |a|)am$. The compatibility conditions of (Rem. 1.2.28) are fulfilled for the module operations on $M$ by $A$ and $A^{op}$. Hence we can use the construction of (Rem. 1.2.28) to make*

[2] Note that we suppress parentheses even though for the well-definedness of the terms we are dealing with they are important. The expression $aa'$ for instance is not defined.

*$M$ into a left module over $A \otimes A^{op}$ with multiplication rule given for homogeneous $a, a' \in A$ and $m \in M$ by*

$$(a \otimes a')m := \chi(|a'|, |m|)ama'.$$

*Note that this construction works also in the other direction, recovering the $A$-bimodule structure of $M$ from its structure as $A \otimes A^{op}$-left module. Since also graded $A$-bimodule homomorphisms on $M$ become graded $A \otimes A^{op}$-left module homomorphisms on $M$, we see that this procedure induces an equivalence between the categories of left $A \otimes A^{op}$-modules and $A$-bimodules. An important example along these lines is that the standard multiplication $\mu : A \otimes A \to A$, by virtue of being an $A$-bimodule homomorphism because of associativity of multiplication, becomes a $A \otimes A^{op}$-left module homomorphism.*
*Since we will encounter this construction more often, we will in the future write $A^e$ instead of $A \otimes A^{op}$ and call $A^e$ the* enveloping algebra *of $A$.*

## 1.3 Limits

We will now shortly recall the notions of *direct* and *inverse* limits of a family of objects. Our exposition follows more or less the ones given in (Atiyah, MacDonald [1], Ex. 2.14-2.23) and (Hatcher [33], chapter 3) . After introducing these constructions, we apply them in the $G$-graded category.

### 1.3.1 Direct limits

#### General definitions

We start by reminding the reader of the notion of a *direct system*:

**Definition 1.3.1** *A set $I$ together with a binary relation $\leq$ is called a* directed set *if the binary relation is reflexive, transitive and if for any $a, b \in I$ there is a common upper bound $c \in I$, i.e. if*

$$\forall a, b \in I\ \exists c \in I : a \leq c \wedge b \leq c.$$

*Suppose now that $\mathcal{C}$ is a category and $\mathcal{A} := \{A_i : i \in I\}$ is a family of objects indexed by a directed set, together with for each pair $(i, j)$ with $i \leq j$ a morphism*

$$f_{ij} : A_i \to A_j$$

*subject to the conditions*

$$f_{jk} \circ f_{ij} = f_{ik}$$

*and*

$$f_{ii} = id$$

*for all $i, j, k \in I$ with $i \leq j \leq k$. Then $\mathcal{A}$ is called a* direct system *of objects in $\mathcal{C}$. We use the notation $(A_i, f_{ij})$ to refer to the direct system given by the objects $A_i$, the directed index set $I$, and the morphisms $f_{ij}$.*

Given a direct system $(A_i, f_{ij})$ of objects, the direct limit over $(A_i, f_{ij})$ can be defined uniquely up to isomorphism but in general without any guarantees of existence by the following universal property:

**Definition 1.3.2** *Let $\mathcal{C}$ be a category and $(A_i, f_{ij})$ a directed system in $\mathcal{C}$. Assume that $A \in \mathcal{C}$ is an object in $\mathcal{C}$ and that there is given for each $A_i$ a morphism $\varphi_i : A_i \to A$ such that for any $i \leq j$ the compatibility condition*

$$\varphi_i = \varphi_j \circ f_{ij}$$

*is fulfilled. If now for every other object $B$ together with a family of morphisms $\psi_i : A_i \to B$ satisfying the same compatibility conditions there exists a unique morphism $\rho : A \to B$ such that*

$$\rho \circ \varphi_i = \psi_i$$

*for all $i \in I$, then $A$ is called the* direct limit *of the directed system $(A_i, f_{ij})$. If it exists, we denote the direct limit of the direct system $(A_i, f_{ij})$ by*

$$\lim_{\rightarrow} A_i.$$

In terms of commutative diagrams, these relations can be expressed also as follows:

$$\begin{array}{ccc} A_i & \xrightarrow{f_{ij}} & A_j \\ \downarrow f_{ik} & & \downarrow f_{jk} \\ A_k & \xrightarrow{id} & A_k \end{array}$$

for the compatibility conditions inside the direct system,

$$\begin{array}{ccc} A_i & \xrightarrow{\varphi_i} & A \\ \downarrow f_{ij} & & \downarrow id \\ A_j & \xrightarrow{\varphi_j} & A \end{array}$$

for compatibility between the $\varphi_i$ and the direct system, of course with analogous diagrams for the $\psi_i$ implied, and finally a condition on $\rho : A \to B$, namely

$$\begin{array}{ccc} A_i & \xrightarrow{\varphi_i} & A \\ id \downarrow & & \rho \downarrow \\ A_i & \xrightarrow{\psi_i} & B. \end{array}$$

We will not talk at length about this category-theoretical definition, as we will never need it in full generality. We will, however, quickly mention that, as is usual for things defined through a universal property, the direct limit is unique up to unique isomorphisms if it is defined. The proof of this follows the usual pattern for verifying such uniqueness properties for objects defined through some universal property: first, one checks that if in the definition the objects

$A$ and $B$ are the same and if $\varphi_i = \psi_i$ for all $i$, then the morphism $\rho : A \to A$ must by unicity be the identity. Then, one assumes that $A$ and $B$ are both universal objects and constructs morphisms $\rho_{AB} : A \to B$ and $\rho_{BA} : B \to A$ according to the definitions given above. Their composition $\sigma := \rho_{BA} \circ \rho_{AB}$ is a morphism $\in Mor(A, A)$ satisfying all compatibility conditions imposed on $\rho$ in the definitions, so by our last observation it must be the identity. Since there is only one morphism $\rho : A \to B$ compatible with the other structures, it is trivial that the compatible isomorphism $A \to B$ is uniquely given.

### The $G$-graded category

We will now look more specifically at direct limits in the category of $G$-graded $k$-algebras. First, we look at the case of graded $k$-modules:

**Lemma 1.3.3** *Assume that $k$ is a commutative ring, that $G$ is a commutative monoid and that $(M_i, f_{ij})$ is a direct system of $G$-graded modules and (degree zero) module morphisms. Then, consider $M' := \oplus_{i \in I} M_i$ and let $M''$ be the module generated by all the elements $a_j - f_{ij}(a_i)$, where we identify the $a_i$ with their images in $M'$ and where we assume everything homogeneous. Setting $M := M'/M''$, we get an explicit representation of the direct limit of the $M_i$.*

**Proof** All relations appearing in the definitions are homogeneous since the $f_{ij}$ are degree zero maps. The maps $\varphi_i : M_i \to M$ are defined in the natural way from the embeddings $M_i \to \oplus_{i \in I} M_i$. It is immediate from the definitions that they are compatible with the maps in the direct system. Assume now that $N$ together with module homomorphisms $\psi_i : M_i \to N$ is also compatible with the $f_{ij}$. Then, we have to prove existence of a morphism $\rho : M \to N$ such that

$$\psi_i = \rho \circ \varphi_i$$

for all $i \in I$. To this end, we prove first that every element $\overline{x} \in M$ has a representative $x \in M$ which is in fact in one of the $M_i$. Suppose $\overline{x} \in M$, with representative $x = \sum_{i \in I} x_i \in M'$. Recall that by the definition of $M'$, $x$ has finite support here, i.e. $Supp(x) := \{i \in I : x_i \neq 0\}$ is finite. Now due to $I$ being a directed set there is a $j \in I$ which is an upper bound to all $i \in Supp(x)$. We see then that $\tilde{x} := \sum_{i \in I} f_{ij}(x_j)$, which is an element of $M_j$, is in the same residue class in $M$ as $x$. This means that each $\overline{x} \in M$ can be represented as some $\varphi_i(x)$ for some $i \in I$ and $x \in M_i$. Hence, with $\overline{x} = \varphi_i(x)$ we can define

$$\rho(\varphi_i(x)) = \psi_i(x)$$

and what is left to prove is only that this is a well-defined, graded linear function.
Assume that we have $\varphi_i(x) = \varphi_j(x')$. Then, by definition of the $\varphi_n$, we see $x \in M_i$ and $x' \in M_j$ and $x - x' \in M''$. Without loss of generality we can assume which without $x' = f_{ij}(x)$ and so by compatibility of the $\psi_i$ with the directed system $\psi_i(x) = \psi_j(x')$. This shows that our map is well-defined. Linearity and degree preservation follow easily from the corresponding properties of the $\varphi_i, \psi_i$, so we are done.

**Remark 1.3.4** *Note that with the above notations, $\varphi_i(x) = 0$ for some $x \in M_i$ implies the existence of some $j \in I$ with $i \leq j$ such that $f_{ij}(x) = 0$ already.*

**Proof** Suppose with the above notations that $x \in M_i$ and with $\varphi_i(x) = 0$, that is to say (with the canonical embedding) $x \in M''$. Then by the definition of $M''$ there is a representation of $x$ of the form

$$x = \sum_{i<j}(a_j - f_{ij}(a_j)) + \sum_{i>j}(b_j - f_{ji}(b_j))$$

where $a_j \in A_i$ and $b_j \in A_j$. The sums appearing by definition contain only a finite number of nonvanishing terms and we adopt the convention that $i < j$ if and only if $i \leq j$ and not $i = j$ and $i > j$ if neither $i = j$ nor $i$ and $j$ incomparable nor $i < j$. Because $x \in M_i$ it is easy to see that in the second sum, the factors $b_j$ must all be zero; the second sum disappears altogether. By the same reasoning, for all $a_j \neq 0$ we find in the second term of the first sum that $f_{ij}(a_j) = 0$ and that $\sum a_j = x$. Choosing an $l \in I$ which is a common upper bound to all $j \in I$ with $a_j \neq 0$, we then get $f_{il}(x) = 0$ as desired.

Since arbitrary elements of $M$ have homogeneous representatives and since such representatives can by application of suitable $f_{ij}$ always be pushed into the same homogeneous component, the previous remark can be summarized as showing that two elements of $M$ which are equal are already equal in some $M_i$.

**Lemma 1.3.5** *Assume that $(A_i, f_{ij})$ is a direct system of $G$-graded $k$-algebras, i.e. the $f_{ij}$ should be homomorphisms of graded $k$-algebras. Then*

$$\lim_{\rightarrow} A_i$$

*exists as a limit of $G$-graded $k$-algebras.*
*The limit object can be obtained by the following construction: consider first*

$$A := \lim_{\rightarrow} A_i$$

*as a limit of $k$-modules. Then define a product on $A$ in the following way: for $x_i \in A_i$ and $x_j \in A_j$ we start by finding some common upper bound $l \in I$ of both $i$ and $j$, then move $\varphi_i(x_i)$ and $\varphi_j(x_j)$ into $\varphi_l(A_l)$ by applying $f_{il}$ and $f_{jl}$ respectively, and finally set*

$$\varphi(x_i)\varphi(x_j) := \varphi(f_{il}(x_i)f_{jl}(x_j)).$$

*$A$ with this multiplication is the desired limit of the direct system $(A_i, f_{ij})$ in the category of $G$-graded $k$-algebras.*
*If the $A_i$ were $(G, \chi)$-commutative with a common $\chi$, then so is their direct limit.*

**Proof** First we have to prove that the multiplication rule given is independent of choice of representatives $x_i, x_j$ and independent of the choice of $l \in I$. As the $f_{ij}$ are supposed algebra homomorphisms now, it is immediate that our definition does not depend on choice of $l$ (for $l, l' \in I$ two different choices, we can move to a common upper bound, which is compatible to the multiplication because of all $f_{ij}$ multiplicative). Assume now that $x, y, x', y' \in M'$ such that $x - x' \in M'', y - y' \in M''$, without loss of generality all in some fixed components of

$M' = \oplus_{i \in I} M_i$. By independence of $l$, we can move these elements if necessary to any component of $M'$ corresponding to an upper bound to their original indices; in particular, we can move them to some component high enough that the images of $x$ and $x'$ as well as of $y$ and $y'$ coincide after the move. This yields well-definedness of our multiplication.
Verification of the direct limit property follows exactly the same path as in the case of graded modules. What remains to be checked is just that all maps appearing are now morphisms of graded $k$-algebras. The graded aspect in this is trivial, as we already verified that they are all degree zero maps. Multiplicativity of the $\varphi_i$ follows immediately from the definition of our multiplication. Multiplicativity of $\rho$ is an easy consequence of the multiplicative property of the maps $\psi_i$ and $\varphi_i$. Also directly from the definitions it can be seen that $A$ as defined has a neutral element and that it is the equivalence class containing all the neutral elements of the $A_i$. From this it follows that all the maps in our construction respect also the ring unit, finishing the proof that they are morphisms of $k$-algebras.
To see that $A$ is $G$-commutative, it suffices to check $G$-commutativity for two arbitrary homogeneous given elements $x, y \in M$, which can be viewed without loss of generality as inhabiting some common $A_i$. $(G, \chi)$-commutativity of $A_i$ then gives the desired result.

We will now look at one particular case of the direct limit construction which will be of use to us later, which is the definition of infinite tensor products via direct limits.

**Example 1.3.6** *(Infinite tensor products) Let $\{A_i : i \in I\}$ be a family of $(G, \chi)$-graded not necessarily $(G, \chi)$-commutative $k$-algebras. Clearly, the finitary power set $\mathcal{P}_F(I)$ containing the finite subsets of $I$ with subset inclusion as ordering relation is an example of a directed set. To each finite $J \subseteq I$ we can associate the finite tensor product $A_J = \bigotimes_{i \in J} A_i$ of $(G, \chi)$-graded $k$-algebras. Using the fact that the tensor product of $(G, \chi)$-colored algebras is up to isomorphism associative and commutative, we can for each pair of finite subsets $J \subseteq J' \subseteq I$ define a canonical morphism of graded $k$-algebras $A_J \to A_{J'}$ by tensoring by copies of $1 \in A_i$. The maps so defined together with the $k$-algebras $A_J$ evidently form a directed system of colored $k$-algebras. We will call the direct limit of this system the graded tensor product of all the $k$-algebras $A_i$, writing*

$$\bigotimes_{i \in I} A_i := \varinjlim_{J \in \mathcal{P}_F(I)} A_J$$

*By the previous remark, this is a $(G, \chi)$-commutative algebra if all of the $A_i$ were $(G, \chi)$-commutative.*

Of course, if the $A_i$ are $(G, \chi)$-commutative polynomial rings, this direct limit construction results in another polynomial ring:

**Example 1.3.7** *Suppose that $\{A_i : i \in I\}$ is a family of $G$-commutative polynomial algebras in the sense of Example 1.2.10 with variable bases $B_i$. Set $B := \dot\cup B_i$, then $\otimes_{i \in I} A_i \cong k[B]$.*

**Proof** The case of finite tensor products inductively reduces to the case of the tensor product of two polynomial algebras. In this case, for two $(G, \chi)$-commutative polynomial algebras with variable bases $B, C$ we see $k[B] \otimes k[C] \cong k[B \dot\cup C]$ simply by constructing the 'canonical' isomorphism between the underlying tensor algebras $T(k^{(B)} \otimes k^{(C)})$ and $T(k^{(B \cup C)})$ and noting that it is compatible with the relations on either side defining our $G$-commutative polynomial algebras. That the direct limit of the finite tensor products is still a $G$-commutative polynomial algebra is a consequence of the slightly more general fact that a direct limit of polynomial algebras ordered by inclusion is still a polynomial algebra, with the variable set a union of suitable variable bases of the algebras in the family.

### 1.3.2 Inverse Limits

We will also need inverse limits. This construction is the category-theoretical dual of the direct limit, meaning that one can obtain all the definitions by taking the category-theoretical definition of a direct limit and inverting the directions of all the arrows. Of course, in doing so one also has to replace the notion of a direct system by that of an inverse system, which will be the first concept we shall now introduce. We will generally keep the discussion of inverse limits briefer than the preceding one on direct limits. Where we omit proofs now, the reader can follow (with reversed arrows) the ideas laid out in the preceding subsection to obtain them.

#### First definitions

**Definition 1.3.8** *Let $(I, \leq)$ be a directed set. Suppose that $\mathcal{C}$ is a category and that $\{A_i : i \in I\}$ is a family of $\mathcal{C}$-objects indexed by $I$. If as additional data we have for each $i \leq j \in I$ a morphism $f_{ij} : A_j \to A_i$ such that the whole family of morphisms is subject to the conditions*

$$f_{ii} = id_{A_i}$$

*for all $i \in I$ and*

$$f_{jk} \circ f_{ij} = f_{ik}$$

*for all $i \leq j \leq k \in I$, we call the family of objects together with the family of morphisms, the indexing set $I$ and the preorder given on $I$ an* inverse system*. We will in this case shortly refer to the inverse system given by this data by $(A_i, f_{ij})$, relying entirely on the context to avoid confusion with direct systems.*

The notion of an inverse limit of an inverse system can now be developed in a manner very similar to that of a direct limit of a direct system of objects. We will give the formal definition now; the reader is invited to check that it is up to reversion of arrows identical to Definition 1.3.2:

**Definition 1.3.9** *Let $\mathcal{C}$ be a category and let $(A_i, f_{ij})$ be an inverse system of objects in $\mathcal{C}$. If then there is an object $A \in \mathcal{C}$ together with a family of morphisms $\pi_i : A \to A_i$ such that*

$$f_{ij} \circ \pi_j = \pi_i$$

*and if for any other object $B$ together with a family of morphisms $\rho_i : B \to A_i$ similarly compatible with the inverse system there is a unique morphism $\tau : B \to A$ such that*

$$\rho_i = \pi_i \circ \tau$$

*for all $i \in I$, we call $A$ an* inverse limit *of the inverse system $(A_i, f_{ij})$. The inverse limit of the inverse system $(A_i, f_{ij})$ is denoted by*

$$\lim_{\leftarrow} A_i.$$

As was the case with direct limits, inverse limits need not always exist in an arbitrary category, but if an inverse limit of some directed family of objects exists, it is unique up to isomorphism. For algebraic objects, there is a concrete construction of the inverse limit. We will need this in the category of $G$-graded $k$-modules (with $k$ a commutative ring) and in the category of cochain complexes of such modules.

### Digression: complications arising from the grading

In some ways, slightly more care will be needed here than in the more familiar category of modules over a commutative ring without additional structure. A large part of the reason can be traced to the fact that in the ungraded category, the inverse limit of a family of modules can be explicitly constructed as a relatively intuitive submodule of the infinite cartesian product of all members of the family. We will in the following recall this construction. Then we will show that the same problem in the graded category cannot have quite so simple a solution.
Suppose now that $k$ is a commutative ring and that $(M_i, f_{ij})$ is an inverse system of $k$-modules with indexing set $I$. Then, we get the following:

**Lemma 1.3.10** *The inverse limit of $(M_i, f_{ij})$ exists and is given by the $k$-module*

$$M := \{(a_i)_{i\in I} : a_i = f_{ij}(a_j) \in M_i \forall i \leq j\}.$$

**Proof** Let $\pi_i : M \to M_i$ be the projection to $M_i$. Then from the definition of $M$, it is obvious that $f_{ij} \circ \pi_j = \pi_i$ for any $i \leq j$. With $N$ another $k$-module and $\rho_i : N \to M_i$ another family of compatible $k$-module homomorphisms, one can define $\tau : N \to M$ through $\tau(v) = (\rho_i(v))_{i\in I}$. This is well-defined because of the compatibility conditions with the inverse system. All other properties which were asked for in the definition of an inverse limit work as well with these definitions.

**Remark 1.3.11** *Suppose we are given a family $\{M_i : i \in I\}$ of $k$-modules over an arbitrary indexing set $I$. The set $J := \mathcal{P}_{fin}(I)$ of finite subsets of $I$ with inclusion as ordering relation forms a directed set then. Set for $F$ a finite subset of $I$*

$$M_F := \bigoplus_{i\in F} M_i.$$

*Denote further for $F_1 \subseteq F_2 \subseteq I$ two finite subsets the natural projection map $M_{F_2} \to M_{F_1}$ by $\pi_{F_1,F_2}$. With these notations, $(M_F, \pi_{F,F'})$ becomes an inverse system of $k$-modules. By the preceding lemma, we get*

$$M := \lim_{\leftarrow} M_F = \{(a_F)_{F\in\mathcal{P}_{fin}(I)} : a_F = \pi_{F,F'}(a_{F'}) \forall F \subseteq F'\}.$$

*Now an $(a_F)$ on the righthand side is uniquely determined by the entries $a_{\{i\}}$ for $i \in I$. This observation leads to an isomorphism*

$$M \cong \prod_{i \in I} M_i,$$

*i.e. with the infinite cartesian product of all the $M_i$.*

Note that if in the definition of being an inverse system one relaxes the requirement that the indexing set be directed to being merely pre-ordered, it becomes possible to write the general category-theoretic definition of a direct product in terms of inverse limits. This is then achieved by endowing the indexing set $I$ of an arbitrary family of objects with the trivial preorder given by $i \leq j \Leftrightarrow i = j$.
In the context of attempting to translate the above proof of the existence of inverse limits of arbitrary inverse systems of $k$-modules into the category of *graded* $k$-modules, one difficulty one encounters is simply that the infinite cartesian product of a family $\{M_i : i \in I\}$ of graded $k$-modules does not naturally carry a grading compatible with the gradings on the modules $M_i$. For instance, the vector space of sequences of polynomials in one variable over a field $K$ does not obviously carry a grading compatible with the standard grading on $K[X]$. For the sake of completeness, we give a stronger counterexample. Our goal is to construct a commutative ring $k$ and a family $\{M_i : i \in I\}$ of $k$-modules graded over an abelian group $G$ such that there is no $G$-grading of $M := \prod_{i \in I} M_i$ which makes the projections $\pi_i : M \to M_i$ into *graded* linear maps.
To reach that goal, we set $G = \mathbb{Z}$. Then, we fix a commutative ring $k$ and consider the set

$$A := Abb(\mathbb{N}, k)$$

of maps from $\mathbb{N}$ to $k$. With componentwise multiplication, this becomes a unital commutative ring. We view it as $\mathbb{Z}$-graded and concentrated in degree zero. Let now $k_{[i]}$ be the $\mathbb{Z}$-graded $A$-module given by taking a copy of $k$, viewing it as a set of elements of degree $i$, and defining scalar multiplication by the rule

$$(\lambda_1, \lambda_2, \ldots).v := \lambda_i v.$$

Then the $A$-module $V := \prod_{i \in \mathbb{Z}} k_{[i]}$ cannot be endowed with a grading such that the $k_{[i]}$ become subsets containing only homogeneous elements of degree $i$. The reason is that if $v \in V$ is any homogeneous element of $V$ and if $r \in A$ is an arbitrary element with finite support, the product $rv$ will be a homogeneous element of the essential submodule

$$V_{sum} := \oplus_{i \in \mathbb{Z}} k_{[i]}.$$

Running through all $r \in A$ with finite support, this shows $v \in k_{[deg(v)]}$. Since $V_{sum}$ is a proper submodule of $V$, this proves the claim that under these conditions there is no way to define a suitable decomposition of $V$ into homogeneous components.

**Inverse limits of graded objects**

**Lemma 1.3.12** *Suppose that $k$ is a commutative ring and that $(M_i, f_{ij})$ is an inverse system of $G$-graded $k$-modules. Denote the component of degree $g \in G$ of $M_i$ by $M_i^g$. Then,*

$$M :=< \{(a_i)_{i\in I} | \forall i \leq j : a_i = f_{ij}(a_j), \forall i.a_i \in M_i^g\} >$$

*is the inverse limit of the $M_i$. In other words, $M$ is generated by functions $f : I \to \cup_{i\in I} M_i$ which choose for each $i \in I$ a homogeneous element $a_i \in M_i^g$. Here $g \in G$ depends upon $f$ but is fixed for each $f$. The graded structure on $M$ is given naturally by the degrees of the generating elements.*

**Proof** Let $\pi_i : M \to M_i$ be the natural projection maps. It is clear that these are compatible with the maps defining the inverse system, i.e. that for all $i \leq j \in I$ we have $f_{ij} \circ \pi_j = \pi_i$. Assume that $N$ is another $G$-graded $k$-module together with maps $\rho_i : N \to M_i$ satisfying $f_{ij} \circ \rho_j = \rho_i$. Set now $g : N \to M$ to

$$g(x) := (\rho_i(x))_{i\in I}$$

To prove that $M$ is our inverse limit, we merely have to show that for any $x \in N$ we have $g(x) \in M$, as $\pi_i \circ g(x) = \rho_i(x)$ is self-evident. Compatibility between the $\rho_i$ and the $f_{ij}$ easily implies that $f_{ij}(g(x)_j) = g(x)_i$ as required for an element of $M$. Also, for a homogeneous element of $N$, say of degree $g \in G$, all entries of $g(x)$ are also homogeneous of degree $g$ as the $\rho_i$ are all degree preserving. Since any element of $N$ is a finite sum of homogeneous elements, this concludes the proof.

We will need also inverse limits of (co)chain complexes of such objects.

**Lemma 1.3.13** *Suppose that $((C^i, \delta^i), f_{ij})$ is an inverse system of chain complexes of $G$-graded $k$-modules. Then, the inverse limit of this system exists and is given by*

$$C_n := \lim_{\leftarrow} C_n^i$$

*with componentwise differentials*

$$\delta_n((x_i)_{i\in I}) := (\delta_n^i(x_i))_{i\in I}$$

*and projection maps $\pi_n^i : C_n \to C_n^i$ given by the projections of the graded module limit.*

**Proof** $(\delta_n^i(x_i))_{i\in I}$ is indeed always an element of $C_{n-1}$, because for any $i \leq j$ the map $f_{ij} : M_j \to M_i$ was supposed to be a chain map, so $f_{ij}(\delta_n^j(x_j)) = \delta_n^i(f_{ij}(x_j)) = \delta_n^i(x_i)$ as desired. It is clear that $C_*$ with the proposed differential forms a chain complex. What remains to be proven is that it is indeed the inverse limit of all the other chain complexes.
That the $\pi^i$ are chain maps is evident from the definition of the limit complex differential. That they are compatible with the chain maps $f_{ij}$ can be seen by considering that in fixed dimension $n$ of the complex the projections are just the projections of the module inverse

limit and hence compatible with the inverse system given by the dimension $n$ parts of all the complexes. Finally, take another chain complex of $G$-graded $k$-modules and call it $D$. Suppose that this chain complex comes with a family of chain maps $\rho^i : D \to C^i$. Then, we can as in the case of graded modules define graded $k$-module homomorphisms $\gamma_n : D_n \to C_n$. That the resulting $\gamma : D \to C$ is a chain map follows from the fact that the differential on $C$ is defined by componentwise action of the differentials $\delta^i$ and that the map $\gamma$ is defined by the componentwise action of the chain maps $\rho^i$. This concludes the proof.

Of course, exactly the same holds with exactly the same proofs for inverse limits of cochain complexes and analogous statements can easily be produced also for direct limits.

### 1.3.3 Limits and (co)homology

The aim of the section was to provide some tools that link taking direct or inverse limits and taking cohomology. The following corollary to the previous lemma will provide one such:

**Corollary 1.3.14** *Suppose that $(C_i, f_{ij})$ is an inverse system of cochain complexes of $G$-graded $k$-algebras. This induces for each dimension $n$ an inverse system of cohomology modules and the inverse limit of that inverse system is equal to the cohomology of the limit of the cochain complexes. In other words, taking cohomology commutes with limits of complexes.*

**Proof** Since chain maps induce homomorphisms on the cohomology level, the statement that the inverse system of cochain complexes induces an inverse system of cohomology modules is obvious. The rest of the corollary follows from the fact that in our category, passing to the limit commutes with the cochain complex structure, i.e. for an inverse system $(C_i, f_{ij})$ of cochain complexes we have according to Lemma 1.3.13

$$\lim_{\leftarrow} C_i^n = (\lim_{\leftarrow} C_i)^n$$

This means that we can without harm view the $\mathbb{Z}$-indexed family of inverse systems of cohomology modules induced by the given inverse system of cochain complexes as one inverse system of cochain complexes with trivial differentials, pass to the limit, and pick up the corollary.

# Chapter 2

# Graded Hochschild cohomology

We will begin this chapter by recalling some notions about Hochschild cohomology of group-graded algebras. We adapt everything immediately to the context of $G$-graded associative $k$-algebras with a coloring given by a bicharacter $\chi : G \times G \to k^*$. Compared to the $G$-graded situation without additional information, this will necessitate some changes, primarily to the differentials of the cochain complexes involved.
There are two reasons for doing these things: first, we will develop a deformation theory for $(G, \chi)$-graded associative algebras in chapter three and will need a version of Hochschild cohomology geared towards the needs of this theory then. Second, we want to develop Hochschild cohomology in such a way that its development parallels that of Harrison theory for $(G, \chi)$-commutative algebras in chapter four.
We point out that in the context of $G$-graded algebras which are not $(G, \chi)$-commutative, the bicharacter does still introduce additional structural information even if it is not used to encode a commutation condition to be enforced on $A$. For instance, its presence has an immediate impact on the structure of the modules of quasihomomorphisms $\mathbf{Hom}_A(M, N)$, on the definition of the opposite algebra $A^{op}$, on the definition of graded tensor products, and on the structure of formal graded associative deformation rings.
As a preparation, we insert two subsections recalling some notions about graded projective resolutions and the $(G, \chi)$ graded $Ext$ functor.

## 2.1 Algebraic preliminaries

For the purpose of this section, assume that $R$ is a graded $(G, \chi)$-colored ring, that $A$ is a $G$-graded $R$-module, and that $B$ is also an $R$-module. All $R$-modules appearing here shall be assumed left-modules unless something to the contrary is directly stated, projectivity will mean projectivity as a left module and so on. We leave it to the reader to work out the analog statements for the bimodule or right module situations.

### 2.1.1 Projective resolutions

**Definition 2.1.1** *Recall that a graded $R$-module $P$ is called* graded projective *if it satisfies the following* lifting property*: for any two graded $R$-modules $M$ and $N$, any degree zero epimorphism $f : N \to M$, and any (degree zero) homomorphism of $R$-modules $g : P \to M$ there exists a homomorphism (as usual of degree zero) $l : P \to N$ such that $f \circ l = g$.*

**Remark 2.1.2** *A graded $R$-module $P$ is graded projective exactly if it is a direct summand in a graded free module.*

**Proof** ($\Rightarrow$) As in the ungraded situation, we can take some homogeneous system of generators $S$ for $P$ and construct the graded free module $F(S)$ over $R$. We then find a degree zero surjective linear map $f : F(S) \to P$ by setting $f(s) = s$ for all $s \in S$ and extending linearly. By graded projectivity, there is a degree zero linear map $h : P \to F(S)$ with $f \circ h = id$. Here one gets $F(S) = Im(h) \oplus Ke(f)$ and $Im(h) \cong P$ as graded modules.
($\Leftarrow$) Assume $P \oplus P' = F$ with a graded free $R$-module $P'$, $f : M \to N$ an epimorphism of graded $R$-modules and $g : P \to N$ a homomorphism of graded $R$-modules. Because of $F = P \oplus P'$, $g$ can be extended to all of $F$ in a natural fashion and if there is $h : F \to M$ such that $f \circ h$ is equal to this extended $g$, then $f \circ h|_P = g$. Therefore it is sufficient to look at the case that $P$ itself is graded free. In this case, one chooses a graded basis $B$ of $P$. Setting $h(b)$ to some homogeneous preimage of $g(b)$ in $F$, one finds an appropriate $h$.

**Remark 2.1.3** *Note that an $R$-module $P$ is graded projective exactly if it is projective as an ungraded $R$-module and carries a grading compatible to that of $R$.*

**Proof** Graded projective modules are projective when viewed as ungraded modules, as by the previous remark they are direct summands of free modules. On the other hand, suppose that $P$ is projective as an ungraded $R$-module, and suppose that $M, N$ are $G$-graded $R$-modules together with a graded epimorphism $p : N \to M$ and a graded module homomorphism $g : P \to M$. By projectivity, there is then $h : P \to N$, not necessarily homogeneous of degree zero, with $p \circ h = g$. Taking $h_0(x) := h(x)_g$ for $x \in P_g$ homogeneous, we find with $h_0$ a graded homomorphism satisfying the same condition.

**Definition 2.1.4** *Let $A$ be a graded $R$-module. An exact sequence*

$$\ldots \to P_{i+1} \to P_i \to P_{i-1} \to \ldots \to P_2 \to P_1 \to A \to 0$$

*of graded $R$-modules $P_i$ is called a* projective (free) resolution *of $A$ if the modules $P_i$ are all graded projective (free). Note that the morphisms in such a sequence are required to be of degree zero.*

**Remark 2.1.5** *Any graded $R$-module $A$ has some graded projective resolution. Indeed, one can construct even a graded free resolution of $A$ with the usual iterative process.*

**Proposition 2.1.6** *Let*

$$\ldots \to^{f_4} P_3 \to^{f_3} P_2 \to^{f_2} P_1 \to^{f_1} A \to 0$$

*and*

$$\ldots \to^{g_4} Q_3 \to^{g_3} Q_2 \to^{g_2} Q_1 \to^{g_1} B \to 0$$

*be two graded projective resolutions of graded $R$-modules $A, B$ and denote the two chain complexes by $P$ and $Q$ for short. Assume that there is a graded module homomorphism $\mu : A \to B$. Then:*

1. *There exists a chain map $\alpha : P \to Q$ between the two complexes, such that $\alpha$ induces in degree zero the map $\mu$.*

2. *Any two such chain maps are chain homotopic.*

**Proof** *ad(1)*: We define the maps $\alpha_i : P_i \to Q_i$ inductively. Since both sequences are exact, we know that the maps $P_1 \to A$ and $Q_1 \to B$ are surjective, so by virtue of $P_1$ being projective, there must be a map $\alpha_1 : P_1 \to Q_1$ such that $g_1 \circ \alpha_1 = \mu \circ f_1$. Now, for all $i > 1$ we have to satisfy the equations

$$g_i \circ \alpha_i = \alpha_{i-1} \circ f_i.$$

In order to achieve this, we want to prove that for all $x \in P_i$ we have $\alpha_{i-1} \circ f_i(x) \in Im(g_i) = Ke(g_{i-1})$, since then we can use projectivity of $P_i$ together with the surjection $g_i : Q_i \to Im(g_i)$ to show the existence of a map of the desired kind. But, by the induction hypothesis we have $g_{i-1}(\alpha_{i-1}(f_i(x))) = \alpha_{i-2}(f_{i-1}(f_i(x))) = 0$, so $\alpha_{i-1}(f_i(x)) \in Im(g_i)$ is evident. This concludes the proof of assertion (1).
*ad(2)*: The process to construct the desired chain homotopy is very similar to the preceding construction. Indeed, denote by $\alpha$ and $\beta$ the two chain maps. Then explicitly what we want to achieve is to find a family of maps

$$\lambda_i : P_i \to Q_{i+1}$$

such that we have

$$g_{i+1}\lambda_i + \lambda_{i-1}f_i = \alpha_i - \beta_i$$

for all i. We will do this inductively again. Start by setting $\lambda_0 := 0$. Then, the equation for $\alpha_0 = \beta_0 = \mu$ is trivially true. The next equation that we need to satisfy is $g_2\lambda_1 = \alpha_1 - \beta_1$. As before, this reduces to $\alpha_1(x) - \beta_1(x) \in Im(g_2) = Ke(g_1)$ for all $x \in P_1$, since then we can use projectivity of $P_1$ together with the fact that tautologically $g_2$ induces a surjective map with target set its image. To this end, we compute:

$$g_1(\alpha_1(x) - \beta_1(x)) = g_1(\alpha_1(x)) - g_1(\beta_1(x)) = \mu f_1(x) - \mu f_1(x) = 0.$$

Finally, we proceed to the induction step. Assume that we have already constructed maps $\lambda_0, \lambda_1, \ldots, \lambda_{i-1}$ satisfying the equations $g_{j+1}\lambda_j + \lambda_{j-1}f_j = \alpha_j - \beta_j$ for all $j < i$. Then, we want to find a $\lambda_i$ such that $g_{i+1}\lambda_i = \alpha_i - \beta_i - \lambda_{i-1}f_i$ is fulfilled. As usual, we have to prove that $\alpha_i(x) - \beta_i(x) - \lambda_{i-1}(x) \in Im(g_{i+1}) = Ke(g_i)$ for all $x$. This is done by the calculation:

$$g_i(\alpha_i(x) - \beta_i(x) - \lambda_{i-1}(f_i(x))) = \alpha_{i-1}(f_i(x)) - \beta_{i-1}(f_i(x)) - \alpha_{i-1}(f_i(x)) + \beta_{i-1}(f_i(x)) = 0.$$

This concludes the proof.

### 2.1.2 $(G, \chi)$-graded *Ext*

We will now assume that $k$ is a commutative ring and that $A$ is a $(G, \chi)$-colored associative $k$-algebra.

**Definition 2.1.7** *Let $M$ and $N$ be $G$-graded $A$-left modules and let*

$$\ldots \to P_3 \to P_2 \to P_1 \to M \to 0$$

*be a (left module, graded) projective resolution of $M$. Consider the induced complex*

$$\ldots \leftarrow \boldsymbol{Hom}_A(P_3, N) \leftarrow \boldsymbol{Hom}_A(P_2, N) \leftarrow \boldsymbol{Hom}_A(P_1, N) \leftarrow 0.$$

*The $n$-th cohomology group of this complex is denoted by $Ext^n_R(M, N)$.*

**Remark 2.1.8** *First, one should verify that indeed a map $\varphi_{i+1} : P_{i+1} \to P_i$ in the projective resolution appearing in Def. 2.1.7 induces a degree zero map $\varphi^*_{i+1} : \boldsymbol{Hom}_A(P_i, N) \to \boldsymbol{Hom}_A(P_{i+1}, N)$ as claimed. Let $f \in \boldsymbol{Hom}_A(P_i, N)$ be a homogeneous element and set $\varphi^*_{i+1}(f) := f \circ \varphi_{i+1}$. It is clear that this map is of the same degree as $f$ and that it is of type $P_{i+1} \to N$ , so we have to check only that it is a quasihomomorphism. Taking a homogeneous $m \in P_{i+1}$ and $r \in A$, one uses the fact that $\varphi_{i+1}$ is degree zero to calculate*

$$\varphi^*_{i+1}(f)(rm) = f(\varphi_{i+1}(rm)) = f(r\varphi_{i+1}(m)) = \chi(|f|, |r|)rf(\varphi_{i+1}(m))$$

*as desired.*

**Remark 2.1.9** *The definition above is independent of the projective resolution used, i.e. the graded Ext functor is well-defined.*

**Proof** Let

$$\ldots \to^{f_4} P_3 \to^{f_3} P_2 \to^{f_2} P_1 \to^{f_1} M \to 0$$

and

$$\ldots \to^{g_4} Q_3 \to^{g_3} Q_2 \to^{g_2} Q_1 \to^{g_1} M \to 0$$

be two projective resolutions of $M$. By Prop. 2.1.6, we know that there exists a chain map $\alpha : P \to Q$ such that $\alpha$ induces the identity over $A$. Also according to Prop. 2.1.6, any two such chain maps are chain homotopic, so as far as this proof is concerned, it matters not which we choose. As a general fact about chain complexes, we note that removal of one term at the right end of two chain complexes leaves intact the properties

- of being a chain map between the two complexes
- of being an isomorphism of the two complexes
- of being a chain homotopy between the two complexes.

Therefore, it will suffice to show that $\alpha^*$ defined as right composition with $\alpha$, i.e.

$$\begin{array}{ccccccc} \ldots & \longleftarrow & \mathbf{Hom}_A(Q_3, N) & \longleftarrow & \mathbf{Hom}_A(Q_2, N) & \longleftarrow & \mathbf{Hom}_A(Q_1, N) \\ \alpha^* \downarrow & & \alpha^* \downarrow & & \alpha^* \downarrow & & \alpha^* \downarrow \\ \ldots & \longleftarrow & \mathbf{Hom}_A(P_3, N) & \longleftarrow & \mathbf{Hom}_A(P_2, N) & \longleftarrow & \mathbf{Hom}_A(P_1, N) \end{array}$$

induces on the level of cohomology an isomorphism of $k$-modules.
That $\alpha^*$ is a chain map is clear. Likewise, it is clear that a chain homotopy $\lambda_i : P_i \to Q_{i+1}$ between chain maps $\alpha$ and $\beta$ between the chain complexes $P$ and $Q$ will similarly lift, with $\lambda^*$ becoming a chain homotopy between the maps $\alpha^*$ and $\beta^*$. For this reason, using Prop. 2.1.6 we realize that it matters not which among the chain maps between $\mathbf{Hom}_A(P, N)$ and $\mathbf{Hom}_A(Q, N)$ of the form $\alpha^*$ we look at in particular; they all induce the same maps on the level of homology. Finally, with $\alpha : P \to Q$ and $\beta : Q \to P$ two chain maps in opposite directions, we observe that $\alpha^* \circ \beta^* = (\beta \circ \alpha)^*$.
Now it is time to put all these observations together towards a proof of our proposition. In particular, because *both* the $P_i$ and the $Q_i$ are projective, we immediately know that we can find chain maps $\alpha : \mathbf{Hom}_A(P, N) \to \mathbf{Hom}_A(Q, N)$ and $\beta : \mathbf{Hom}_A(Q, N) \to \mathbf{Hom}_A(P, N)$. Also, because all the chain maps of the form $\gamma^*$ between the complexes concerned are chain homotopic, it is clear that the maps $\beta^* \circ \alpha^*$ and $\alpha^* \circ \beta^*$ will induce the same maps in homology as the identity maps on both complexes respectively. In other words, the induced maps on cohomology are inverse to each other, yielding the desired isomorphism.

## 2.2 Hochschild cohomology

**Definition/Proposition 2.2.1** *Let again $A$ be a $(G, \chi)$-colored associative algebra over a commutative ring $k$ and let $M$ be an $A$-bimodule. Then, we set $C^n(A, M) := \boldsymbol{Hom}_k(A^{\otimes n}, M)$ and define for $\varphi \in C^{n-1}(A, M)$ homogeneous*

$$\begin{array}{rcl} \beta_0(\varphi)(a_1 \otimes \ldots \otimes a_n) & := & \chi(|\varphi|, |a_1|)a_1\varphi(a_2 \otimes a_3 \otimes a_3 \otimes \ldots \otimes a_n) \\ \beta_i(\varphi)(a_1 \otimes a_2 \otimes \ldots \otimes a_n) & := & \varphi(a_1 \otimes a_2 \otimes a_3 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_n) \\ \beta_n(\varphi)(a_1 \otimes a_2 \otimes \ldots \otimes a_n) & := & \varphi(a_1 \otimes a_2 \otimes \ldots \otimes a_{n-1})a_n \end{array}$$

*where all the $a_i$ appearing in the definition are supposed homogeneous and with extension of the functions appearing to non-homogeneous inputs as usual. Under these conditions, the map*

$$\beta := \sum_{i=0}^{n} (-1)^i \beta_i$$

*defines a differential on $\oplus_{i=0}^{\infty} C_i(A, M)$. If there is potential of confusion, we will write ${}_n\beta_i$ for $\beta_i : C^n(A, M) \to C^{n+1}(A, M)$. We will call the cochain complex thus defined the* Hochschild cochain complex of $A$ with coefficients in $M$. *We will refer to the cohomology of this complex as the* Hochschild cohomology of $A$ with coefficients in $M$. *Finally the $n$-th Hochschild homology group of $A$ with coefficients in $M$ will be denoted by $H^n(A, M)$; and, we will set $HH^n(A) := H^n(A, A)$. Accordingly, $Z^n(A, M)$ will denote the graded $k$-module of $n$-Hochschild cocycles and $B^n(A, M)$ will be the $k$-module of $n$-Hochschild coboundaries.*

**Proof** We have to check that $\beta$ in the modified Hochschild complex given above does define a differential, i.e. that we have $\beta^2 = 0$. As usual, we assume $\varphi \in C^n(A, M)$ and $a_i \in A$ to be homogeneous and prove the general statement by extension from generators to arbitrary cochains. It is clear that $\beta$ as well as all the face maps $\beta_i$ are $k$-linear degree zero maps. Our goal is to prove that, as in the classical situation, we have

$$_{n+1}\beta_{i+1}\ {}_n\beta_j = {}_{n+1}\beta_j\ {}_n\beta_i$$

for all $0 < j < i \leq n+1$. Having obtained this, one can conclude as usual that $\beta^2 = 0$. We first check $\beta_1\beta_0 = \beta_0\beta_0$, by calculating:

$$\begin{aligned} & \beta_1\beta_0(\varphi)(a_0 \otimes \ldots \otimes a_{n+1}) = \beta_0(\varphi)(a_0a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}) \\ = \ & \chi(|\varphi|, |a_0| + |a_1|)a_0a_1\varphi(a_2 \otimes \ldots \otimes a_{n+1}) \end{aligned}$$

and

$$\beta_0^2(\varphi)(a_0 \otimes \ldots \otimes a_{n+1}) = \chi(|\varphi|, |a_0|)\chi(|\varphi|, |a_1|)a_0a_1\varphi(a_2 \otimes \ldots \otimes a_{n+1})$$

as desired. We next check the proposed equation for $j = 0$ and $0 < i \leq n$. Here one finds

$$\begin{aligned} & \beta_{i+1}\beta_0(\varphi)(a_0 \otimes \ldots \otimes a_{n+1} = \beta_0(\varphi)(a_0 \otimes \ldots \otimes a_ia_{i+1} \otimes \ldots \otimes a_{n+1}) \\ = \ & \chi(|\varphi|, |a_0|)a_0\varphi(a_1 \otimes \ldots \otimes a_ia_{i+1} \otimes \ldots \otimes a_{n+1}) \end{aligned}$$

and

$$\begin{aligned} & \beta_0\beta_i(\varphi)(a_0 \otimes \ldots \otimes a_{n+1}) = \chi(|\varphi|, |a_0|)a_0\beta_i(\varphi)(a_1 \otimes \ldots \otimes a_{n+1}) \\ = \ & \chi(|\varphi|, |a_0|)a_0\varphi(a_1 \otimes \ldots \otimes a_ia_{i+1} \otimes \ldots \otimes a_{n+1}). \end{aligned}$$

Also the case $i = n+1$ and $j = 0$ needs to be treated. We calculate:

$$\beta_{n+2}\beta_0(\varphi)(a_0 \otimes \ldots \otimes a_{n+1}) = \chi(|\varphi|, |a_0|)a_0\varphi(a_1 \otimes \ldots \otimes a_n)a_{n+1}$$

and

$$\begin{aligned} & \beta_0\beta_{n+1}(\varphi)(a_0 \otimes \ldots \otimes a_{n+1}) = \chi(|\varphi|, |a_0|)a_0\beta_{n+1}(\varphi)(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) \\ = \ & \chi(|\varphi|, |a_0|)a_0\varphi(a_1 \otimes \ldots \otimes a_n)a_{n+1}. \end{aligned}$$

The cases covering the remaining $\beta_i, \beta_j$ work exactly as in the classical case because the corrective term distinguishing our version of the Hochschild differential from its ungraded counterpart does not appear in them. This concludes the proof.

The main goal of this subsection will be to prove a result linking $(G, \chi)$-graded Hochschild cohomology of graded projective algebras with graded *Ext*. At the end, we will use this to calculate graded Hochschild cohomology in arbitrarily high dimension on an example. To get there, we first have to introduce a different but related complex and prove some simple statements about it:

**Definition 2.2.2** *Recall that in Ex. 1.2.29 we fixed the notation $A^e := A \otimes A^{op}$ and remarked that the category of $A^e$ left modules is equivalent to the category of $A$-bimodules. Consider now the complex of left $A^e$ modules*

$$\ldots \to A^{\otimes(n+1)} \to A^{\otimes n} \to A^{n-1} \to \ldots \to A^{\otimes 3} \to A^{\otimes 2}$$

*where $A^{\otimes 2}$ is in degree zero and where the differentials are given by*

$$b' := \sum_{i=0}^{n-1} (-1)^i d_i$$

*where*

$$d_i(a_0 \otimes a_1 \otimes \ldots \otimes a_n) := a_0 \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n$$

*This complex is called the* bar complex *of $A$. The complex obtained by adjoining the multiplication map $\mu$,*

$$\ldots \to A^{\otimes 3} \to A^{\otimes 2} \to^{\mu} A$$

*we will call the* augmented bar complex. *The components of the bar complex will be denoted by $C_n^{bar}(A)$, i.e.*

$$C_n^{bar}(A) := A^{\otimes(n+2)}.$$

**Remark 2.2.3** *The bar complex is a resolution as an $A^e$-module of the $k$-algebra $A$.*

**Proof** First, one needs to check that all differentials are indeed $A^e$-linear. This part of the proof is also the only one that is influenced by the presence of $(G, \chi)$-graded structure. Since it is clear that all the $d_i$ are additive and degree zero, and since it suffices to check everything on homogeneous generators, one needs only check that

$$d_i((r_1 \otimes r_2).(a_0 \otimes a_1 \otimes \ldots \otimes a_n)) = (r_1 \otimes r_2).d_i(a_0 \otimes a_1 \otimes \ldots \otimes a_n).$$

Both sides of this equation evaluate to

$$\chi(|r_2|, \sum_{i=0}^{n} |a_i|) r_1 a_0 \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n r_2$$

which gives linearity.
That the bar complex is a chain complex can be checked by a direct calculation similar to the one we used to verify that the Hochschild cohomology complex is a cochain complex.
Now we see that

$$\ldots \to A^{\otimes 3} \to A^{\otimes 2} \to A$$

is exact at $A^{\otimes 2}$, because we have

$$\mu(\sum_{i=1}^{n} a_i \otimes b_i) = 0 \Leftrightarrow \sum_{i=1}^{n} a_i b_i = 0 \Leftrightarrow \sum_{i=1}^{n} a_i \otimes b_i = b'(\sum_{i=1}^{n} 1 \otimes a_i \otimes b_i)$$

whereas $\mu \circ b' = 0$ is obvious. What remains to be proven is that the homology of the complex vanishes in all other degrees, too. To this end, we consider the map

$$s : A^{\otimes n} \to A^{\otimes(n+1)}, s(a_1 \otimes a_2 \otimes \ldots \otimes a_n) := 1 \otimes a_1 \otimes \ldots \otimes a_n$$

With this $s$, one obviously has $d_0 s = id$ and $d_i s = s d_{i-1}$ for all $i > 0$, so

$$b' \circ s + s \circ b' = id$$

and hence $s$ is a homotopy between the identity and the zero map (in other words, a *contracting* homotopy) on $C_*^{bar}$, thus yielding the conclusion we want.

Finally, we prove two little lemmata before arriving at the desired result:

**Lemma 2.2.4** *Let $A$ be a $G$-graded $k$-algebra, unital and a homogeneous submodule of a graded free $k$-module $A'$, and let $F$ be another graded free $k$-module. Then, $A \otimes F$ with the natural left multiplication is free as a graded $A$-algebra.*

**Proof** Choose a homogeneous $k$-basis $\{b_i : i \in I\}$ of $M$ and a homogeneous $k$-basis $\{a_j : j \in J\}$ of $A'$. Then, we will prove that the set $B := \{1 \otimes b_i : i \in I\}$ is an $A$-basis of $A \otimes F$ (homogeneity of this basis is obvious). It is immediate that $B$ is a generating subset, so what has to be proven is only independence. Hence, let

$$\sum_{i \in I} r_i \otimes b_i = 0$$

as always with only finitely many $r_i \neq 0$. Decomposing the $r_i$ over our $k$-basis of $A'$, we find

$$\sum_{i \in I} (\sum_{j \in J} \lambda_{ij} a_j) \otimes b_i = \sum_{i \in I, j \in J} \lambda_{ij} (a_j \otimes b_i) = 0$$

from which it follows that all the $\lambda_{ij}$ and hence all the $r_i$ must be zero, because the $a_j \otimes b_i$ form a $k$-independent subset of $A' \otimes F$. This concludes the proof.

**Lemma 2.2.5** *Let $A$ be a (unitary) graded $k$-algebra which is a submodule of a graded free $k$-module $A'$, and $M$ be an $k$-projective graded $A$-bimodule. Then, $A \otimes M$ graded is a projective $A$-module.*

**Proof** Choose a $k$-module $M'$ such that $F = M \oplus M'$ is a graded free $k$-module. By Lemma 2.2.4, the $A$-module $A \otimes F$ is free, and obviously we have $A \otimes F = A \otimes M \oplus A \otimes M'$. Hence, $A \otimes M$ is a direct summand in an $A$-free $A$-module, so projective over $A$.

We now come to the result connecting the *Ext* functor to Hochschild homology which we promised at the beginning of the subsection. It will allow us to calculate Hochschild homology in terms of *Ext* if the $k$-algebra $A$ is graded projective over $k$. This is an advantage over working with the raw definition of Hochschild homology insofar as in the calculation of *Ext* the choice of a projective resolution of the target algebra gives one considerable degrees of freedom. We will quickly illustrate this point by computing an example which, while simple, does already not appear to yield easily to attempts of calculation using only the definitions.

**Proposition 2.2.6** *Let $A$ be a unital projective $(G,\chi)$-graded $k$-algebra and let $M$ be an $A$-bimodule. Then, there is an isomorphism*

$$H_n(A,M) \cong Ext^n_{A^e}(A,M).$$

**Proof** $A$ is projective over $k$, so $A^{\otimes n}$ is also projective over $k$ for any $n \in \mathbb{N}$ (the tensor product of any two projective $k$-modules is a projective $k$-module: with $P_1 \subseteq_\oplus F_1, P_2 \subseteq_\oplus F_2$ we get $P_1 \otimes P_2 \subseteq_\otimes F_1 \otimes F_2$ simply by multiplying out the product $F_1 \otimes F_2$ with suitable decompositions of $F_1, F_2$ into direct sums).
It remains to be verified that the modules $A^{\otimes n} \otimes A^e \cong_k A^{\otimes(n+2)}$ are all projective as $A^e$-modules. To this end, we note that $A^e$ itself is projective as $k$-module, for it carries the same $k$-module structure as $A^{\otimes 2}$; hence, it is in particular a submodule of a free module over $k$. As $A^{\otimes n}$ is projective over $k$, all conditions of Lemma 2.2.5 are satisfied, and so $A^e \otimes A^{\otimes n}$ is an projective $A^e$-module.
Hence, the bar complex is a projective resolution of $A$ as an $A^e$-module. Now, to obtain the proposition, we simply pass to $\mathbf{Hom}_{A^e}$:

$$\ldots \leftarrow \mathbf{Hom}_{A^e}(A^{\otimes 4}, M) \leftarrow \mathbf{Hom}_{A^e}(A^{\otimes 3}, M) \leftarrow \mathbf{Hom}_{A^e}(A^{\otimes 2}, M).$$

Note here that $\mathbf{Hom}_{A^e}(A^{\otimes(n+2)}, M) \cong \mathbf{Hom}_k(A^{\otimes n}, M)$ as a $k$-module via the mapping

$$\psi : \mathbf{Hom}_k(A^{\otimes n}, M) \to \mathbf{Hom}_{A^e}(A^{\otimes(n+2)}, M)$$
$$\psi(\varphi)(a_0 \otimes a_1 \otimes \ldots \otimes a_{n+1}) := \chi(|\varphi|, |a_0|)a_0\varphi(a_1 \otimes \ldots \otimes a_n)a_{n+1}$$

as usual with everything appearing in the definition supposed homogeneous. One needs to check that this map is well-defined, $k$-linear, and that it is bijective. Checking well-definedness boils down to carrying out a calculation confirming that all of the definitions going into this work together well. One needs to verify that $\chi(|\varphi|, |a_0|)a_0\varphi(a_1 \otimes \ldots \otimes a_n)a_{n+1}$ depends only on $a_0 \otimes \ldots \otimes a_{n+1}$ when $\varphi$ is fixed, and that $\psi(\varphi) \in \mathbf{Hom}_{A^e}(A^{\otimes n}, M)$.
We consider the former as clear but check the latter. Only left $A^e$-linearity needs proving. On the one hand, we have

$$\begin{aligned} \psi(\varphi)((r \otimes s).(a_1 \otimes \ldots \otimes a_n)) &= \chi(|s|, \sum_{i=1}^n |a_i|)\psi(\varphi)(ra_1 \otimes a_2 \otimes \ldots \otimes a_{n-1} \otimes a_n s) \\ &= \chi(|\varphi|, |ra_1|)\chi(|s|, \sum_{i=1}^n |a_i|)ra_1\varphi(a_2 \otimes \ldots \otimes a_{n-1})a_n s \end{aligned}$$

for $r, s, a_i, \varphi$ homogeneous by the definitions. On the other hand, we see also

$$\begin{aligned} (r \otimes s)\psi(\varphi)(a_1 \otimes \ldots \otimes a_n) &= (r \otimes s)\chi(|\varphi|, |a_1|)a_1\varphi(a_2 \otimes \ldots \otimes a_{n-1})a_n \\ &= \chi(|\varphi|, |a_1|)\chi(|s|, |\varphi| + \sum_{i=1}^n |a_i|)ra_1\varphi(a_1 \otimes \ldots \otimes a_{n-1})a_n s \end{aligned}$$

and therefore

$$\psi(\varphi)((r \otimes s).(a_1 \otimes \ldots \otimes a_n)) = \chi(|\varphi|, |r| + |s|)(r \otimes s)\psi(\varphi)(a_1 \otimes \ldots \otimes a_n)$$

as desired. That $\psi$ is $k$-linear is obvious. To understand bijectivity, one should consider the fact that $\varphi \in \mathbf{Hom}_{A^e}(A^{\otimes(n+2)}, M)$ is uniquely determined by its values on terms of the form

$$1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1$$

which immediately enables us to construct $\psi^{-1}$. Now the two complexes involved have by definition as cohomology exactly the Hochschild groups and the *Ext* groups respectively. What remains to be shown is therefore only that the isomorphism is a chain map, i.e. compatibility with the differentials on both sides.
Denoting by $\beta'_i$ the *Ext* cohomology complex differential

$$\beta'_i : \mathbf{Hom}_{A^e}(A^{\otimes(i+2)}, M) \to \mathbf{Hom}_{A^e}(A^{\otimes(i+3)}, M)$$

induced by the bar resolution, we need to show $\psi \circ \beta = \beta' \circ \psi$. Taking $a_0, a_1, \ldots, a_{n+2}$ to be homogeneous elements of $A$, and $\varphi \in \mathbf{Hom}_k(A^{\otimes n}, M)$ to be a homogeneous order $n$ Hochschild cochain, we get

$$\begin{aligned}
& (\psi \circ \beta)(\varphi)(a_0 \otimes \ldots \otimes a_{n+2}) = \chi(|\varphi|, |a_0|) a_0 \beta(\varphi)(a_1 \otimes \ldots \otimes a_{n+1}) a_{n+2} \\
= \; & \chi(|\varphi|, |a_0| + |a_1|) a_0 a_1 \varphi(a_2 \otimes \ldots \otimes a_{n+1}) a_{n+2} + \chi(|\varphi|, |a_0|) a_0 \varphi(a_1 a_2 \otimes a_3 \otimes \ldots \otimes a_{n+1}) a_{n+2} \\
+ \; & \ldots + (-1)^n \chi(|\varphi|, |a_0|) a_0 \varphi(a_1 \otimes \ldots \otimes a_n a_{n+1}) a_{n+2} \\
+ \; & (-1)^{n+1} \chi(|\varphi|, |a_0|) a_0 \varphi(a_1 \otimes \ldots \otimes a_n) a_{n+1} a_{n+2}
\end{aligned}$$

and

$$\begin{aligned}
& (\beta' \circ \psi)(\varphi)(a_0 \otimes \ldots \otimes a_{n+2}) = \psi(\varphi)(a_0 a_1 \otimes \ldots \otimes a_{n+2}) - \psi(\varphi)(a_0 \otimes a_1 a_2 \otimes \ldots \otimes a_{n+2}) \\
+ \; & \ldots + (-1)^n \psi(\varphi)(a_0 \otimes a_1 \otimes \ldots \otimes a_n a_{n+1} \otimes a_{n+2}) + (-1)^{n+1} \psi(\varphi)(a_0 \otimes \ldots \otimes a_{n+1} a_{n+2}) \\
= \; & \chi(|\varphi|, |a_0| + |a_1|) a_0 a_1 \varphi(a_2 \otimes \ldots \otimes a_{n+1}) a_{n+2} - \chi(|\varphi|, |a_0|) a_0 \varphi(a_1 a_2 \otimes \ldots \otimes a_{n+1}) a_{n+2} \\
+ \; & \ldots + (-1)^n \chi(|\varphi|, |a_0|) a_0 \varphi(a_1 \otimes \ldots \otimes a_n a_{n+1}) a_{n+2} \\
+ \; & (-1)^{n+1} \chi(|\varphi|, |a_0|) a_0 \varphi(a_1 \otimes \ldots \otimes a_n) a_{n+1} a_{n+2}
\end{aligned}$$

as desired.

### Hochschild cohomology of the dual numbers

For the purpose of illustration, we will now look at a simple application of these observations, computation of graded Hochschild cohomology of the dual numbers of a field for all color-commutative structures that are compatible with the multiplicative structure of these algebras. There are two reasons why we treat this admittedly very simple example. On the one hand, we believe that it is for pedagogical reasons good to illustrate the abstract techniques presented before by calculation on a specific structure. On the other hand, this example has the merit of being simple enough that one can with reasonable effort both classify all color-commutative structures on them and compute their Hochschild cohomologies to arbitrary order. The beginnings of a more difficult project of the same kind, utilizing a more complicated class of algebras, can be found in Appendix B.

**Commutative structures on** $K[\epsilon]$ Let $K$ be a field and $A := K[\epsilon]$ be the ring of dual numbers over $K$. Our first task is then to determine which color-commutative structures may exist on $K$. We will use for this the same conventions discussed in (Appendix B). In short, we will only consider gradings of the underlying algebra $A$ which have the property that $Supp(A) := \{g \in G : A_g \neq 0\}$ is a generating subset of $G$, and we will not distinguish between graded structures which can be transformed into each other by a group automorphism $\varphi : G \to G$ as in Def. A.0.3.
Intuitively, the result one is likely to expect here is that there are essentially three different cases: first, the commutative structure with trivial grading; second, supercommutative gradings with one even and one odd nonzero component generated by 1 and $\epsilon \in A$ respectively; and third, commutative gradings with the same underlying decomposition of $A$. We will see that this intuitive expectation is almost true, the only exception appearing in characteristic two.
We get the following:

**Remark 2.2.7** *Suppose that $A$ has been endowed with a decomposition $A = \oplus_{g\in G} A_g$ making it into a $G$-graded algebra. Suppose further that $\chi : G \times G \to K^*$ is a bicharacter on $G$ making $A$ together with the $G$-grading into a color-commutative algebra. Finally, suppose that $G$ is generated by $Supp(A)$. Then, we have to distinguish the following cases:*

1. *$char(K) \neq 2$: in this case, 1 and $\epsilon$ always form a homogeneous $k$-basis of $A$. $\chi$ is either trivial or induces a supercommutative structure. If $\chi$ is trivial, $G$ can be any $\mathbb{Z}_n$ or can be $\mathbb{Z}$. If $\chi$ is not trivial, $\mathbb{Z}_2$ has to be a subgroup of $G$ and $G$ has to be cyclic. The degree of $\epsilon$ is in any of these cases given by a generator of $G$, and up to equivalence by $deg(\epsilon) = 1$.*

2. *$char(K) = 2$: in this case, the bicharacter is always trivial. $G$ can be zero or a finite or infinite cyclic group. In the latter case, we have up to equivalence either $\epsilon$ homogeneous of degree one or $j := 1 + \epsilon$ homogeneous of degree one.*

**Proof** We begin with some general observations. First, since $dim_K(A) = 2$, if $A$ is graded over a nonzero cyclic group $G$ in such a way that $Supp(A)$ generates $G$, there will be exactly one nonzero component apart from the zero component and the index of that component will be a generator of $G$. Without loss of generality we can then assume $G \in \{\mathbb{Z}_2, \mathbb{Z}_3, \dots, \mathbb{Z}\}$ and $A$ concentrated in the components of degrees one and zero.
Second, $G$ must be cyclic as with a grading of an appropriate kind over a noncyclic group we would have at least three nonzero graded components, namely the component in degree zero and at least two for a system of generators of $G$, contradicting $dim(A) = 2$.
Suppose now $G \neq 0$. Then, we have to treat the cases $char(K) = 2$ and $char(K) \neq 2$ in turn:

1. $char(K) \neq 2$: In this case, we have $A_0 = K$ and $A_1 = (\epsilon)$. $A_0 = K$ is clear from $1 \in A_0$ and dimension counting. To see $A_1 = (\epsilon)$, suppose that $x := \lambda + \mu\epsilon \in A_1$. It is clear that $\mu \neq 0$. If $\lambda \neq 0$, we have $x^2 \neq 0$, and therefore $x^2 = \lambda^2 + 2\lambda\mu\epsilon \in A_0$ by dimension counting. But by subtracting a suitable multiple of the unit we can then show $2\lambda\mu\epsilon \in A_0$, and thereby $\epsilon \in A_0$, contradicting $dim_K(A_0) = 1$.
   With the admissible gradings thereby determined, we see without difficulty that the only admissible bicharacters are those that give a supercommutative and a commutative

structure on $A$ respectively and that $G$ must be of even order if $\chi$ prescribes a supercommutative structure.

2. $char(K) = 2$: Following the same line of reasoning, one finds one additional case corresponding to a homogeneous element of degree one of the form $j := \lambda + \mu\epsilon$ with $\lambda \neq 0$. We have $j^2 = \lambda^2$ and can if necessary normalize this so that $j^2 = 1$. $K[\epsilon]$ is in this case the same as the graded algebra $K[j]$ with $j$ homogeneous of degree one.

**Calculation of Hochschild cohomology** We will now look at the graded Hochschild cohomologies of these algebras. It will be sufficient to look at three cases, corresponding to $G = \mathbb{Z}$ with commutative respectively supercommutative structures, and to the exceptional case $A \cong K[j]$ with $char(K) = 2$, $j^2 = 1$, $deg(j) = 1$, and a grading over $\mathbb{Z}_2$. The other cases are very similar to these.

**Example 2.2.8** *Suppose $G = \mathbb{Z}$, $A_0 = K$, $A_1 = (\epsilon)$, and $\chi(n,m) = 1$ and recall that for $M$ a $G$-graded $K$-module and an element $g \in G$, the shifted module $M[g]$ was defined through $M[g]_h := M_{g+h}$. Then, we have*

$$HH^n(A) \cong \begin{cases} A^e & \text{if } n = 0 \\ K[n] & \text{if } n \text{ odd} \\ K[n+1] & \text{if } n \text{ even} \end{cases}$$

*if $char(K) \neq 2$ and*

$$HH^n(A) \cong A[n]$$

*if $char(K) = 2$.*

**Proof** We will first calculate a free resolution of $A$ over $A^e$. Applying the definitions, we get $A^e = K[X,Y]/(X^2,Y^2)$, where $X$, $Y$ are homogeneous of degree one and operation as left module on $A$ of this is given through the identification $A \cong A^e/(X-Y)$. We construct step by step the following 'periodic' resolution:

$$\ldots \to A^e[-3] \to^{\cdot(X-Y)} A^e[-2] \to^{\cdot(X+Y)} A^e[-1] \to^{\cdot(X-Y)} A^e \to^{\mu} A \to 0.$$

Recall in this context that, for a module $M$ graded over an abelian group $G$ and an element $g \in G$, the shifted module $M[g]$ is defined by setting $M[g]_h = M_{g+h}$. The maps in the above exact sequence are then understood to take the standard free generator of the $A^e$-module $A^e[-n]$ to the element indicated in the superscript at the end of each arrow, living in $A^e[-(n-1)]_n$. All maps appearing are clearly $A^e$-linear and degree zero, and all modules are free with one generator. Also it is clear that any two of the maps compose to zero. That the image of each map coincides with the kernel of the next remains to be proven. We will do this only in the super case, since the calculation there is quite similar but a bit more involved than here.
We will in the following denote the map $A^e[2n+1] \to^{\cdot(X-Y)} A^e[2n]$ by $\varphi_{2n+1}$ and the map $A^e[2n] \to^{\cdot(X+Y)} A^e[2n-1]$ by $\psi_{2n}$.
We now use this resolution to calculate $Ext$. We see

$$\mathbf{Hom}_{A^e}(A^e[-n], A) \cong A[n]$$

because an $A^e$-linear map $f : A^e[-n] \to A$ is uniquely determined by the image $r \in A$ of the standard generator of $A^e[-n]$ and with $r$ an element of degree $d \in G$, the resulting homogeneous map $A^e[-n] \to A$ is of degree $-n+d$. The derived complex hence takes the form

$$0 \to A \to^{\varphi_1^*} A[1] \to^{\psi_2^*} A[2] \to^{\varphi_3^*} A[3] \to^{\psi_4^*} \dots$$

where $\varphi_{2n+1}^*$ and $\psi_{2n}^*$ denote the induced maps of $\varphi_{2n+1}$ and $\psi_{2n}$. Recall that explicitly the identification between elements of $\mathbf{Hom}_{A^e}(A^e[-n], A)$ and $A[n]$ used above maps $r \in A$ to $f(x \otimes y) = xry$. With this in mind, we see for any such homogeneous

$$f \in \mathbf{Hom}_{A^e}(A^e[-2n-1], A)$$

or respectively $r \in A$ that

$$\varphi_{2n+1}^*(f)(x \otimes y) = f((1 \otimes \epsilon - \epsilon \otimes 1)(x \otimes y)) = f(x \otimes \epsilon y - \epsilon x \otimes y) = xr\epsilon y - \epsilon xry = 0$$

and on the other hand

$$\psi_{2n}^*(f)(x \otimes y) = 2\epsilon xry$$

by similar calculations. Now this means that in characteristic two, all maps in the derived complex are zero and hence it is identical with cohomology, i.e. we have $HH^n(A) = A[n]$ for all $n \in \mathbb{N}$. If $char(K) \neq 2$, we have $Im(\psi_{2n}^*) = (\epsilon)$, hence

$$HH^n(A) \cong \begin{cases} A^e & \text{if } n = 0 \\ K[n] & \text{if } n \text{ odd} \\ K[n+1] & \text{if } n \text{ even} \end{cases}$$

as desired.

We will now look at the second case, i.e. at the supercommutative case with the same grading.

**Example 2.2.9** *Suppose $G = \mathbb{Z}$, $A_0 = K$, $A_1 = (\epsilon)$, and $\chi(n, m) = (-1)^{nm}$, and naturally $char(K) \neq 2$. Then we get*

$$HH^n(A) \cong A^e[n].$$

**Proof** We reuse the notations of the previous proof. Note that $A^e$ has a different multiplicative structure in the supercommutative case than before, given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 b_2.$$

and that the left module action of $A^e$ on $A$ is now for homogeneous elements given by

$$(a \otimes b)x = \chi(|b|, |x|)axb$$

even though in the case of this particular example the righthand side always simplifies to $axb$ since it is zero if both $b$ and $x$ are degree one. We can identify $A^e$ with the supercommutative polynomial ring $K[|X, Y]$ and obtain the free resolution

$$\dots A^e[-3] \to^{\cdot(X-Y)} A^e[-2] \to^{\cdot(X-Y)} A^e[-1] \to^{\cdot(X-Y)} A^e \to^{\mu} A \to 0$$

of $A$ over $A^e$. This time, we denote the map $A^e[-n] \to A^e[-n+1]$ in this resolution by $\varphi_n$. It is known from Ex. 1.2.29 that $\mu$ is $A^e$-linear. For the other maps in the sequence, $A^e$-linearity and degree preservation are easy to check. Also, all $A^e$-modules appearing in the sequence are graded free by construction. We will now prove that all kernels and images in this sequence match.
First, we show by calculation in $A^e$ that

$$(X-Y)(X-Y) = -XY - YX = 0$$

because $X$ and $Y$ anticommute, so we have $Im(\varphi_{n+1}) \subseteq Ke(\varphi_n)$ for all $n \in \mathbb{N}$. A similar calculation shows that $Ke(\mu)$ is generated as a $K$-module by $\{X-Y, XY\}$ and hence as $A^e$-module by $X-Y$. Finally, we have to calculate $Ke(\varphi_n)$. To this end, we compute

$$(X-Y)(\lambda_0 + \lambda_1 X + \lambda_2 Y + \lambda_3 XY) = \lambda_0(X-Y) + (\lambda_1 + \lambda_2)XY$$

for any $\lambda_i \in K$ and conclude that $Ke(\varphi_n)$ is generated by $X-Y$ times the free generator of $A^e[-n]$ as desired.
To calculate $Ext$, assume that $f \in Hom_{A^e}(A^e[-n], A)$ is a homogeneous function given by $f(e_1) = r \in A^e[n]$, where $e_1$ is the standard generator of $A^e[-n]$. The degree of this map is $|r| - n$. We then see for $a, b \in A$ homogeneous that

$$\begin{aligned} \varphi_n^*(f)(a \otimes b) &= f((1 \otimes \epsilon - \epsilon \otimes 1)(a \otimes b)) \\ &= (-1)^{|a|} f(a \otimes \epsilon b) - f(a\epsilon \otimes b) \\ &= (-1)^{|a|}(-1)^{|f|(|a|+|b|+1)} a r \epsilon b - (-1)^{|f|(|a|+|b|+1)} a \epsilon r b. \end{aligned}$$

If here $a$ or $b$ are of degree one, both summands are zero. If on the other hand $a$ is of degree zero, they cancel out. Either way, we see $\varphi_n^* = 0$ for all $n \in \mathbb{N}$ and therefore $HH^n(A) \cong A^e[n]$ for all $n$.

Finally, we deal with the exceptional case $char(K) = 2$ with a homogeneous element $j$ of degree one with $j^2 = 1$:

**Example 2.2.10** *Suppose $G = \mathbb{Z}_2$, $\chi(n,m) = 1$, and $A_0 = K$, and $A_1 = (j)$, where $j \in A$ is an element not equal to 1 with $j^2 = 1$. In this case, we get*

$$HH^n(A) = A^e[n].$$

**Proof** In general, $K[j]$ has a free resolution over $A^e$ of the form

$$\ldots \to A^e[-3] \to^{\cdot(1\otimes j - j \otimes 1)} A^e[-2] \to^{\cdot(1\otimes j + j\otimes 1)} A^e[-1] \to^{\cdot(1\otimes j - j\otimes 1)} A^e \to^{\mu} A \to 0.$$

Since we are in characteristic two, signs do not matter for us. We will denote the map $A^e[-n] \to A^e[-n+1]$ by $\varphi_n$. As in the previous case, we get $\varphi_n^* = 0$ for all $n$, hence $HH^n(A) \cong A^e[n]$.

**Remark 2.2.11** *The calculations of cohomology can of course be easily generalized to the case where $K$ is not a field, but just an arbitrary commutative ring. As well, one can generalize these calculations to arbitrary truncated polynomial rings in principle. The main complications arising with both of these possibilities are not so much related to the procurement of projective resolutions or the subsequent calculation of the Ext modules, but with classifying the admissible color-commutative structures in the step before. The problem that arises in this respect is that even a free module over an arbitrary commutative ring can admit direct sum decompositions with more members than the number of elements in the basis. We do not believe that interesting phenomena appear due to these effects in this example and have not investigated this more closely.*

## 2.3 Universal derivations, interpretation of $H^1(A)$

Suppose as usual that $G$ is an abelian group, that $\chi$ is a bicharacter on $G$, and that $A$ is a $(G,\chi)$-graded algebra over the commutative ring $k$. Suppose further that $M$ is a $(G,\chi)$-graded $A$-bimodule, not necessarily $(G,\chi)$-symmetric. Recall that we adopted the convention that all algebras be unital and that as a graded ring $k$ is treated as concentrated in zero degree.

**Definition 2.3.1** *A* derivation *on $A$ with coefficients in $M$ is a $k$-linear map $D \in \boldsymbol{Hom}_k(A,M)$ such that any homogeneous component $D_g$ of $D$ satisfies for $a,b \in A$ homogeneous elements the equations*

$$D_g(ab) = \chi(|D_g|,|a|)aD_g(b) + D_g(a)b.$$

*The graded $A$-module of derivations on $A$ with coefficients in $M$ will be denoted by $Der(A,M)$, with $Der(A) := Der(A,A)$.*

**Remark 2.3.2** *Suppose that $A$ is as before and that $M$ is a graded $A$-bimodule. Define for any homogeneous $s \in A$ and $r \in M$ a function $ad(r) : A \to M$ by linear extension of*

$$ad(r)(s) := rs - \chi(|m|,|s|)sr = [r,s]$$

*and continue this definition linearly in $r$ to define $ad(r)$ also for nonhomogeneous $r$. Then clearly $ad(r)$ is $k$-linear for any $r \in M$ and the homogeneous components of $ad(r)$ are given by the functions $ad(r_g)$, where $r = \sum_{g\in G} r_g$ is the decomposition of $r$ into homogeneous parts. If $r,a,b \in A$ are three homogeneous elements, we see that*

$$\chi(|r|,|a|)a \cdot ad(r)b + ad(r)(a)b = rab - \chi(|r|,|ab|)abr = ad(r)(ab)$$

*so in fact $ad(r) \in Der(A,M)$ and it is clear that the map*

$$ad : M \to Der(A,M)$$

*so defined is $k$-linear and degree zero, so a graded $k$-module homomorphism.*

Note also that there is the following easy link with the first Hochschild cohomology module:

**Example 2.3.3** *One easily computes*

$$Z^1(A,M) =< \{\varphi \in \boldsymbol{Hom}_k(A,M) : \varphi(ab) = \chi(|\varphi|,|a|)a\varphi(b) + \varphi(a)b\} >$$

*and $B^1(A,M)$ as the module of functions of the form*

$$\varphi(a) := am - ma$$

*for arbitrary $m \in M$. Thus, $H^1(A,M)$ corresponds to the module of $(G,\chi)$ graded derivations modulo derivations induced by elements of $A$ itself.*

We will also take the opportunity to talk shortly about *universal* derivations of a color-commutative algebra. We will for the rest of this section assume that $A$ is a $(G,\chi)$-commutative algebra and will consider any modules over $A$ by default as $(G,\chi)$-symmetric bimodules.

**Definition 2.3.4** *Let $M$ and $A$ be as above and let $d : A \to M$ be a derivation. Then, $d$ is called* universal *if for any other derivation $\delta : A \to N$ on $A$ there exists one and only one $\phi \in \boldsymbol{Hom}_k(M,N)$ such that*

$$\delta = \phi \circ d.$$

It is clear that for universal derivations $d_1 : A \to M$ and $d_2 : A \to N$ we have uniquely defined two bijective maps $\Phi \in \mathbf{Hom}_k(M,N)$ and $\Psi \in \mathbf{Hom}_k(N,M)$ which transform one derivation into the other. As for the existence of the universal derivation, we prove the following:

**Proposition 2.3.5** *Set $I := Ker(\mu : A \otimes A \to A)$. Recall that $A \otimes A$ becomes a color-commutative algebra with multiplication defined by extension of the multiplication rule*

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := \chi(|b_1|,|a_2|)a_1a_2 \otimes b_1b_2$$

*from products of homogeneous elements $a_i$ and $b_i$ to arbitrary elements of $A \otimes A$. Consider the $A$-bimodule $M := I/I^2$ and the map*

$$d : A \to M, d(x) := 1 \otimes x - x \otimes 1$$

*Then, $d$ is a universal derivation on $A$. We will in the future denote the module $M$ by $\Omega^1_{A|k}$ and use the notation $dx$ for the residue class of $1 \otimes x - x \otimes 1$ in $I/I^2$. The module $\Omega^1_{A|k}$ will also be called the* module of Kähler differentials on $A$ over $k$.

**Proof** First, we notice that $I$ is a homogeneous submodule of $A \otimes A$ since $\mu$ is a degree zero map and clearly $k$-linear. Likewise, $I^2$ can easily be seen to have a homogeneous generating subset. We will now prove that $I/I^2$ is a $(G,\chi)$-symmetric $A$-module. To this end, notice that $I$ is generated over $A$ by the elements $1 \otimes x - x \otimes 1$ with homogeneous $x$: for it is clear that $1 \otimes x - x \otimes 1 \in I = Ker(\mu)$ for all $x \in A$, and if on the other hand the term $\sum_{i=1}^n a_i \otimes b_i \in Ker(\mu)$, we obtain $\mu(\sum_{i=1}^n a_i \otimes b_i) = \sum_{i=1}^n a_i b_i = 0$, and so

$$\sum_{i=1}^n a_i \otimes b_i = \sum_{i=1}^n a_i(1 \otimes b_i) = \sum_{i=1}^n a_i(1 \otimes b_i) - (\sum_{i=1}^n a_i b_i) \otimes 1 = \sum_{i=1}^n a_i(1 \otimes b_i - b_i \otimes 1)$$

which is in $< \{1 \otimes x - x \otimes 1 : x \in A\} >$ as promised. Hence, it suffices to test the hypothesis that $I/I^2$ is a symmetric $A$-module for elements of the form $1 \otimes x - x \otimes 1$ with homogeneous $x$. Here we obtain

$$\begin{aligned} & a(1 \otimes x - x \otimes 1) - \chi(|a|, |x|)(1 \otimes x - x \otimes 1)a = a \otimes x - 1 \otimes ax - ax \otimes 1 + \chi(|a|, |x|)x \otimes a \\ = & -(1 \otimes a - a \otimes 1)(1 \otimes x - x \otimes 1) \in I^2 \end{aligned}$$

and so we have verified that $I/I^2$ is symmetric. It is easily verified by direct calculation that $d$ is indeed a derivation, and of degree zero as such. Now, what remains to be proven is universality. To this end, assume that $\delta : A \to N$ is another derivation. Consider the map $\phi : M \to N, 1 \otimes x - x \otimes 1 \mapsto \delta(x)$. This defines an $A$-linear map: as was shown before, the elements $1 \otimes x - x \otimes 1$ generate $I$ as an $A$-module, and so $I^2$ is generated as an $A$-module by elements of the form $(1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1)$. Therefore, to show that our map is well-defined, we only have to prove that all these will have image zero. This is true:

$$\phi((1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1)) = \phi((1 \otimes y - y \otimes 1)x - x(1 \otimes y - y \otimes 1)) = x(\delta(y) - \delta(y)) = 0$$

because of $A$-linearity of $\varphi$. Also because $I$ is generated by elements of the form $1 \otimes x - x \otimes 1$, the values on these elements together with $A$-linearity define $\phi$ for all elements of $I/I^2$, so we are done proving well-definedness.
By definition of $\phi$, the relation $\delta = \phi \circ d$ is clear. Any other map satisfying this equation also has to produce the same values as $\phi$ on the elements of the form $1 \otimes x - x \otimes 1$, so we have uniqueness as well and are done.

## 2.4 Color Pre Lie systems

Later, we will link Hochschild cohomology of a $(G, \chi)$-graded $k$-algebra $A$ with its deformation theory. In this context, certain algebraic tools will be useful which we intend to introduce here. Primarily, our aim in this section is the introduction of *color pre-Lie systems*, which are the colored analogs of the pre-Lie systems originally introduced by Gerstenhaber in (Gerstenhaber [27]). We will also introduce the notion of a *color pre-Lie algebra* and recall the definition of a color Lie algebra:

**Definition 2.4.1** *Suppose that $V$ is a $G$-graded $k$-module, that $\chi : G \times G \to k^*$ is a bicharacter and that $[\cdot, \cdot] : V \times V \to V$ is a $k$-bilinear map satisfying*

$$\begin{aligned} [a, b] &= -\chi(|a|, |b|)[b, a] \quad \textit{(graded skew symmetry)} \\ 0 &= \chi(|c|, |a|)[a, [b, c]] + \chi(|a|, |b|)[b, [c, a]] + \chi(|b|, |c|)[c, [a, b]] \quad \textit{(Jacobi identity)} \end{aligned}$$

*for homogeneous $a, b, c \in V$. To avoid certain anomalies that may be caused by the possible presence of torsion in $A$ we also ask that $[a, a] = 0$ if $\chi(|a|, |a|) = 1$ and $[a, [a, a]] = 0$ for any homogeneous $a$. Then $(V, [\cdot, \cdot])$ is called a* color Lie algebra.

**Remark 2.4.2** *The $(G, \chi)$-graded Jacobi condition is the same as the slightly more digestible identity*

$$[a, [b, c]] = \chi(|a|, |b|)[b, [a, c]] + [[a, b], c].$$

**Example 2.4.3** *Let $A$ be a $(G,\chi)$-graded $k$-algebra and suppose that $Der(A)$ is the module of $(G,\chi)$-graded derivations on $A$. Then, $Der(A)$ is closed under the bracket defined on homogeneous $f,g$ by*

$$[f,g] := f\circ g - \chi(|f|,|g|)g\circ f$$

*and becomes a color Lie algebra with that bracket.*

**Proof** First, we have to prove that for any two graded derivations $f,g\in Der(A)$ the function given by $f\circ g-\chi(|f|,|g|)g\circ f$ is indeed a graded derivation of $A$. It suffices to show this for homogeneous $f$ and $g$, because the decomposition into homogeneous components clearly distributes through the proposed bracket. Assume then that $f,g\in Der(A)$ are homogeneous derivations and that $a,b\in A$ are homogeneous elements of $A$. It is clear that $[f,g]$ is a map of degree $|f|+|g|$. We calculate:

$$\begin{aligned}
[f,g](ab) &= f(g(ab)) - \chi(|f|,|g|)g(f(ab))\\
&= \chi(|g|,|a|),\chi(|f|,|a|)af(g(b)) + \chi(|g|,|a|)f(a)g(b) + \chi(|f|,|g(a)|)g(a)f(b)\\
&+ f(g(a))b - \chi(|f|,|g|)\chi(|f|,|a|)\chi(|g|,|a|)ag(f(b)) - \chi(|f|,|g|)\chi(|f|,|a|)g(a)f(b)\\
&- \chi(|f|,|g|)\chi(|g|,|f(a)|)f(a)g(b) - \chi(|f|,|g|)g(f(a))\\
&= \chi(|g|+|f|,|a|)af(g(b)) + \chi(|g|,|a|)f(a)g(b) + f(g(a))b\\
&- \chi(|f|,|g|+|a|)\chi(|g|,|a|)ag(f(b)) - \chi(|g|,|a|)f(a)g(b) - \chi(|f|,|g|)g(f(a))\\
&= \chi(|g|+|f|,|a|)af(g(b)) - \chi(|f|,|g|+|a|)\chi(|g|,|a|)ag(f(b))\\
&+ f(g(a))b - \chi(|f|,|g|)g(f(a))b\\
&= \chi(|f|+|g|,|a|)a(f(g(b)) - \chi(|f|,|g|)g(f(b))) + (f(g(a)) - \chi(|f|,|g|)g(f(a)))b\\
&= \chi(|f|+|g|,|a|)a[f,g](b) + [f,g](a)b
\end{aligned}$$

as desired. As $k$-bilinearity of the bracket and $[f,f]=0$ for all even $f$ are obvious, only the graded Jacobi identity is left to prove. We calculate for $f,g,h\in Der(A)$ homogeneous:

$$[f,[g,h]] = f\circ g\circ h - \chi(|g|,|h|)f\circ h\circ g - \chi(|f|,|g|+|h|)g\circ h\circ f + \chi(|f|,|g|+|h|)\chi(|g|,|h|)h\circ g\circ f$$

and by cyclic permutation of this and comparison of the three resulting terms the Jacobi identity follows.

Examples of color Lie algebras can be obtained in particular from certain algebras which come from a weakening of associativity relative to color associative algebras and which we call *color pre-Lie algebras*. They are analogs of the *pre-Lie algebras* introduced by Gerstenhaber in (Gerstenhaber [27]):

**Definition 2.4.4** *Suppose that $k$ is a commutative ring, that $A$ is a $G$-graded $k$-module and that $\circ : A\otimes A\to A$ is a $k$-bilinear map of degree zero on $A$. If then the condition*

$$(f\circ g)\circ h - \chi(|g|,|h|)(f\circ h)\circ g = f\circ(g\circ h) - \chi(|g|,|h|)f\circ(h\circ g)$$

*is satisfied for any homogeneous $f,g,h\in A$ we call $(A,\circ)$ a* $(G,\chi)$-pre-Lie-algebra *or* color pre-Lie-algebra*.*

We will not say much about color-pre-Lie-algebras. Our interest in them is mainly due to the following result:

**Proposition 2.4.5** *Suppose that $A$ is a $(G, \chi)$-pre-Lie algebra over the commutative ring $k$. Then, we can define another multiplication on $A$ by setting*

$$[f, g] := f \circ g - \chi(|f|, |g|) g \circ f$$

*for homogeneous elements $f, g \in A$ and extending bilinearly. With this bracket $(A, [\cdot, \cdot])$ becomes a color Lie algebra.*

**Proof** It is easily verified that the bracket is color skew symmetric and that for $f \in A$ even we have $[f, f] = 0$. To see $[f, [f, f]] = 0$ for any homogeneous $f \in A$, we note first that $\chi(|f|, |f| + |f|) = \chi(|f|, |f|)^2 = 1$ for any such $f$. Expanding the definition of the bracket this gives

$$[f, [f, f]] = f \circ (f \circ f) - \chi(|f|, |f|) f \circ (f \circ f) - (f \circ f) \circ f + \chi(|f|, |f|)(f \circ f) \circ f$$

By the defining 'associativity' condition of a $(G, \chi)$-pre-Lie algebra we can transform the first two terms of this into $(f \circ f) \circ f - \chi(|f|, |f|)(f \circ f) \circ f$, after which everything cancels out. We are hence left with the task of proving the color Jacobi identity. To this end, let $f, g, h$ be three arbitrary homogeneous elements of $A$. The equation to be proven, namely

$$[f, [g, h]] = \chi(|f|, |g|)[g, [f, h]] + [[f, g], h]$$

is equivalent to

$$\begin{aligned}
& f \circ (g \circ h) - \chi(|g|, |h|) f \circ (h \circ g) - \chi(|f|, |g| + |h|)(g \circ h) \circ f \\
& +\chi(|f|, |g| + |h|)\chi(|g|, |h|)(h \circ g) \circ f \\
= \ & \chi(|f|, |g|) g \circ (f \circ h) - \chi(|f|, |g| + |h|) g \circ (h \circ f) - \chi(|g|, |h|)(f \circ h) \circ g \\
+ \ & \chi(|f| + |g|, |h|)(h \circ f) \circ g + f \circ (g \circ h) - \chi(|g|, |h|) f \circ (h \circ g) \\
- \ & \chi(|f|, |g| + |h|)(g \circ h) \circ f + \chi(|f|, |g| + |h|)\chi(|g|, |h|)(h \circ g) \circ f
\end{aligned}$$

Now we use the color pre-Lie property to transform all the terms of the form

$$a \circ (b \circ c) - \chi(|b|, |c|) a \circ (c \circ b)$$

into terms of the form

$$(a \circ b) \circ c - \chi(|b|, |c|)(a \circ c) \circ b$$

at which point everything cancels out, proving the proposition.

Of particular importance to us will be a specific way to construct a color pre-Lie-ring which we will describe now. It relies on yet another type of algebraic structure, namely on *color pre Lie systems*. The primary motivation behind this construction in the context of this thesis will be that the space of Hochschild cochains of a $(G, \chi)$-colored associative algebra can in a useful way be given a color pre-Lie system structure. We define a color pre-Lie system in the following way:

**Definition 2.4.6** *Suppose that $k$ is a commutative ring, that $G$ is an abelian group, and that $\chi : G \times G \to k^*$ is a bicharacter. Suppose that we are given a $\mathbb{Z}\times G$-graded $k$-module $V = \oplus_{z\in\mathbb{Z}} V_z$ and a family of degree zero bilinear maps $\{\circ_i : V \otimes V \to V | i \in \mathbb{Z}\}$ such that we have the association laws*

$$(f \circ_i g) \circ_j h = \begin{cases} \chi(|g|,|h|)(f \circ_j h) \circ_{i+p} g & \text{for} \quad 0 \le j < i \\ f \circ_i (g \circ_{j-i} h) & \text{for} \quad i \le j \le n+i \end{cases} \tag{2.1}$$

*where $f \in V_m, g \in V_n, h \in V_p$ are supposed homogeneous and where by $|f|, |g|, |h|$ we mean the degrees of the corresponding elements within these $G$-graded submodules of $V$. Then, we call $V$ together with the maps $\circ_i$ a* color pre-Lie system.

We first show a small lemma deducing a third identity which is sometimes important from the two given in the above definition:

**Lemma 2.4.7** *With the same notations as above, the relation*

$$(f \circ_i g) \circ_j h = \chi(|g|,|h|)(f \circ_{j-n} h) \circ_i g$$

*is automatically true given $n + i + 1 \le j \le n + m$.*

**Proof** Suppose $n + i + 1 \le j$ and set $j' := i, i' := j - n$. Then we can write

$$(f \circ_i g) \circ_j h = (f \circ_{j'} g) \circ_{i'+n} h$$

where the latter term can be transformed by the first part of the association rule for a color pre Lie system into

$$\chi(|g|,|h|)(f \circ_{i'} h) \circ_{j'} g$$

if we have $0 \le j' \le i' - 1$. Adding $n+1$ to all terms in this inequality shows that this condition is the same as $i + n + 1 \le j$, which gives us the stated identity.

The following result is important as it describes how a color pre-Lie system $(V, \circ_i)$ gives rise to a color pre-Lie ring structure on $V$:

**Proposition 2.4.8** *Suppose that $(V, \circ_i, G, \chi)$ is a color pre-Lie system. For $f \in V_m, g \in V_n$ homogeneous elements we set*

$$f \circ g := \sum_{i=0}^{m} (-1)^{in} f \circ_i g$$

*and extend linearly to arbitrary elements in $V$. Using the bicharacter $\tilde{\chi} : (\mathbb{Z} \times G)^2 \to k^*$ given by*

$$\tilde{\chi}((n_1, g_1), (n_2, g_2)) = (-1)^{n_1 n_2} \chi(g_1, g_2)$$

*we get then a structure of $(\mathbb{Z} \times G, \tilde{\chi})$-pre-Lie algebra on $(V, \circ)$.*

**Proof** First, we fix some notations which we will use for this proof. Given a homogeneous element $f \in V$, we will denote by $deg_{\mathbb{Z}}(f)$ and $deg_G(f)$ the projections of $deg(f)$ to $\mathbb{Z}$ and $G$ respectively. By $|f|$ we will mean $deg(f)$ or $deg_G(f)$ depending on the context.
Now, we have to show that

$$(f \circ g) \circ h - \tilde{\chi}(|g|, |h|)(f \circ h) \circ g = f \circ (g \circ h) - \tilde{\chi}(|g|, |h|) f \circ (h \circ g)$$

for any homogeneous $f, g, h \in V$. Suppose more specifically $f \in V_m, g \in V_n, h \in V_p$. Then expanding the definition of $\circ$, we calculate

$$\begin{aligned} (f \circ g) \circ h &= \sum_{i=0}^{m} \sum_{j=0}^{m+n} (-1)^{in+jp} (f \circ_i g) \circ_j h \\ &= \sum_{i=0}^{m} \sum_{j=0}^{i-1} (-1)^{in+jp} (f \circ_i g) \circ_j h + \sum_{i=0}^{m} \sum_{j=i}^{n+i} (-1)^{in+jp} (f \circ_i g) \circ_j h + \\ & \sum_{i=0}^{m} \sum_{j=i+n+1}^{n+m} (-1)^{in+jp} (f \circ_i g) \circ_j h \end{aligned} \tag{2.2}$$

and

$$f \circ (g \circ h) = \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{i(n+p)+jp} f \circ_i (g \circ_j h) \tag{2.3}$$

Now using the association rule of a color pre-Lie system and shifting start and end of the summation appropriately, we can transform the middle term of (Eq. 2.2) into

$$\sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{in+(j+i)p} f \circ_i (g \circ_j h)$$

which is the same as the righthand side of (Eq. 2.3). In $(f \circ g) \circ h - f \circ (g \circ h)$ the middle term of $(f \circ g) \circ h$ is therefore seen to cancel out and we get

$$(f \circ g) \circ h - f \circ (g \circ h) = \sum_{i=0}^{m} \sum_{j=0}^{i-1} (-1)^{in+jp} (f \circ_i g) \circ_j h + \sum_{i=0}^{m} \sum_{j=i+n+1}^{n+m} (-1)^{in+jp} (f \circ_i g) \circ_j h \tag{2.4}$$

and substituting $g$ for $h$ and vice-versa in this equation gives on the other hand

$$(f \circ h) \circ g - f \circ (h \circ g) = \sum_{i=0}^{m} \sum_{j=0}^{i-1} (-1)^{ip+jn} (f \circ_i h) \circ_j g + \sum_{i=0}^{m} \sum_{j=i+p+1}^{m+p} (-1)^{ip+jn} (f \circ_i h) \circ_j g$$

Now we can use Lemma 2.4.7 and the first part of (Eq. 2.1) to rewrite the righthand side of

(Eq. 2.4) to

$$\begin{aligned}
&\chi(|g|,|h|)(\sum_{i=0}^{m}\sum_{j=0}^{i-1}(-1)^{in+jp}(f\circ_j h)\circ_{i+p} g+\sum_{i=0}^{m}\sum_{j=i+n+1}^{n+m}(-1)^{in+jp}(f\circ_{j-n} g)\circ_i g)\\
=\ &\chi(|g|,|h|)(\sum_{j=p}^{m+p}\sum_{i=0}^{j-1-p}(-1)^{(j-p)n+ip}(f\circ_i h)\circ_j g+\sum_{j=0}^{m}\sum_{i=j+1}^{m}(-1)^{jn+(i+n)p}(f\circ_i h)\circ_j g)\\
=\ &\chi(|g|,|h|)(\sum_{i=0}^{m}\sum_{j=i+1+p}^{m+p}(-1)^{jn+(i+n)p}(f\circ_i h)\circ_j g+\sum_{i=0}^{m}\sum_{j=0}^{i-1}(-1)^{jn+(i+n)p}(f\circ_i h)\circ_j g)
\end{aligned}$$

and so get

$$(f\circ g)\circ h-f\circ(g\circ h)=(-1)^{pn}\chi(|g|,|h|)((f\circ h)\circ g-f\circ(h\circ g))$$

in the end, which is the same as the defining identity of a color-pre-Lie algebra with bicharacter $\tilde{\chi}$, namely

$$(f\circ g)\circ h-\tilde{\chi}(|g|,|h|)(f\circ h)\circ g=f\circ(g\circ h)-\tilde{\chi}(|g|,|h|)f\circ(h\circ g)$$

This concludes the proof.

## 2.5 The color Gerstenhaber bracket

We already saw that derivations with target $A$ can be identified with Hochschild 1-cocycles with coefficients in $A$. We also saw that the space of graded derivations with coefficients in $A$ over a $(G,\chi)$-colored associative algebra $A$ carries a structure of color Lie algebra. The following proposition may be viewed as a generalization of this observation to arbitrary dimension:

**Definition/Proposition 2.5.1** *Suppose that $A$ is a $(G,\chi)$-colored associative algebra over the commutative ring $k$. Then, for any $f\in C^{n+1}(A)$ and $g\in C^{m+1}(A)$ one defines an $i$-th composition by linear extension of*

$$\begin{aligned}
&(f\circ_i g)(a_0\otimes\ldots\otimes a_{n+m}):=\\
&\chi(|g|,\sum_{j=0}^{i-1}|a_j|)f(a_0\otimes a_1\otimes\ldots\otimes a_{i-1}\otimes g(a_i\otimes\ldots\otimes a_{i+m})\otimes a_{i+m+1}\otimes\ldots\otimes a_{n+m}).
\end{aligned}$$

*We then set*

$$(f\circ g)(a_0\otimes\ldots\otimes a_{n+m}):=\sum_{i=0}^{n}(-1)^{in}(f\circ_i g)(a_1\otimes\ldots\otimes a_{n+m-1})$$

*and finally*

$$[f,g]:=f\circ g-(-1)^{nm}\chi(|f|,|g|)g\circ f$$

*again for homogeneous elements and extend everything to arbitrary linear combinations of those. Denote further for a $\mathbb{Z}$-graded $k$-module $V$ by $V[1]$ the $\mathbb{Z}$-graded module shifted by one, i.e. $V[1]_n:=V_{n+1}$. Then, we get the following statements:*

1. *$C^*(A)[1]$ together with the bracket $[\cdot,\cdot]$ and the grading given naturally over $\mathbb{Z}\times G$ and the bicharacter*

$$\tilde{\chi}((n_1, g_1), (n_2, g_2)) := (-1)^{n_1 n_2}\chi(g_1, g_2)$$

*becomes a color Lie algebra.*

2. *We have $\beta(f) = (-1)^{n-1}[\mu, f]$ for any Hochschild cochain $f \in C^n(A)$, where $\mu : A\otimes A \to A$ is the multiplication in $A$.*

3. *The given bracket induces a color Lie algebra structure also on the Hochschild cohomology modules.*

4. *On $H^1(A)$ the Lie algebra structure so given coincides with the color Lie algebra structure on the module of derivations on $A$.*

*We will refer to both the bracket $[\cdot,\cdot]$ so defined on $C^*(A)[1]$ and the induced bracket in cohomology as the* color Gerstenhaber bracket*. The composition operation $\circ$ of the underlying color pre-Lie structure on $C^*(A)[1]$ will be called the* color Gerstenhaber composition.

**Proof** 1. We will show the first part by proving that partial composition induces a color pre-Lie system structure on $C^*(A)[1]$. The $\circ$ operation is then easily seen to be the corresponding color pre-Lie algebra multiplication, and the given bracket the color Lie algebra bracket induced by this color pre-Lie algebra. To see the color pre-Lie property, suppose that $f, g, h$ are homogeneous Hochschild cochains and that $f \in C^{m+1}(A), g \in C^{n+1}(A), h \in C^{p+1}(A)$. Then, we will have to prove the following composition rule:

$$(f \circ_i g) \circ_j h = \begin{cases} \chi(|g|, |h|)(f \circ_j h) \circ_{i+p} g & \text{for} \quad 0 \le j < i \\ f \circ_i (g \circ_{j-i} h) & \text{for} \quad i \le j \le n+i \end{cases}$$

for all $a_i$ homogeneous elements. Suppose first that $0 \le j < i$. Then writing $f(\ldots)$ for

$$f(a_0 \otimes \ldots \otimes h(a_j \otimes \ldots \otimes a_{j+p}) \otimes \ldots \otimes a_{m+n+p})$$

we find

$$(f \circ_i g) \circ_j h(a_0 \otimes \ldots \otimes a_{m+n+p}) = \chi(|h|, \sum_{k=0}^{j-1} |a_k|)\chi(|g|, |h| + \sum_{k=0}^{i+p-1} |a_i|)f(\ldots)$$

and

$$(f \circ_j h) \circ_{i+p} g = \chi(|g|, \sum_{k=0}^{i+p-1} |a_k|)\chi(|h|, \sum_{k=0}^{j-1} |a_k|)f(\ldots)$$

giving the desired identity in the case at hand. Consider now the other case, $i \le j \le n+i$. Then we write $f(\ldots)$ for

$$f(a_0 \otimes \ldots \otimes g(a_i \otimes \ldots \otimes h(a_j \otimes \ldots) \otimes \ldots) \otimes \ldots)$$

to find

$$(f \circ_i g) \circ_j h(a_0 \otimes \ldots \otimes a_{m+n+p}) = \chi(|h|, \sum_{k=0}^{j-1} |a_k|)\chi(|g|, \sum_{k=0}^{i-1} |a_k|) f(\ldots) = f \circ_i (g \circ_{j-i} h)$$

as desired.
2. We prove that the stated equation holds for a homogeneous cochain $f$ and on homogeneous inputs. The general statement follows then by linear extension. Hence, suppose that $a_i \in A$ are homogeneous elements of $A$ and that $f \in C`n + 1(A)$ is a homogeneous cochain. Then we calculate

$$[\mu, f](a_1 \otimes \ldots \otimes a_{n+1}) = (-1)^{n-1}(\chi(|f|, |a_1|)a_1 f(a_2 \otimes \ldots \otimes a_{n+1})$$
$$+(-1)^{n+1} f(a_1 \otimes \ldots \otimes a_n)a_{n+1} - \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}))$$

which is clearly identical to $(-1)^{n-1}\beta(f)(a_1 \otimes \ldots \otimes a_{n+1})$ as claimed.
3. In particular, we can use this observation to prove that the bracket $[\cdot, \cdot]$ remains well-defined if we pass to cohomology. First, we show that the product of two coycles $f \in C^m(A), g \in C^n(A)$ is again a cocycle. We use the color Jacobi identity

$$[\mu, [f, g]] = \tilde{\chi}(|f|, |g|)[f, [\mu, g]] + [[\mu, f], g]$$

where on the righthand side both terms are zero because $f, g$ were supposed cocycles.
Second, we show that the bracket product is on the set of cocycles invariant under addition of coboundaries. To prove this, it is sufficient to show that $[f, g]$ and $[g, f]$ is a coboundary whenever $f$ is a coboundary coming from a homogeneous cochain and $g$ is a homogeneous Hochschild cocycle. This is again proved by the color Jacobi identity which implies for $f$ any homogeneous cochain and $g$ a homogeneous cocycle that $[[\mu, f], g] = [\mu, [f, g]]$ and $[g, [\mu, f]] = \tilde{\chi}(|g|, |f|)[\mu, [f, g]]$.
4. This is now immediate from the definitions of the bracket on derivations respectively Hochschild cocycles.

The color Gerstenhaber bracket shares the following important property with its uncolored cousin:

**Lemma 2.5.2** *Suppose that $A$ is a $(G, \chi)$-graded algebra over the commutative ring $k$ and that $\mu : A \otimes A \to A$ is a $k$-bilinear map of degree zero. Assume further that $A$ is 2-torsionfree. Then $\mu$ induces an associative multiplication on $A$ if and only if $[\mu, \mu] = 0$.*

**Proof** As $|\mu| = 0$ by definition, we see for any triple of homogeneous elements $a, b, c \in A$ that

$$[\mu, \mu](a, b, c) = 2(\mu \circ \mu)(a, b, c) = 2(\mu(\mu(a, b), c) - \mu(a, \mu(b, c))).$$

The claim follows immediately from 2-torsionfreeness of $A$.

# Chapter 3

# Deformation theory in the $(G, \chi)$-graded category

In this chapter, we will discuss the deformation theory of $(G, \chi)$-graded algebras. The main differences to the ordinary case of deformations of associative algebras are the presence of certain restrictions on deformations coming from the requirement that the deformation ring be graded over the same group as the original and the possibility to use mildly noncentral deformation parameters. Also, as we will see, the trivial deformation itself is a more complicated object in this category than in the category of associative algebras.
We will from now on assume that $A$ is a $(G, \chi)$-graded algebra over the commutative ring $k$.

## 3.1 Power series rings

The purpose of the present section is to adapt the notion of a power series ring to the colored context. The goal is to define from a $(G, \chi)$-graded $k$-algebra $A$ a $G$-graded $k$-algebra $A[[X]]$ equipped additionally with the same bicharacter $\chi$ as the original such that the following heuristic conditions are met:

- $X$ is considered a homogeneous element of some fixed degree $g \in G$ and obeys the commutation rules $aX = \chi(|a|, |X|)Xa$ for homogeneous $a \in A$
- $A[[X]]$ itself is in a natural way a $G$-graded algebra
- the product on $A[[X]]$ should be modeled as closely as possible on the Cauchy product on power series algebras over ungraded rings
- $A/(X) \cong A$ as a graded algebra.

Of course, in the case of $G = 0$ our notion of power series ring should recover the usual concept for ungraded algebras.
Given these minimal conditions, there is still one important choice to make, namely whether or not we should ask the formal parameter $X$ to commute with itself in the same way as it does

with elements of $A$. In the case that $A$ is a superalgebra over e.g. a characteristic zero field and if we are to construct the power series ring in one *odd* variable $X$, this amounts to asking whether we the formal variable $X$ should truly be in the supercenter of $A[[X]]$ and hence satisfy the relation $X^2 = 0$, or whether there should be an infinite succession of powers $X^n$ of $X$ all linearly independent over the base ring as in the case of an *even* variable.
Note that if $A$ was a supercommutative algebra, then the second of these possibilities would prevent the to-be-constructed power series ring $A[[X]]$ from also being supercommutative. As in general for color-commutative algebras we will want the trivial deformation of the color-commutative algebra in question to still be in the category of color-commutative algebras, we will decide to enforce these commutation conditions among powers of $X$ in our definitions.
Having defined such an extension of the original algebra $A$, we will be able to talk about deformations of $A$ in the $(G, \chi)$-graded category. We start by setting the following

**Definition/Proposition 3.1.1** *Let $k$ be a commutative ring, $G$ an abelian group, $\chi : G \times G \to k^*$ a bicharacter and $A$ be a $(G, \chi)$-graded algebra. Fix an element $g \in G$ and set $\lambda := \chi(g, g)$. Let $V$ be the $A$-module generated (under pointwise multiplication) by maps $f : \mathbb{N} \to A$ with the property that there exists a $g_f \in G$, dependent on $f$, such that for any $n \in \mathbb{N}$ we have $f(n) \in A_{g_f - ng}$, and consider such $f$ to be of degree $g_f$ if not zero. Let $W'$ be the submodule of $V$ containing the functions $f \in V$ with $f(0) = 0, f(1) = 0$ and $W := (1 - \lambda)W'$. Set then $A[[X]] := V/W$ as an $A$-module and write $\sum_{i \geq 0} \lambda_i X^i$ for the equivalence class of the function given by $f(n) = \lambda_n \in A$. Then, introducing for $f = \sum_{i \geq 0} \lambda_i X^i$ and $f' = \sum_{i \geq 0} \mu_i X^i$ elements of $A[[X]]$ with $\lambda_i$ and $\mu_i$ homogeneous the multiplication law*

$$f \cdot f' := \sum_{n \geq 0} \sum_{i+j=n} \chi(g, |\mu_j|)^i \lambda_i \mu_j X^n \tag{3.1}$$

*makes $A[[X]]$ into a $G$-graded associative ring. If $A$ was color-commutative with bicharacter $\chi$, then so is $A[[X]]$.*
*We call $A[[X]]$ the power series ring over $A$* in degree $g$. *Note that the construction presented really depends on the choice of $g$, and by the given notations, $X$ is an element of degree $g$ in $A[[X]]$.*
*The same multiplication gives a ring structure also on $V$ itself. We refer to that ring as $A[[\tilde{X}]]$ or as the* alternative *power series ring over $A$ in degree $g$.*

**Proof** It is clear from the definitions that $V$ is a $G$-graded module and that $W'$ is a homogeneous submodule of $V$. Since $1 - \lambda$ is a degree zero element of $A$, also $W$ is a homogeneous submodule of $V$ and the quotient $A[[X]] = V/W$ is well-defined as a $G$-graded two-sided $A$-module. What remains to be proven are the properties of the proposed Cauchy product, i.e. well-definedness, associativity, compatibility with the grading and in the case of color-commutative $A$ also color-commutativity.
We first show that the given product defines an associative multiplication of degree zero on $V$, where well-definedness is not yet an issue. Suppose for this that $f, f', f'' \in V$ are three arbitrary homogeneous elements with

$$f = \sum_{i \geq 0} \lambda_i X^i, f' = \sum_{j \geq 0} \mu_j X^j, f'' = \sum_{k \geq 0} \rho_k X^k$$

where we use the natural notations. Then, we see

$$(f \cdot f') \cdot f'' = \sum_{n \geq 0} \sum_{i+j+k=n} \chi(|X|, |\mu_j|)^i \chi(|X|, |\rho_k|)^{i+j} \lambda_i \mu_j \rho_k X^n$$

and

$$f \cdot (f' \cdot f'') = \sum_{n \geq 0} \sum_{i+j+k=n} \chi(|X|, |\mu_j| + |\rho_k|)^i \chi(|X|, |\rho_k|)^j \lambda_i \mu_j \rho_k X^n$$

and therefore associativity. That the proposed multiplication is degree zero follows with the same $f$ and $f'$ by calculating that the terms $\chi(|X|, |\mu_j|)^i \lambda_i \mu_j X^{i+j}$ are of degree

$$|\lambda_i| + |\mu_j| + (i+j)|X| = |\lambda_i| + i|X| + |\mu_j| + j|X| = |f| + |f'|$$

for homogeneous $f$ and $f'$.
To see well-definedness, we remark that the submodule $W$ is a graded ideal of the graded ring $(V, \cdot)$. From this it follows immediately that the multiplication given induces a $G$-graded ring structure also on the quotient.
Finally, we have to show that $A[[X]]$ is color-commutative whenever $A$ was. It is sufficient to show this for homogeneous generators, so we can again take

$$f = \sum_{i \geq 0} \lambda_i X^i, f' = \sum_{j \geq 0} \mu_j X^j$$

with $\lambda_i \in A_{|f| - i|X|}, \mu_j \in A_{|f'| - j|X|}$. By the definitions we have

$$f \cdot f' = \sum_{n \geq 0} \sum_{i+j=n} \chi(|X|, |\mu_j|)^i \lambda_i \mu_j X^n$$

and we calculate

$$f' \cdot f = \sum_{n \geq 0} \sum_{i+j=n} \chi(|X|, |\lambda_i|)^j \chi(|\mu_j|, |\lambda_i|) \lambda_i \mu_j X^n$$

We will compare corresponding coefficients to prove color-commutativity. Indeed, we find using the relations $|f| = |\lambda_i| + i|X|$ and $|f'| = |\mu_j| + j|X|$ that

$$\chi(|X|, |\lambda_i|)^j \chi(|\mu_j|, |\lambda_i|) = \chi(|f'|, |f|) \chi(|X|, |f'|)^i$$

and

$$\chi(|X|, |\mu_j|)^i = \chi(|X|, |f'|)^i \chi(|X|, |X|)^{ij} = \chi(|X|, |f'|)^i \lambda^{ij}$$

whereupon coefficient comparison shows that the coefficients of $f'f$ can be obtained from the coefficients of $ff'$ by multiplying with a factor of the form $\chi(|f|, |f'|)\lambda^{ij}$, where $i + j =: n$ is the power of $X$ associated to that coefficient. This however means that $ff' = \chi(|f|, |f'|)f'f$ as desired: for if $n < 2$ we have $i = 0$ or $j = 0$ in $i + j = n$, and if $n \geq 2$ the factor $\lambda^{ij}$ is multiplied by $X^n$. In the first case, we have $\lambda^{ij} = 1$. In the second case, $\lambda$ by the definition of $A[[X]]$ acts on $X^n$ like unity, so the factor $\lambda^{ij}$ poses no problem also then. This concludes the proof.

**Remark 3.1.2** *Using the notation $X^n a$ to denote $\chi(|X|, a)^n a X^n$ for homogeneous $a \in A$, one gets another description of the same product by the formula*

$$(\sum_{i \geq 0} X^i \lambda_i)(\sum_{i \geq 0} X^i \mu_i) = \sum_{n \geq 0} X^n \sum_{i+j=n} \chi(|\lambda_i|, |X|)^j \lambda_i \mu_j.$$

*We will mostly use this notation in the future.*

We will now look at some cases where the construction given in Def./Prop. 3.1.1 recovers more or less familiar notions:

**Example 3.1.3** *When $G = 0$, we have automatically $g = 0$ and $\lambda = 1$ and all elements of $A$ are of course degree zero. An arbitrary formal sum $\sum_{i \geq 0} X^i \lambda_i$ with $\lambda_i \in A$ then satisfies the requirements of being homogeneous of degree zero, i.e. $\lambda_i \in A_{-ig}$, and so every power-series in the classical sense becomes in this case a power series as per our definitions and vice versa. In the definition of the Cauchy product, all additional factors vanish in this case, recovering the ordinary product rule for power series rings.*

**Example 3.1.4** *More generally, suppose that $G$ is an arbitrary abelian group, but that $g = 0$. In this case, our construction purports to construct the power series ring* in degree zero *over $A$. Let us try to see what this means. We have automatically $\lambda = 1$, and therefore no commutation conditions on the powers of $X$. An element $f := \sum_{i \geq 0} X^i \lambda_i$ is by the definitions considered homogeneous if and only if all $\lambda_i$ are homogeneous and of same degree, say $\lambda_i \in A_{g_f}$ for some $g_f \in G$. Since $g = 0$, the factors $\chi(g, |\mu_j|)$ in the defining (Eq. 3.1) of the Cauchy product in our category are all equal one, so again we recover the ordinary Cauchy product. However, an arbitrary element of $A[[X]]$ in the sense of our definition is a* finite *sum of homogeneous elements, and therefore as an ungraded ring our $A[[X]]$ is in this case in general* not *isomorphic to the power series ring in one variable over $A$ viewed as an ungraded algebra. For instance, if $G = \mathbb{Z}$ and $A = k[h]$ is the polynomial ring in one variable with standard $\mathbb{Z}$-grading, then $A[[X]]$ will not contain the element $\sum_{i \geq 0} X^i h^i$ as there is no way of writing this as a finite sum of homogeneous terms. Note that this problem can not appear here if $G$ is finite.*

**Example 3.1.5** *Suppose, on the other hand, that $G$ is an arbitrary abelian group and that $g \neq 0$, but that $A$ is concentrated in degree zero. One important example of this situation is the graded power series ring in degree $g \neq 0$ over the commutative base ring $k$. As in this case all coefficients appearing in the power series are degree zero, the corrective factors in (Eq. 3.1) vanish again, giving the ordinary Cauchy product. However, the structure of $A[[X]]$ depends in this case strongly on the order of $g$ and on $\lambda = \chi(g, g)$. Assume for the sake of simplicity that $\lambda = 1$. Then one has to distinguish principally between the cases of $ord(g)$ finite and $ord(g)$ infinite.*
*If $g$ generates an infinite subgroup of $G$, then the requirement that in a homogeneous element $f := \sum_{i \geq 0} X^i \lambda_i$ each coefficient $\lambda_i$ should be in $A_{deg(f)-ig}$ together with the absence of coefficients of nonzero degree means that the only homogeneous terms in $A[[X]]$ are of the form $X^i \lambda_i$. As arbitrary elements of $A[[X]]$ are finite sums of those, and as the product on $A[[X]]$ is given by the ordinary Cauchy product, we see that here $A[[X]]$ has in fact the algebraic structure of the*

*polynomial ring in one variable over the ring A.*
*If, on the other hand, g is of finite order $n \in \mathbb{N}$, we can decompose any formal sum $f = \sum_{i\geq 0} X^i\lambda_i$ into a finite sum of the form*

$$\sum_{i=0}^{n-1}\sum_{j\geq 0} X^{jn+i}\lambda_{jn+i}$$

*wherein every summand of the outer sum is a homogeneous power series. In this case, therefore, $A[[X]]$ is identical, after forgetting the graded structure, to the ordinary power series ring over A.*

**Example 3.1.6** *As a final example, suppose that $G = \mathbb{Z}$ and that A is a $\mathbb{Z}$-graded, commutative ring such that $A_i = 0$ for all $i < 0$ and $A_i \neq 0$ for all $i \geq 0$. We have already looked at the case $g = 0$. Assume now that $g \neq 0$ but that the bicharacter $\chi : G \times G \to k^*$ is trivial, i.e. $\chi(n, m) = 1$. We recover again the normal Cauchy product. However, nonetheless the algebraic structure of $A[[X]]$ depends in this case on g, most notably on the* sign *of g.*
*We look at the case $g > 0$ first. By definition, homogeneous elements are of the form $f := \sum_{i\geq 0} X^i\lambda_i$ with $|\lambda_i| + ig = c_f$ for all nonzero $\lambda_i$ with some constant $c_f \in \mathbb{Z}$ depending on f. If $g > 0$, these sums are necessarily finite. Since on the other hand $X^i a$ is a finite sum of homogeneous terms for any $i \in \mathbb{Z}$ and $a \in A$, we get again the polynomial ring in one variable over A.*
*Consider on the other hand the case $g < 0$. Then, due to $A_i \neq 0$ for all $i \geq 0$, there is no shortage of possible coefficients of degree $-ig$ for any such i. Graded power series over A are in this case still more restricted however than ungraded power series in one interesting sense which we will shortly describe. If one sets $\mathfrak{a} := \sum_{i\geq 1} A_i$, it is well known that A becomes an ultrametric topological ring when endowed with the metric induced by the $\mathfrak{a}$-adic topology. Suppose now that A is complete with respect to this topology, take any homogeneous graded power series $f = \sum_{i\geq 0} X^i\lambda_i$ and set $c := |f|$. Then f can be identified with a map $f : A \to A$ obtained by setting $f(x) = \sum_{i\geq 0} x^i\lambda_i$, because $\lambda_i \in A_{ig}$ means that $x^i\lambda_i \in \mathfrak{a}^{ig}$, which because of the aforementioned properties of the $\mathfrak{a}$-adic topology implies convergence of the infinite series.*

Summing up, we notice that there seem to be several reasonable notions of a power series ring over a $(G, \chi)$-graded algebra A. First, we have a choice as to the degree g of the formal variable. Second, we can choose between imposing and not imposing the commutation relations given by $\chi$ on the powers of X.
In the context of formal deformation theory, the power series ring $A[[X]]$ in a single variable over an associative algebra A gives the trivial deformation of A. The fact that in our category we have many mutually different and not obviously unreasonable choices to make in constructing the power series ring means that we will obtain, for each of these choices, another theory of deformations of our algebra A. With a view towards obtaining a cohomological description of deformation theory, it is reasonable to expect that similar to the classical case, classification of formal deformations of $(G, \chi)$-graded algebras will be linked to graded Hochschild cohomology. Since the Hochschild cohomology modules as defined in chapter two are themselves G-graded,

it stands to reason to believe that they will by themselves be able to take care of keeping track of differences in deformation theory arising from taking as trivial deformation power series in differing degrees $g$. We will see that this hope is *largely* true. However, the difference between choosing as trivial deformation a power series ring $A[[X]]$ over $A$ or an alternative power series ring $A[[\tilde{X}]]$ is real and will not be immediately captured by cohomology. Still the theory arising from $A[[\tilde{X}]]$ is so similar to the one arising from $A[[X]]$ in the case of $X$ an even formal variable that we will give it no separate treatment.

**Remark 3.1.7** *Using similar arguments as in the proof of Prop./Def. 3.1.1, one can also see that $A[[X]]$ as defined becomes a left $k[[X]]$-module (where $k$ is viewed as a $G$-graded ring concentrated in degree zero and where $X$ is supposed to be of the same degree as in $A[[X]]$) with the multiplication defined for homogeneous generators by*

$$(\sum_{i\geq 0} X^i \lambda_i)(\sum_{i\geq 0} X^i \alpha_i) := \sum_{n\geq 0} X^n (\sum_{i+j=n} \lambda_i \alpha_j).$$

*Here $\sum_{i\geq 0} X^i \lambda_i \in k[[X]]$ and $\sum_{i\geq 0} X^i \alpha_i \in A[[X]]$.*
*Likewise, $A[[X]]$ becomes a right $k[[X]]$-module with the multiplication*

$$(\sum_{i\geq 0} X^i \alpha_i)(\sum_{i\geq 0} X^i \lambda_i) = \sum_{n\geq 0} X^n \sum_{i+j=n} \chi(|\alpha_i|, |X|)^j \alpha_i \lambda_j$$

*for $\sum_{i\geq 0} X^i \alpha_i \in A[[X]]$ and $\sum_{i\geq 0} X^i \lambda_i \in k[[X]]$ homogeneous generators of $A[[X]]$ and $k[[X]]$ respectively. $A[[X]]$ is $(G, \chi)$-symmetric as $k[[X]]$-bimodule.*

**Remark 3.1.8** *Similarly, the tensor product $A[[X]] \otimes_k A[[X]]$ becomes an $k[[X]]$-bimodule by left multiplication on the left factor and right multiplication on the right factor. Well-definedness of this operation can be seen by considering that for any $\Lambda \in k[[X]]$, the map $A[[X]] \times A[[X]] \to A[[X]]$ taking $\alpha_1, \alpha_2$ to $(\Lambda\alpha_1)\alpha_2$ is $k$-bilinear and induces therefore a well-defined linear map $A[[X]] \otimes A[[X]] \to A[[X]]$ and by going through the same argument for right multiplication.*

**Remark 3.1.9** *Note that, being a special case of Ex. 3.1.5, $k[[X]]$ can look quite different from what one might expect naively from experience with ungraded power series rings. Indeed, if for instance $G = \mathbb{Z}$, $|X| = 1$ and $\chi$ trivial, one gets by reasoning as in Ex. 3.1.5 $k[[X]] \cong k[X]$ as graded algebras, where $k[X]$ denotes the ordinary polynomial ring over $k$ in one variable of degree one.*

An obvious consequence of the preceding remark is that in these $(G, \chi)$-graded power series rings, power series are not in general invertible whenever the entry in order zero is invertible, as is the case for ordinary power series. As this issue has some relevancy for instance for the definition of formal isomorphisms between deformed products, we provide the following remark:

**Remark 3.1.10** *Suppose that $f = \sum_{i\geq 0} X^i \alpha_i \in A[[X]]$ is a* homogeneous *element, i.e. that for all $i$ we have $\alpha_i \in A_{deg(f)-i|X|}$. Then $f$ is invertible if and only if $\alpha_0$ is invertible in $A$.*

**Proof** Suppose $f$ is invertible with inverse $f' := \sum_{i\geq 0} X^i\beta_i$. According to the definitions, we have

$$(\sum_{i\geq 0} X^i\alpha_i)(\sum_{i\geq 0} X^i\beta_i) = \sum_{n\geq 0} X^n \sum_{i+j=n} \chi(|\alpha_i|, |X|)^j \alpha_i\beta_j.$$

The condition $ff' = 1$ is then the same as the infinite system of equations for the $\beta_i$ given below:

$$\begin{aligned}
&\alpha_0\beta_0 = 1\\
&\chi(|\alpha_0|, |X|)\alpha_0\beta_1 + \alpha_1\beta_0 = 0\\
&X^2(\chi(|\alpha_0|, |X|)^2\alpha_0\beta_2 + \chi(|\alpha_1|, |X|)\alpha_1\beta_1 + \alpha_2\beta_0) = 0\\
&\dots\\
&X^n \sum_{i+j=n} \chi(|\alpha_i|, |X|)^j \alpha_i\beta_j = 0\\
&\dots
\end{aligned}$$

and as $\alpha_0$ is invertible, this system of equations can be solved recursively for $\beta_i$ even without the factors $X^i$. However one needs to check that the result is homogeneous. Set now recursively

$$\beta_n := \chi(|X|, |\alpha_0|)^n \alpha_0^{-1}(\sum_{i+j=n, j<n} \chi(|\alpha_i|, |X|)^j \alpha_i\beta_j).$$

This means that $|\beta_n| = -|\alpha_0| + |\alpha_i| + |\beta_j|$. By recursion, this implies

$$|\beta_n| = -|f| + |f| - i|X| - |f| - j|X| = -|f| - n|X|,$$

so $g$ as constructed recursively is homogeneous of degree $-|f|$ as we wanted to show. Similarly, one can construct a left inverse $f''$ of $f$. As our ring is associative, $f'$ and $f''$ coincide, proving that $f$ is invertible.

## 3.2 Graded formal deformations

### 3.2.1 Preliminaries and definitions

We will now introduce $(G, \chi)$-graded formal deformations. The aim of this section is to get in place the basic concepts of the deformation theory of associative algebras in the form they take when adapted to the category which we are presently working in. We will later link them to the cohomological tools developed in the previous chapter.
We start with the following well-known observations:

**Lemma 3.2.1** *Let $k$ be a commutative ring with unit, $M$ a $k$-module, and $e \in k$ an idempotent. Then $M = eM \oplus (1-e)M$.*

**Proof** For any $r \in k$, $rM$ is a submodule of $M$ since $k$ is commutative. For any $m \in M$, we have $m = em + (1-e)m$, so $M = eM + (1-e)M$. Finally, if $m = em' = (1-e)m''$, then $m = 0$ since $m = e^2m' = e(1-e)m'' = 0$. Therefore $M = eM \oplus (1-e)M$.

**Corollary 3.2.2** *The same holds true if $M$ is a $G$-graded $k$-module over the commutative ring $k$, since by definition $e$ and $e' := 1 - e$ are elements of degree zero, meaning that $eM$ and $e'M$ are* graded *submodules of $M$.*

**Corollary 3.2.3** *Suppose $k$ a commutative ring and $e$ an idempotent, $e' = 1 - e$ as before, and $A$ a $k$-algebra. Then $eA$ is a unital $k$-algebra in its own right, with unit $e$. The decomposition $A = eA \oplus e'A$ becomes with these notations a decomposition of $k$-algebras. Componentwise multiplication on the decomposition recovers the multiplication on $A$. The same holds for the graded situation.*

**Corollary 3.2.4** *Suppose $e$ and $e'$ as before and $A$ a $k$-algebra, possibly $(G, \chi)$-graded. We have $A^{\otimes n} = eA^{\otimes n} \oplus e'A^{\otimes n}$ since $eA \otimes e'A$ and any other 'mixed' powers appearing in $A^{\otimes n} = (eA \oplus e'A)^{\otimes n}$ are zero (multiplication by $e$ acts bijectively on the left but like zero on the right).*

**Corollary 3.2.5** *Suppose that $M, N$ are two $k$-modules, that $\varphi : M \to N$ is a $k$-linear map between them, and that $e$ and $e'$ are as before. Then $\varphi = e\varphi + e'\varphi$, and $e\varphi(e'M) = 0$, $e'\varphi(eM) = 0$. As a result, one gets a decomposition $Hom_k(M, N) \cong Hom_k(eM, eN) \oplus Hom_k(e'M, e'N)$. The same holds in the graded situation for $Hom_k(M, N)$ and $\boldsymbol{Hom}_k(M, N)$.*

Note that the two preceding corollaries taken together imply that in particular the Hochschild differential $\beta : C^n(A, M) \to C^{n+1}(A, M)$ splits along these lines. One therefore gets

$$H^n(A, M) \cong H^n(eA, eM) \oplus H^n(e'A, e'M).$$

This works for graded Hochschild cohomology just as well as for the ungraded cohomology. We introduce now for the rest of the chapter the following standing assumptions: $A$ is a $(G, \chi)$-graded associative algebra, $|X| = g$, $\lambda = \chi(g, g)$, and 2 is invertible both in $A$ and in $k$. We continue providing corollaries of Lemma 3.2.1:

**Corollary 3.2.6** *In particular, with $2 \in k$ invertible and $\lambda$ as above, the lemma is applicable with $e_\lambda = \frac{1}{2}(1 + \lambda)$ and $e'_\lambda = \frac{1}{2}(1 - \lambda)$. It shows then that $A = e_\lambda A \oplus e'_\lambda A$ and $A[[X]] = e_\lambda A[[X]] \oplus e'_\lambda A[[X]]$. By the construction of $A[[X]]$, the second summand in the last equation is zero in order $\geq 2$.*

We will continue using the notations $e_\lambda := \frac{1+\lambda}{2}$ and $e'_\lambda := \frac{1-\lambda}{2}$ for these idempotents. As an application of the previous observations, the following Lemma follows:

**Lemma 3.2.7** *For $a \in A$ any element, the conditions $X^n a = 0$ for $n \geq 2$ and $e_\lambda a = 0$ are equivalent.*

**Proof** By the definitions, $X^n a = 0$ means $a \in e'A$. By the direct sum decomposition $A = eA \oplus e'A$ the statement follows, since $e$ acts like unity on $eA$.

**Proposition 3.2.8** *Let $A$ be a $(G,\chi)$-graded algebra, $|X| = g \in G$ and $\varphi : A \to A[[X]]$ as well as $\psi : A \otimes A \to A[[X]]$ maps in $Hom_k(A, A[[X]])$ and $Hom_k(A \otimes A, A[[X]])$. We can write $\varphi$ and $\psi$ in the form*

$$\varphi(a) = \sum_{i \geq 0} X^i \varphi_i(a) \text{ and } \psi(a,b) := \sum_{i \geq 0} X^i \psi_i(a,b)$$

*respectively, with the $\varphi_i \in \boldsymbol{Hom}_k(A, A)$ being linear maps and $\psi_i \in \boldsymbol{Hom}_k(A \otimes A, A)$ being bilinear on $A$.*

**Proof** Suppose that $\varphi \in Hom_k(A, A[[X]])$. Then it is evident that there exist maps $\tilde{\varphi}_i : A \to A$ such that

$$\varphi = \sum_{i \geq 0} X^i \tilde{\varphi}_i.$$

In orders zero and one the $\tilde{\varphi}_i$ are themselves $\in \mathbf{Hom}_k(A, A)$. Now look at a term $X^i \tilde{\varphi}_i$ with $i \geq 2$. It is evident from Lemma 3.2.7 that the maps $e_\lambda \tilde{\varphi}_i$ are $k$-linear. Since $\varphi$ is degree zero, $e_\lambda \tilde{\varphi}_i$ must be zero or of degree $-ig$, and so a homogeneous member of $\mathbf{Hom}_k(A, A)$. Indeed, $\tilde{\varphi}_i$ can always be chosen graded linear, because if necessary we can replace $\tilde{\varphi}_i$ by $e_\lambda \tilde{\varphi}_i$ itself since $X^i e_\lambda = X^i$. Finally, $\psi$ can be treated in the same way. This concludes the proof.

**Proposition 3.2.9** *Whenever linear maps $\varphi$ and $\psi$ of this kind are given in the way described in Prop. 3.2.8, degree zero $k[[X]]$-linear maps $\varphi' : A[[X]] \to A[[X]]$ and $\psi' : A[[X]] \otimes A[[X]] \to A[[X]]$ are induced by $\varphi$ and $\psi$ through*

$$\varphi'(\sum_{i \geq 0} X^i \lambda_i) = \sum_{n \geq 0} X^n \sum_{i+j=n} \varphi_i(\lambda_j)$$

*and*

$$\psi'(\sum_{i \geq 0} X^i \lambda_i, \sum_{i \geq 0} X^i \mu_i) = \sum_{n \geq 0} X^n \sum_{i+j+k=n} \chi(|\lambda_i|, |X|)^j \psi_k(\lambda_i, \mu_j)$$

*respectively.*

**Proof** Suppose that $\varphi : A \to A[[X]]$ and $\psi : A \otimes A \to A[[X]]$ are given by

$$\varphi = \sum_{i \geq 0} X^i \varphi_i$$

and

$$\psi = \sum_{i \geq 0} X^i \psi_i$$

with degree $-ig$ linear maps $\varphi_i$ and degree $-ig$ bilinear maps $\psi_i$. We have to prove that the maps $\varphi'$ and $\psi'$ are well-defined, homogeneous of degree zero, and linear. We will begin with $\varphi'$. Assuming that $\varphi'$ as given is well-defined, $k$-linearity follows from linearity of the maps $\varphi_i$. Likewise, $deg(\varphi') = 0$ follows by taking a homogeneous element $\sum_{i \geq 0} X^i \lambda_i$ and checking the

degrees of the terms on the righthand side of the definition of $\varphi'$ using the fact that $\varphi_i$ is a map of degree $-ig$ which is $-|X^i|$ if $X^i \neq 0$. What remains to be proven are therefore only well-definedness and $k[[X]]$-linearity.
We begin with well-definedness. Since we have $k$-linearity already, it is sufficient for this to show that with $\sum_{i\geq 0} X^i\lambda_i = 0$ the righthand side of the defining equation for $\varphi'$ gives zero. Now $\sum_{i\geq 0} X^i\lambda_i = 0$ by definition is nothing else than $\lambda_0 = 0, \lambda_1 = 0$ and $e_\lambda\lambda_i = 0$ for all $i \geq 2$. By the defining equation, we find

$$\varphi'(\sum_{i\geq 0} X^i\lambda_i) = \sum_{n\geq 0} X^n \sum_{i+j=n} \varphi_i(\lambda_j) = \varphi_0(\lambda_0) + X(\varphi_0(\lambda_1) + \varphi_1(\lambda_0)) + \sum_{n\geq 2} X^n \sum_{i+j=n} \varphi_i(\lambda_j).$$

Since $\lambda_0 = \lambda_1 = 0$, the first two terms here are zero. For the same reason, we see $\sum_{i+j=n} \varphi_i(\lambda_j) = \sum_{i+j=n, j\geq 2} \varphi_i(\lambda_j)$. Now for any $\rho \in \mathbf{Hom}_k(A, A)$ and any $a \in A$ we have $e_\lambda\rho(a) = \rho(e_\lambda a)$ because of $k$-linearity, so for $n \geq 2$ also $X^2\rho(a) = 0$ if and only if $X^2 a = 0$. Since $X^2\lambda_j = 0$ for all $j \geq 2$, well-definedness of $\phi'$ follows.
Next we have to prove $k[[X]]$-linearity of $\varphi'$. This is done by calculation. We assume that $\sum_{i\geq 0} X^i\lambda_i \in k[[X]]$ and $\sum_{i\geq 0} X^i\alpha_i \in A[[X]]$ are arbitrary elements. Then, we compute

$$\varphi'((\sum_{i\geq 0} X^i\lambda_i)(\sum_{j\geq 0} X^i\alpha_i)) = \sum_{n\geq 0} X^n \sum_{i+j+k=n} \lambda_i\varphi_j(a_k) = (\sum_{i\geq 0} X^i\lambda_i)\varphi'(\sum_{j\geq 0} X^j\alpha_j)$$

as desired.
We look now at the corresponding statements for $\psi'$. The arguments are very similar to the preceding ones for all points with the exception of $k[[X]]$-linearity, so we do not repeat them here. To see that $\psi'$ induces a two-sided $k[[X]]$-linear map, we take two homogeneous elements $\sum_{i\geq 0} X^i\alpha_i, \sum_{i\geq 0} X^i\alpha'_i \in A[[X]]$ and a homogeneous element $\sum_{i\geq 0} X^i\lambda_i$ and compute:

$$\begin{aligned} \psi'(\sum_{i\geq 0} X^i\lambda_i \sum_{i\geq 0} X^i\alpha_i, \sum_{i\geq 0} X^i\alpha'_i) &= \psi'(\sum_{n\geq 0} X^n \sum_{i+j=n} \lambda_i\alpha_j, \sum_{i\geq 0} X^i\alpha'_i) \\ = \sum_{n\geq 0} X^n \sum_{i+j+k+l=n} \chi(|\alpha_j|, |X|)^l \lambda_i\psi_k(\alpha_j, \alpha'_l) \end{aligned}$$

and

$$\begin{aligned} (\sum_{i\geq 0} X^i\lambda_i)\psi'(\sum_{i\geq 0} X_i^\alpha, \sum_{i\geq 0} X^i\alpha'_i) &= (\sum_{i\geq 0} X^i\lambda_i)(\sum_{n\geq 0} X^n \sum_{i+j+k=n} \chi(|\alpha_i|, |X|)^j \psi_k(\alpha_i, \alpha'_j)) \\ = \sum_{n\geq 0} X^n(\sum_{i+j+k+l=n} \chi(|\alpha_j|, |X|)^l \lambda_i\psi_k(\alpha_j, \alpha'_l)) \end{aligned}$$

which shows that the $k$-linear map induced by this on $A[[X]] \otimes A[[X]]$ is $k[[X]]$-linear with respect to left multiplication. A corresponding calculation shows the same property for the right operation of $k[[X]]$ on $A[[X]] \otimes A[[X]]$. This completes the proof.

**Remark 3.2.10** *Note that the second statement of Prop. 3.2.9 is different from claiming that for arbitrary $\alpha \in A[[X]]$ the map given by $\psi'(\cdot, \alpha)$ would be $k[[X]]$-linear in the sense of being* *in* $\boldsymbol{Hom}_{k[[X]]}(A[[X]], A[[X]])$. *Indeed, this would not be true. The problem with this statement compared to the one we just proved would be that a map* $\varphi := \psi'(\cdot, \alpha)$ *would have to verify, in order to be a member of* $\boldsymbol{Hom}_{k[[X]]}(A[[X]], A[[X]])$, *the equation*

$$\varphi(\Lambda\alpha') = \chi(|\alpha|, |\Lambda|)\Lambda\varphi(\alpha')$$

*when* $\Lambda \in k[[X]]$ *and* $\alpha \in A[[X]]$ *are supposed homogeneous. On the other hand, being a member of* $Hom_{k[[X]]}(A[[X]] \otimes A[[X]], A[[X]])$ *means*

$$\varphi(\Lambda(\alpha_1 \otimes \alpha_2)) = \Lambda\varphi(\alpha_1 \otimes \alpha_2).$$

To simplify notations, we set the following definition:

**Definition 3.2.11** *Let* $\varphi : A[[X]] \to A[[X]]$ *and* $\psi : A[[X]] \otimes A[[X]] \to A[[X]]$ *be* $k[[X]]$*-linear maps. Chose a way to write the induced maps* $\tilde{\varphi} : A \to A[[X]]$ *and* $\tilde{\psi} : A \otimes A \to A[[X]]$ *in the form*

$$\tilde{\varphi} = \sum_{i\geq 0} X^i \tilde{\varphi}_i$$

*with graded linear maps* $\tilde{\varphi}_i \in \boldsymbol{Hom}_k(A, A)$ *and* $deg(\tilde{\varphi}_i) = -ig$ *and*

$$\tilde{\psi} = \sum_{i\geq 0} X^i \tilde{\psi}_i$$

*with* $\tilde{\psi}_i \in \boldsymbol{Hom}_k(A \otimes A, A)$ *and* $deg(\tilde{\psi}_i) = -ig$. *The infinite sums giving the induced linear maps* $\tilde{\varphi}$ *and* $\tilde{\psi}$ *respectively are then called* representations *of* $\varphi$ *and* $\psi$. *By abuse of notation, we will sometimes write*

$$\varphi = \sum_{i\geq 0} X^i \tilde{\varphi}_i \quad \text{and} \quad \psi = \sum_{i\geq 0} X^i \tilde{\psi}_i$$

*or even just*

$$\varphi = \sum_{i\geq 0} X^i \varphi_i, \quad \psi = \sum_{i\geq 0} X^i \psi_i$$

*in this case.*

One can prove the following, justifying to some extent the preceding definition:

**Proposition 3.2.12** *Suppose that* $\varphi : A[[X]] \to A[[X]]$ *and* $\psi : A[[X]] \otimes A[[X]] \to A[[X]]$ *are* $k[[X]]$*-linear maps with representations* $\tilde{\varphi}$ *and* $\tilde{\psi}$ *respectively. Then,* $\varphi$ *is the extension of* $\tilde{\varphi}$ *and* $\psi$ *is equal to the extension of* $\tilde{\psi}$ *both according to the procedure described in Prop. 3.2.9.*

**Proof** We start by proving the first claim. Let $\sum_{i\geq 0} X^i \alpha_i \in A[[X]]$ and denote by $\varphi'$ the extension of $\tilde{\varphi}$. Inserting the relevant definitions, we have to prove that

$$\sum_{n\geq 0} X^n \sum_{i+j=n} \tilde{\varphi}_i(\alpha_j) = \varphi(\sum_{i\geq 0} X^i \alpha_i).$$

The idea is to check equality by splitting off a term which contains all contributions up to some fixed but arbitrary order $N \in \mathbb{N}$. Indeed, we see

$$\begin{aligned}\varphi(\sum_{i\geq 0} X^i\alpha_i) &= \varphi(\alpha_0) + X\varphi(\alpha_1) + \ldots + X^N\varphi(\alpha_N) + \sum_{i\geq N+1} X^{N+1}\varphi(X^{i-N-1}\alpha_i) \\ &= \tilde{\varphi}_0(\alpha_0) + X(\tilde{\varphi}_1(\alpha_0) + \tilde{\varphi}_0(\alpha_1)) + \ldots + X^N(\sum_{i+j=N} \tilde{\varphi}_i(\alpha_j)) + X^{N+1}(\ldots)\end{aligned}$$

as desired.
We will now carry out the same procedure for $\psi$. First, we calculate that

$$\psi(Xa, b) = X\psi(a, b)$$

and

$$\psi(a, X^n b) = \chi(|X|, |b|)^n\psi(a, bX) = \chi(|X|, |b|)^n\chi(|a|+|b|, |X|)^n X\psi(a, b) = \chi(|a|, |X|)^n X^n\psi(a, b)$$

for any homogeneous $a, b \in A$ and $n \in \mathbb{N}$ by $k[[X]]$-linearity of $\psi$. Consequently, we see for $\alpha = \sum_{i\geq 0} X^i\alpha_i$ and $\alpha' = \sum_{i\geq 0} X^i\alpha'_i$ homogeneous elements of $A[[X]]$ that

$$\begin{aligned}\psi(\alpha, \alpha') &= \sum_{n=1}^{N}(\sum_{i+j=n} \psi(X^i\alpha_i, X^j\alpha'_j)) + X^{N+1}(\ldots) \\ &= \sum_{n=1}^{N} X^n(\sum_{i+j=n} \chi(|\alpha_i|, |X|)^j\psi(\alpha_i, \alpha'_j)) + X^{N+1}(\ldots) \\ &= \sum_{n=1}^{N} X^n(\sum_{i+j+k=n} \chi(|a_i|, |X|)^j\tilde{\psi}_k(\alpha_i, \alpha'_j)) + X^{N+1}(\ldots)\end{aligned}$$

where the last term shows equivalence to $\psi'$ as defined in Prop. 3.2.9 up to order $N$.

We will now define graded formal deformations of a $(G,\chi)$-graded $k$-algebra $A$.

**Definition 3.2.13** *Let $A$ be a $(G,\chi)$-graded algebra over the commutative ring $k$. A* graded formal deformation in degree $g$ *of $A$ is an associative product $\mu : A[[X]] \times A[[X]] \to A[[X]]$ on the ring $A[[X]]$ of formal power series in a variable $X$ of degree $g \in G$ arising from a sequence of homogeneous maps $\mu_i \in \boldsymbol{Hom}_k(A \otimes A, A)$ of degrees $deg(\mu_i) = -ig$ in the manner explained in Prop. 3.2.9, under the additional condition that $\mu_0$ be the original multiplication on $A$. That is to say, we have a map $\mu' : A \otimes A \to A[[X]]$ given by*

$$\mu' = \sum_{i\geq 0} X^i\mu_i$$

*and extension of this to $A[[X]]$ as in Prop. 3.2.9 is used to define the graded $k$-linear map $\mu : A[[X]] \otimes A[[X]] \to A[[X]]$.*

When $\mu$ is a formal graded deformation of the product on $A$, and if no confusion as to the identity of this deformation can arise, we will sometimes write $f \star g$ for $\mu(f,g)$. Note that with these notations, the deformed algebra $(A[[X]], \star)$ needs not have a unit, even though we assume associative algebras to be unital by default otherwise.
With $\mu_1$ and $\mu_2$ two graded formal deformations in identical degree over $A$, it is possible for the algebraic structures $(A[[X]], \mu_1)$ and $(A[[X]], \mu_2)$ to be isomorphic. One goal of any theory of formal graded deformations will be to recognize such cases. To this end, we define the following notion of equivalence:

**Definition 3.2.14** *Suppose that $A$ is as before and that $(A[[X]], \mu)$ and $(A[[X]], \mu')$ are two deformations of $A$. Then $\mu$ and $\mu'$ are considered equivalent if there exists a* formal isomorphism *between them, i.e. if there is a sequence of linear maps $\phi_i \in \boldsymbol{Hom}_k(A, A)$ with $deg(\phi_i) = -ig$ such that the map*

$$\phi := id + \sum_{i \geq 1} X^i \phi_i$$

*commutes with the products, that is to say*

$$\phi \circ \mu = \mu' \circ \phi.$$

*Here the definition of $\phi$ is to be understood in the sense of the extension procedure established in Prop. 3.2.9.*

We will now quickly verify that indeed this definition induces an equivalence relation. We start with the following

**Remark 3.2.15** *Suppose that $\varphi : A[[X]] \to A[[X]]$ and $\psi : A[[X]] \to A[[X]]$ are linear maps with representations*

$$\varphi = \sum_{i \geq 0} X^i \varphi_i \text{ and } \psi = \sum_{i \geq 0} X^i \psi_i$$

*respectively, with $\varphi_i \in \boldsymbol{Hom}_k(A, A)$ of degree $-i|X|$. Then $\varphi \circ \psi$ is induced by the $A \to A[[X]]$-map given by*

$$\rho := \sum_{n \geq 0} X^n \sum_{i+j=n} \varphi_i \circ \psi_j.$$

**Proof** First we note that the proposed map is of the right form, i.e. we have $|\varphi_i \circ \psi_j| = -n$ as desired. Inserting the definitions, it is clear that $(\varphi \circ \psi)|_A = \rho$. Using Prop. 3.2.12, this proves the remark.

Next, we remark that a formal isomorphism is invertible with a formal inverse. This is proven exactly as the degree zero case of Rem. 3.1.10. This remark together with the preceding Rem. 3.2.15 prove that the existence of a formal isomorphism between two formal deformations of a $(G, \chi)$-graded algebra $A$ is an equivalence relation.
We finish this part of the discussion with the following remark:

**Remark 3.2.16** *We can use Prop. 3.2.12 to conclude that if $\mu$ and $\mu'$ are two deformations of the multiplication on $A$, and if $\varphi$ is a potential formal isomorphism between them, it is sufficient to check the equation $\varphi \circ \mu = \mu' \circ \varphi$ for homogeneous elements $a, b \in A$.*

### 3.2.2 The deformation equation

The goal of the present subsection is to derive the deformation equation for graded formal deformations of a $(G, \chi)$-graded $k$-algebra $A$. First we have to prove yet another preliminary:

**Lemma 3.2.17** *Suppose that* $\psi : A[[X]] \otimes A[[X]] \to A[[X]]$ *is a degree zero graded linear map such that the induced map* $\tilde{\psi} : A \otimes A \to A[[X]]$ *is given by*

$$\tilde{\psi} = \sum_{i \geq 0} X^i \tilde{\psi}_i$$

*as in Prop. 3.2.9. Then* $\psi$ *induces an associative degree preserving* $k[[X]]$*-bilinear multiplication on* $A[[X]]$ *if and only if we have*

$$\psi(\psi(a,b),c) = \psi(a, \psi(b,c)) \tag{3.2}$$

*for all homogeneous* $a, b, c \in A$.

**Proof** That associativity of $\psi$ implies (Eq. 3.2) is obvious. Assume on the other hand (Eq. 3.2). Then we first note that for any $n \in \mathbb{N}$ and any homogeneous elements $a, b, c \in A$ we have

$$\psi(\psi(X^n a, b), c) = X^n \psi(\psi(a,b),c) = X^n \psi(a, \psi(b,c)) = \psi(X^n a, \psi(b,c)).$$

In the same way, we see also $\psi(\psi(a,b), cX^n) = \psi(a, \psi(b, cX^n))$. Finally, we calculate

$$\psi(\psi(a, X^n b), c) = \chi(|X|, |b|)^n \psi(\psi(a,b)X^n, c) = \chi(|a|, |X|)^n X^n \psi(\psi(a,b),c)$$

and similarly

$$\psi(a, \psi(X^n b, c)) = \chi(|a|, |X|)^n X^n \psi(a, \psi(b,c)).$$

By $k$-trilinearity of the associator, it follows from the three relations just derived that the associator of $\psi$ is zero up to any fixed order, hence zero.

We will now look more closely at the problem of classifying graded associative deformations of $(G, \chi)$-graded algebras. We state the following proposition:

**Proposition 3.2.18** *Suppose that* $\mu : A[[X]] \otimes A[[X]] \to A[[X]]$ *is a* $k[[X]]$*-linear map, with representation* $\mu = \sum_{i \geq 0} \mu_i$*. Then* $\mu$ *defines a graded formal deformation of* $A$ *if and only if the following equations hold for all homogeneous* $a, b, c \in A$ *and* $n \geq 2$*:*

$$\mu_0(a,b) = ab \tag{3.3}$$

$$\chi(|a|, |X|) a \mu_1(b,c) + \mu_1(a, bc) = \mu_1(ab, c) + \mu_1(a,b)c \tag{3.4}$$

$$e_\lambda \sum_{i+j=n} \chi(|a|, |X|)^i \mu_j(a, \mu_i(b,c)) = e_\lambda \sum_{i+j=n} \mu_j(\mu_i(a,b),c). \tag{3.5}$$

**Proof** The only thing to check is associativity. Following Lemma 3.2.17, it is sufficient to check this for triples of homogeneous elements $a, b, c \in A$. Writing $a \star b$ for $\mu(a, b)$, we calculate:

$$a \star (b \star c) = a \star \sum_{i \geq 0} X^i \mu_i(b, c) = \sum_{n \geq 0} X^n \sum_{i+j=n} \chi(|a|, |X|)^i \mu_j(a, \mu_i(b, c))$$

and similarly

$$(a \star b) \star c = \sum_{i \geq 0} X^i \mu_i(a, b) \star c = \sum_{n \geq 0} X^n \sum_{i+j=n} \mu_j(\mu_i(a, b), c).$$

Comparison of coefficients together with Lemma 3.2.7 gives the proposition.

Similarly, the notion of equivalence of deformations can be characterized order by order:

**Proposition 3.2.19** *Suppose that $\mu$ and $\mu'$ are graded formal deformed multiplications on $A$, with $\varphi$ a formal isomorphism between $(A[[X]], \mu)$ and $(A[[X]], \mu')$. Then, we have*

$$\mu'_1(a, b) + \chi(|a|, |X|) a \varphi_1(b) + \varphi_1(a) b = \mu_1(a, b) + \varphi_1(ab) \tag{3.6}$$

*and*

$$e_\lambda \sum_{i+j+k=n} \chi(|a|, |X|)^j \mu'_k(\varphi_i(a), \varphi_j(b)) = e_\lambda \sum_{i+j=n} \varphi_i(\mu_j(a, b)) \tag{3.7}$$

*for all homogeneous $a, b \in A$. The converse is also true, i.e. if $\mu$ and $\mu'$ are graded formal deformed multiplications on $A$ and if*

$$\varphi = \sum_{i \geq 0} \varphi_i$$

*is a formal map with $\varphi_0 = id$ satisfying (3.6) and (3.7), then $\varphi$ gives a formal isomorphism between $(A[[X]], \mu)$ and $(A[[X]], \mu')$.*

**Proof** Write $a \star b$ for $\mu(a, b)$ and $a \star' b$ for $\mu'(a, b)$. Equivalence of $\mu$ and $\mu'$ is then the same as $\varphi(a \star b) = \varphi(a) \star' \varphi(b)$ for all homogeneous $a, b \in A$, due to Prop. 3.2.12. We calculate

$$\begin{aligned} \varphi(a \star b) &= \varphi\Big(\sum_{i \geq 0} X^i \mu_i(a, b)\Big) \\ &= \sum_{n \geq 0} X^n \sum_{i+j=n} \varphi_i(\mu_j(a, b)) \end{aligned}$$

and

$$\begin{aligned} \varphi(a) \star' \varphi(b) &= (\sum_{i \geq 0} X^i \varphi_i(a)) \star' (\sum_{i \geq 0} X^i \varphi_i(b)) \\ &= \sum_{n \geq 0} X^n \sum_{i+j+k=n} \chi(|\varphi_i(a)|, |X|)^j \mu'_k(\varphi_i(a), \varphi_j(b)) \\ &= \sum_{n \geq 0} X^n \sum_{i+j+k=n} \chi(|a| - i|X|, |X|)^j \mu'_k(\varphi_i(a), \varphi_j(b)) \\ &= \sum_{n \geq 0} X^n \sum_{i+j+k=n} \lambda^{-ij} \chi(|a|, |X|)^j \mu'_k(\varphi_i(a), \varphi_j(b)) \\ &= \sum_{n \geq 0} X^n \sum_{i+j+k=n} \chi(|a|, |X|)^j \mu'_k(\varphi_i(a), \varphi_j(b)) \end{aligned}$$

where the last step uses the fact that in order one, $\lambda^{-ij} = 1$, whereas in order $n \geq 2$, we have $\lambda X^n = X^n$ by definition.
In order one, we now obtain immediately equation (3.6)

$$\mu_1(a,b) + \chi(|a|,|X|)a\varphi_1(b) + \varphi_1(a)b = \mu_1'(a,b) + \varphi_1(ab).$$

In order $n \geq 2$, comparison of coefficients yields by Lemma 3.2.7 equation (3.7)

$$e_\lambda \sum_{i+j+k=n} \chi(|a|,|X|)^j \mu_k'(\varphi_i(a),\varphi_j(b)) = e_\lambda \sum_{i+j=n} \varphi_i(\mu_j(a,b))$$

as desired. This concludes the proof.

**Infinitesimal deformations**

We will now classify infinitesimal deformations of $A$ by Hochschild cohomology. By an infinitesimal deformation of $A$ we mean here, as is usual, a deformation of the original product which is associative up to terms of order two. Two such deformations are regarded equivalent if there is a formal isomorphism which transforms one into the other up to terms of second or higher order. In other words, an infinitesimal deformation should induce an associative multiplication on $A[[X]]/(X^2)$ and two of them are equivalent if they induce formally isomorphic structures on $A[[X]]/(X^2)$.
Note that in the case $\lambda = -1$, higher powers of the formal paramater vanish automatically, so any formal deformation is in this case just an infinitesimal one.
As an application of Prop. 3.2.18, Prop 3.2.19, the following proposition is obtained:

**Proposition 3.2.20** *The equivalence classes of infinitesimal deformations in degree $g \in G$ correspond one to one with elements of $HH^2(A)$ of degree $-g$.*

**Proof** Suppose $\mu = \mu_0 + \mu_1$ an infinitesimal deformation of the multiplication of $A$. By reasoning as in Prop. 3.2.18, we see

$$\chi(|a|,g)a\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0$$

for all homogeneous $a,b,c \in A$. Since $|\mu_1| = -g$, this means

$$\beta(\mu_1) = \chi(|\mu_1|,|a|)\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0$$

implying that $\mu_1$ is a graded Hochschild cocycle and of degree $-g$. Suppose now that $\mu$ and $\mu'$ are two graded infinitesimal deformed products in degree $g$ of $A$. Then by Prop. 3.2.19 we see

$$\mu_1(a,b) - \mu_1'(a,b) = \chi(|\varphi_1|,|a|)a\varphi_1(b) - \varphi_1(ab) + \varphi_1(a)b = \beta(\varphi_1)(a,b)$$

and so the difference between the two is a Hochschild coboundary. This concludes the proof.

### 3.2.3 Obstruction theory

We will now look at the obstructions to extending a deformed product on a $(G,\chi)$-graded algebra $A$ from order $N$ to order $N+1$. In other words, one assumes that a solution to (Eq. 3.4) and (Eq. 3.5) is given up to some fixed order and then one asks under which conditions such a solution can be extended to order $N+1$, i.e. when a $\mu_{N+1} \in \mathbf{Hom}_k(A \otimes A, A)$ can be found which solves (Eq. 3.5) in order $N+1$. Solving (Eq. 3.5) for terms containing $\mu_{N+1}$, we find for $N+1 \geq 2$ that

$$\begin{aligned} & e_\lambda(\chi(|a|,|X|)^{N+1} a\mu_{N+1}(b,c) + \mu_{N+1}(a,bc) - \mu_{N+1}(ab,c) - \mu_{N+1}(a,b)c) \\ = \; & e_\lambda \sum_{\substack{i+j=N+1 \\ i\leq N, j\leq N}} (\mu_j(\mu_i(a,b),c) - \chi(|a|,|X|)^i \mu_j(a,\mu_i(b,c))) \end{aligned}$$

for any homogeneous $a,b,c \in A$. Now we have $\chi(|a|,|X|)^n = \chi(|\mu_n|,|a|)$ for all $n \in \mathbb{N}$, so we see that the lefthand side of the above equation is just $\beta(\mu_{N+1})(a,b,c)$. In effect, this gives

$$e_\lambda \beta(\mu_{N+1})(a,b,c) = e_\lambda \sum_{\substack{i+j=N+1 \\ i\leq N, j\leq N}} (\mu_j(\mu_i(a,b),c) - \chi(|a|,|X|)^i \mu_j(a,\mu_i(b,c))). \tag{3.8}$$

In other words, a solution for $\mu_{N+1}$ can in this situation be found if and only if the term on the righthand side of our equation is of the form $e_\lambda$ times a third Hochschild coboundary. Extension is possible if and only if the righthand side is a multiple by $e_\lambda$ of something exact. This is an analog in the present theory to the statement from the ungraded theory that the corresponding extension problem there has a solution if and only if the righthand side of the corresponding deformation equation is an exact third Hochschild cochain. In the case $X$ even, precisely this statement is recovered.

In the ungraded theory, it is now possible to further characterize the righthand side of (Eq. 3.8) as being automatically closed. This leads in the end to identification of the obstructions to extension with elements of third Hochschild cohomology. We give an analog in our situation of this result as well. First, we will to this end need the following remark:

**Remark 3.2.21** *Suppose that $\mu : A[[X]] \otimes A[[X]] \to A[[X]]$ is a $k[[X]]$-linear map. Then $\mu$ may be viewed as a Hochschild cochain on the $k$-algebra $A[[X]]$ with coefficients in $A[[X]]$. It is homogeneous of degree $(1,0)$ in terms of the color Lie algebra structure on the set of Hochschild cochains defined in Prop./Def. 2.5.1. Due to the way color Lie algebras were introduced Def. 2.4.1, we can conclude immediately from the color Lie property that $[\mu,[\mu,\mu]] = 0$. By Lemma 2.5.2, $\mu$ under these conditions induces an associative multiplication on $A[[X]]$ if and only if $[\mu,\mu] = 0$.*
*The same remarks hold also for the truncated power series algebras $A[[X]]/(X^n)$.*

Writing

$$R_{N+1}(a,b,c) := \sum_{\substack{i+j=N+1 \\ i\leq N, j\leq N}} (\mu_j(\mu_i(a,b),c) - \chi(|a|,|X|)^i \mu_j(a,\mu_i(b,c)))$$

for homogeneous $a,b,c \in A$, (Eq. 3.8) becomes $(1+\lambda)(\beta(\mu_{N+1}) - R_{N+1}) = 0$. If $\mu : A[[X]] \otimes A[[X]] \to A[[X]]$ is a $k[[X]]$-linear map, the equation $[\mu,\mu] = 0$ is by construction order for order exactly the deformation equation, i.e. $X^{N+1}(\beta(\mu_{N+1}) - R_{N+1}) = 0$ or equivalently $e_\lambda(\beta(\mu_{N+1}) - R_{N+1}) = 0$ in order $N+1$.
Assume now that $\mu : A[[X]] \otimes A[[X]] \to A[[X]]$ is associative to order $N$, then $[\mu,\mu]$ disappears for all orders $\leq N$. By Rem. 3.2.21 we also have $[\mu,[\mu,\mu]] = 0$. Since $[\mu,\mu]$ disappears in order $\leq N$, the only contribution to order $N+1$ of $[\mu,[\mu,\mu]]$ comes from $X^{N+1}[\mu_0, \beta(\mu_{N+1}) - R_{N+1}]$, the vanishing of which is the same as

$$e_\lambda \beta(\beta(\mu_{N+1}) - R_{N+1}) = 0.$$

As $\beta(\beta(\mu_{N+1})) = 0$, it follows that $e_\lambda \beta(R_{N+1}) = 0$, or equivalently that $e_\lambda R_{N+1}$ is a third Hochschild cocycle.
Finally, we recall from the beginning of the section that the splitting $A = e_\lambda A \oplus e'_\lambda A$ induced by the idempotents $e_\lambda$ and $e'_\lambda$ gave a corresponding splitting on the Hochschild cochain complex of the form

$$C^n(A,A) = C^n(e_\lambda A, e_\lambda A) \oplus C^n(e'_\lambda A, e'_\lambda A)$$

wherein $e_\lambda$ acts like unity on the first component and like multiplication by zero on the second. The obstructing term $e_\lambda R_{N+1}$ is now seen to be a cocycle in $C^n(e_\lambda A, e_\lambda A)$. For the obstructions to extension to vanish, it has to be exact. We have hence proven the following proposition:

**Proposition 3.2.22** *Let $A$ be a $(G,\chi)$-graded associative algebra over a commutative ring $k$ with $2 \in k$ invertible and let $(A[[X]]/(X^N), \mu)$ be an associative deformation in degree $g \in G$ given up to order $N$. Then the obstructions to extending $\mu$ associatively to order $N+1$ can be identified with the elements of $HH^3(e_\lambda A)_{-(N+1)g}$.*

**Remark 3.2.23** *We return here to the question, raised in the first section of this chapter, how much the deformation theory of $A$ is influenced by the various choices going into the construction of the power series ring over $A$. Prop. 3.2.20 works regardless of any of these choices, so $HH^2(A)$ controls infinitesimal deformations arising from both power series rings and alternative power series rings over $A$ in any degree. However, the situation with respect to obstruction theory is slightly more complicated. From the previous proposition, we see that the obstructions to extending to higher order deformations arising from deformations in degree $g \in G$ are controlled by $HH^3(e_\lambda A)$. This depends on $\lambda = \chi(g,g)$.*
*Obstructions to continuing a deformation in degree $g$ arising from the alternative power series ring $A[[\tilde{X}]]$ from order $N$ to order $N+1$ are controlled by $HH^3(A)_{-(N+1)g}$. This is proven by the same calculations as the even degree case of Prop. 3.2.22.*

### Digression: a more compact way to write the deformation equation

We will show here that, in analogy to the classical case, there is a way to express equation (Eq. 3.8) in a particularly compact form using the Gerstenhaber bracket. What is interesting about this is that apparently the same procedure does not work for the deformation theory arising from the alternative power series ring $A[[\tilde{X}]]$.

We recall that for two homogeneous second degree Hochschild cochains $\mu_1, \mu_2 \in \mathbf{Hom}_k(A \otimes A, A)$ the color Gerstenhaber composition is defined through

$$(\mu_1 \circ \mu_2)(a, b, c) = \mu_1(\mu_2(a, b), c) - \chi(|\mu_2|, |a|)\mu_1(a, \mu_2(b, c))$$

for any homogeneous $a, b, c \in A$ due to the definitions part of Prop/Def. 2.5.1. Also, the color Gerstenhaber bracket of two such homogeneous cochains was in the same place defined as

$$[\mu_1, \mu_2] = \mu_1 \circ \mu_2 + \chi(|\mu_1|, |\mu_2|)\mu_2 \circ \mu_1$$

which with $|\mu_1| = -j|X|$, $|\mu_2| = -i|X|$ is nothing else than

$$[\mu_1, \mu_2] = \mu_1 \circ \mu_2 + \lambda^{ij}\mu_2 \circ \mu_1.$$

This allows us to show the following remark, essentially a rewriting of (Eq. 3.8):

**Remark 3.2.24** *The deformation equation in order $N+1$ can for $N \geq 1$ be written as*

$$(1+\lambda)\beta(\mu_{N+1}) = \frac{1+\lambda}{2} \sum_{\substack{i+j=N+1 \\ i \leq N, j \leq N}} [\mu_j, \mu_i]. \tag{3.9}$$

**Proof** With the same notations as in (Eq. 3.8), we have again $\chi(|a|, |X|)^i = \chi(|\mu_i|, |a|)$ and therefore we see

$$\begin{aligned}(1+\lambda)\beta(\mu_{N+1})(a, b, c) &= (1+\lambda) \sum_{\substack{i+j=N+1 \\ i \leq N, j \leq N}} (\mu_j(\mu_i(a, b), c) - \chi(|\mu_i|, |a|)^i \mu_j(a, \mu_i(b, c))) \\ &= \sum_{\substack{i+j=N+1 \\ i \leq N, j \leq N}} (\mu_j \circ \mu_i)(a, b, c).\end{aligned}$$

We then continue calculating

$$\begin{aligned}(1+\lambda) \sum_{\substack{i+j=N+1 \\ i \leq N, j \leq N}} \mu_j \circ \mu_i &= (1+\lambda) \sum_{\substack{i+j=N+1 \\ i < j \leq N}} (\mu_j \circ \mu_i + \mu_i \circ \mu_j) + (1+\lambda) \sum_{i=1}^{N} \mu_i \circ \mu_i \\ &= \frac{1+\lambda}{2} \sum_{\substack{i+j=N+1 \\ i \leq N, j \leq N, i \neq j}} (\mu_j \circ \mu_i + \mu_i \circ \mu_j) + \frac{1+\lambda}{2} \sum_{i=1}^{N} (\mu_i \circ \mu_i + \mu_i \circ \mu_i) \\ &= \frac{1+\lambda}{2} \sum_{\substack{i+j=N+1 \\ i \leq N, j \leq N, i \neq j}} (\mu_j \circ \mu_i + \lambda^{ij}\mu_i \circ \mu_j) + \frac{1+\lambda}{2} \sum_{i=1}^{N} (\mu_i \circ \mu_i + \lambda^{i^2}\mu_i \circ \mu_i) \\ &= \frac{1+\lambda}{2} \sum_{\substack{i+j=N+1 \\ i \leq N, j \leq N}} [\mu_j, \mu_i].\end{aligned}$$

This completes the proof. Note that we used again the fact that multiplication by $1+\lambda$ 'absorbs' arbitrary powers of $\lambda$.

In the deformation theory arising from $A[[\tilde{X}]]$, (Eq. 3.9) would still hold, but it would not be the equation that controls extensions of deformations to higher order.

# Chapter 4

# Harrison cohomology for color-commutative algebras

We will start this chapter by giving a very short account of what has happened so far. In chapter one, when we introduced the category of $(G, \chi)$-graded algebras, a large part of the motivating examples came from algebras which without additional structure seem very much noncommutative, but which in the appropriate graded category can be treated in a way similar to ordinary commutative algebras. In chapter two, we constructed a version of Hochschild cohomology which is appropriate for this category and showed how some classical results about Hochschild cohomology and related structures generalize into this widened context. In the previous chapter, we analyzed in which form certain notions from deformation theory behave in the $(G, \chi)$-graded category.

Now it is time to return to the motivating examples from the first chapter, i.e. to the relatively well-behaved subclass of colored associative algebras which are color-commutative. The goal of this chapter is to develop a cohomology theory of color-commutative algebras. We will then show that this theory behaves well enough to allow extension of at least some results about commutative algebra cohomology to the color-commutative context.

The theory we will present is the color-commutative analog of *Harrison cohomology*, originally introduced by Harrison in (Harrison [31]). In the commutative context, Harrison cohomology is useful for example in the study of cohomological properties of polynomial rings, commutative algebra extensions, regular local rings (Harrison [31]) and commutative deformations of commutative algebras (Fronsdal [23], [24]), (Fronsdal, Kontsevich [25]). For $k$ a ring containing $\mathbb{Q}$ and $A$ a $k$-flat commutative $k$-algebra, Harrison cohomology can be recovered as the 'lowest' component in the Gerstenhaber-Schack decomposition of Hochschild cohomology (Gerstenhaber, Schack [28]) and coincides with another cohomology theory for commutative algebras, namely André-Quillen cohomology (Quillen [48]).

To the best of our knowledge, a study of Harrison-like cohomologies has not been attempted before in the general color-commutative case. The supercommutative case is partially treated e.g. in (Fronsdal, Kontsevich [25]).

We will now fix some notations. $G$ will denote a fixed abelian group, $k$ will be a commutative ring. $\chi : G \to k^*$ will denote a bicharacter with values in $k^*$. $A$ will by default be a $(G, \chi)$-

commutative $k$-algebra. Modules over $A$ will by default be considered $(G, \chi)$ -symmetric. $K$ will as usual denote a field.

## 4.1 The shuffle product

The first step towards understanding Harrison cohomology is to understand a particular $k$-algebra structure which can be defined on the underlying $k$-module of the tensor algebra $T^*(A)$ of an $(G, \chi)$-commutative algebra $A$. Recall, in this context, that with $A$ a $G$-graded algebra we have

$$T(A) = \oplus_{n \in \mathbb{N}} A^{\otimes n}$$

where the tensor product itself is to be understood as the tensor product of graded $k$-modules as given in Def. 1.1.8. Note that there, we defined two different notions of degree on the tensor algebra of graded modules, namely one internal given for products of homogeneous elements through

$$deg_i(a_1 \otimes a_2 \otimes \ldots \otimes a_n) := \sum_{i=1}^{n} deg(a_i)$$

and also a tensor degree, which was simply given by the defining decomposition of the tensor algebra into submodules of tensors of fixed length.
As was said above, $A$ is supposed $(G, \chi)$-commutative, but this will be used in our construction only at few points. Indeed, for the purposes of the first result to follow, $A$ could be any $G$-graded module over the base ring $k$. However, we will assume color-commutativity of $A$ from the beginning, since the construction presented here fails to achieve its final purpose when $A$ fails to be color-commutative, as the interplay between the shuffle product we are going to present and the $(G, \chi)$-graded Hochschild differential depends crucially on color-commutativity. We will discuss along the way to which extent this can be relaxed without fundamental changes to the construction.
Before we can do any of this, however, we have to introduce some definitions:

**Definition 4.1.1** *A permutation $\sigma \in S_{p+q}$ is called a $(p, q)$-*Shuffle *if it satisfies the conditions*

$$\sigma(1) < \sigma(2) < \sigma(3) < \sigma(4) < \ldots < \sigma(p)$$

*and*

$$\sigma(p+1) < \sigma(p+2) < \sigma(p+3) < \ldots < \sigma(p+q)$$

*The set of all $(p, q)$-Shuffles is denoted by $Sh(p, q)$. We also define* padded below *and* padded above *Shuffles of length $p + q + r$ to be shuffles of the forms*

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & p & p+1 & \ldots & p+q & p+q+1 & \ldots & p+q+r \\ 1 & 2 & 3 & \ldots & p & \sigma(p+1) & \ldots & \sigma(p+q) & \sigma(p+q+1) & \ldots & \sigma(p+q+r) \end{pmatrix}$$

*or*

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & p & p+1 & \ldots & p+q & p+q+1 & \ldots & p+q+r \\ \sigma(1) & \sigma(2) & \sigma(3) & \ldots & \sigma(p) & \sigma(p+1) & \ldots & \sigma(p+q) & p+q+1 & \ldots & p+q+r \end{pmatrix}$$

*respectively, where the order of the elements is supposed to be unpermuted in the natural places. More generally, we define a* $(p,q,r)$-(tri)shuffle *to be an element* $\sigma \in S_{p+q+r}$ *such that* $\sigma(i) < \sigma(j)$ *whenever* $i < j$ *and* $1 \leq i < j \leq p$ *or* $p < i < j \leq p+q$ *or* $p+q < i < j \leq p+q+r$. *We will denote the set of* $(p,q,r)$*-shuffles by* $Sh(p,q,r)$*, the set of* $(q,r)$*-shuffles padded below with* $p$ *entries by* $Sh_p(q,r)$*, and the set of* $(p,q)$*-shuffles padded above by* $r$ *entries by* $Sh(p,q)^r$. *We will, finally, refer to shuffles padded below or above occasionally simply as pb-shuffles or pa-shuffles. For a* $\sigma \in S_n$*,* $V$ *a* $k$*-module endowed with a* $G$*-grading and* $a = a_1 \otimes a_2 \otimes \ldots \otimes a_n \in V^{\otimes n}$ *a product of homogeneous elements we will use the notation* $a_\sigma := a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \ldots \otimes a_{\sigma(n)}$.

Let $A$ now be a $(G,\chi)$-commutative $k$-algebra and consider $V := T^*(A)$. Let $\sigma \in Sh(p,q)$ and $a := a_1 \otimes a_2 \otimes \ldots \otimes a_p \in A^{\otimes p} = T^p(A)$ with homogeneous entries. Then we define the operation of $\sigma$ on $a$ by

$$\sigma.a := a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \ldots \otimes a_{\sigma^{-1}(p)}$$

It is clear that this is well-defined. Now, we are ready for the following:

**Definition/Proposition 4.1.2** *Let* $a := a_1 \otimes a_2 \otimes \ldots \otimes a_p \in T^p(A)$ *a tensor product of homogeneous factors and* $a' := a_{p+1} \otimes a_{p+2} \otimes \ldots \otimes a_{p+q} \in T^q(A)$ *with likewise homogeneous factors. Let* $aa'$ *denote the result of concatenation of* $a$ *and* $a'$*, and set*

$$F(\sigma, a_1 \otimes a_2 \otimes \ldots \otimes a_{p+q}) := \prod_{\substack{i<j \\ \sigma^{-1}(i) > \sigma^{-1}(j)}} \chi(|a_{\sigma^{-1}(j)}|, |a_{\sigma^{-1}(i)}|)$$

*for any* $\sigma \in S_n$*. This map obeys the product rule*

$$F(\rho\sigma, c) = F(\sigma, c)F(\rho, c_{\sigma^{-1}})$$

*for any* $\rho, \sigma \in S_p$ *and* $c = a_1 \otimes \ldots \otimes a_p$ *a product of homogeneous factors. Distributive extension of the multiplication rule*

$$a \diamond a' := \sum_{\sigma \in Sh(p,q)} sign(\sigma) F(\sigma, aa')\sigma.aa'$$

*gives a multiplication*

$$\diamond : T^*(A) \otimes T^*(A) \to T^*(A)$$

*which makes* $T^*(A)$ *into an associative* $k$*-algebra. Setting* $H = \mathbb{Z} \times G$*, we see that* $T^*(A)$ *with the multiplication so given is naturally* $H$*-graded. Taking* $\tilde{\chi}((n_1,g_1),(n_2,g_2)) := (-1)^{n_1 n_2}\chi(g_1,g_2)$ *we obtain a bicharacter on* $H$ *making* $(T^*(A), \diamond)$ *an* $(H,\tilde{\chi})$*-commutative algebra.*

**Proof** The additive structure of $(T^*(A), \diamond)$ is given and we regard it as clear that the product is well-defined and by construction distributive. This being so, we start by proving the product rule for $F$. To this end, let $a_1, \ldots, a_p \in A$ be homogeneous elements of $A$ and let $\sigma, \rho \in S_p$. Let

$c := a_1 \otimes \ldots \otimes a_p$. From the definitions, we see

$$F(\sigma, c)F(\rho, c_{\sigma^{-1}}) = \prod_{\substack{i<j \\ \sigma^{-1}(i)>\sigma^{-1}(j)}} \chi(|a_{\sigma^{-1}(j)}|, |a_{\sigma^{-1}(i)}|) \prod_{\substack{i<j \\ \rho^{-1}(i)>\rho^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{(\rho\sigma)^{-1}(i)}|)$$

$$= \prod_{\substack{\rho^{-1}(i)<\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{(\rho\sigma)^{-1}(i)}|) \prod_{\substack{i<j \\ \rho^{-1}(i)>\rho^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{(\rho\sigma)^{-1}(i)}|)$$

$$= \prod_{\substack{i<j \\ \rho^{-1}(i)<\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{(\rho\sigma)^{-1}(i)}|) \prod_{\substack{i>j \\ \rho^{-1}(i)<\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{\rho\sigma)^{-1}(i)}|)$$

$$\cdot \prod_{\substack{i<j \\ \rho^{-1}(i)>\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)<(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{(\rho\sigma)^{-1}(i)}|) \prod_{\substack{i<j \\ \rho^{-1}(i)>\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{(\rho\sigma)^{-1}(i)}|)$$

$$= \prod_{\substack{i<j \\ \rho^{-1}(i)<\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{(\rho\sigma)^{-1}(i)}|) \prod_{\substack{i>j \\ \rho^{-1}(i)<\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{\rho\sigma)^{-1}(i)}|)$$

$$\cdot \prod_{\substack{i>j \\ \rho^{-1}(i)<\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(i)}|, |a_{(\rho\sigma)^{-1}(j)}|) \prod_{\substack{i<j \\ \rho^{-1}(i)>\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{(\rho\sigma)^{-1}(i)}|)$$

$$= \prod_{\substack{i<j \\ \rho^{-1}(i)<\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{(\rho\sigma)^{-1}(i)}|) \prod_{\substack{i<j \\ \rho^{-1}(i)>\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{(\rho\sigma)^{-1}(i)}|)$$

where in the last step we use the antisymmetry of $\chi$ to see cancellation of the two middle terms in our product. The final term immediately simplifies to

$$\prod_{\substack{i<j \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{(\rho\sigma)^{-1}(i)}|) = F(\rho\sigma, c)$$

giving the product formula.
We will now check associativity of the modified shuffle product. As usual, it suffices to check this on generators. To this end, consider three elements $a := a_1 \otimes \ldots \otimes a_p \in T^p(A)$, $a' := a_{p+1} \otimes a_{p+2} \otimes \ldots \otimes a_{p+q} \in T^q(A)$ and $a'' := a_{p+q+1} \otimes \ldots \otimes a_{p+q+r} \in T^r(A)$, as always with all the $a_i$ being homogeneous. Set $c := a_1 \otimes a_2 \otimes \ldots \otimes a_{p+q+r}$. By straightforward calculation, we obtain:

$$a \diamond (a' \diamond a'') = \sum_{\rho \in Sh(p,q+r), \sigma \in Sh_p(q,r)} sign(\sigma\rho) F(\sigma, c) F(\rho, c_{\sigma^{-1}}) c_{(\rho\sigma)^{-1}}$$

and

$$(a \diamond a') \diamond a'' = \sum_{\sigma \in Sh(p,q)^r, \rho \in Sh(p+q,r)} sign(\sigma\rho) F(\sigma, c) F(\rho, c_{\sigma^{-1}}) c_{(\rho\sigma)^{-1}}$$

so it is sufficient to prove that the two sets

$$M_1 := \{\rho\sigma : \rho \in Sh(p, q+r), \sigma \in Sh_p(q,r)\}$$

and

$$M_2 := \{\rho\sigma : \rho \in Sh(p+q, r), \sigma \in Sh(p,q)^r\}$$

are equal and that the associated signs in both sums are equal. Assuming for a moment that indeed $M_1 = M_2$, the latter follows easily from the product formula: let $\sigma \in Sh_p(q,r), \rho \in Sh(p, q+r), \tau \in Sh(p,q)^r, \eta \in Sh(p+q, r)$ such that $\rho\sigma = \eta\tau$, then clearly we have $sign(\rho\sigma) = sign(\eta\tau)$, but also

$$F(\sigma, c)F(\rho, c_{\sigma^{-1}}) = F(\rho\sigma, c) = F(\eta\tau, c) = F(\tau, c)F(\eta, c_{\tau^{-1}}).$$

We will now show $M_1 = M_2$. The argumentation here is identical to the commutative case. To prove their equality, we simply compute generic elements of both sets: let $\rho \in Sh(p, q+r), \sigma \in Sh_p(q,r)$, then

$$\rho = \begin{pmatrix} 1 & 2 & \dots & p & p+1 & \dots & p+q+r \\ \rho_1 & \rho_2 & \dots & \rho_p & \rho_{p+1} & \dots & \rho_{p+q+r} \end{pmatrix}$$

with $\rho_1 < \rho_2 < \ldots < \rho_p$, $\rho_{p+1} < \ldots < \rho_{p+q+r}$ and likewise

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & p & p+1 & \dots & p+q & p+q+1 & \dots & p+q+r \\ 1 & 2 & \dots & p & \sigma_{p+1} & \dots & \sigma_{p+q} & \sigma_{p+q+1} & \dots & \sigma_{p+q+r} \end{pmatrix}$$

where $\sigma_{p+1} < \sigma_{p+2} < \ldots < \sigma p + q$ and $\sigma_{p+q+1} < \ldots < \sigma_{p+q+r}$. This means that for $\rho\sigma$ we get

$$\rho\sigma = \begin{pmatrix} 1 & 2 & \dots & p & p+1 & \dots & p+q & p+q+1 & \dots & p+q+r \\ \rho_1 & \rho_2 & \dots & \rho_p & \rho_{\sigma_{p+1}} & \dots & \rho_{\sigma_{p+q}} & \rho_{\sigma_{p+q+1}} & \dots & \rho_{\sigma_{p+q+r}} \end{pmatrix}$$

with conditions on the entries $\rho_1 < \ldots < \rho_p$, $\rho_{\sigma_{p+1}} < \ldots < \rho_{\sigma_{p+q}}$ and $\rho_{\sigma_{p+q+1}} < \ldots < \rho_{\sigma_{p+q+r}}$. In other words, what we get is just $M_1 = Sh(p,q,r)$. By exactly the same logic, we get also $M_2 = Sh(p,q,r)$, and so have proven $M_1 = M_2$.

Next, we prove that $T^*(A)$ with this multiplication and with respect to the natural $H$-grading is $(H, \tilde{\chi})$-commutative. Recall for that purpose that we had defined $H := \mathbb{Z} \times G$ and $\tilde{\chi}((n_1, x_1), (n_2, x_2)) := (-1)^{n_1 n_2} \chi(x_1, x_2)$ and

$$deg(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = (n, \sum_{i=1}^{n} |a_i|)$$

for homogeneous $a_i$. It is an easy calculation to verify that $\tilde{\chi}$ is a bicharacter. That the shuffle product is compatible with the grading of $T(A)$ given by $H$ is immediate from the definitions. What we need to prove is just that for homogeneous $a, a' \in T(A)$ we have

$$a' \diamond a = \tilde{\chi}(|a'|, |a|) a \diamond a'$$

Again it is sufficient to verify this on generators. Using $a$ and $a'$ as above and $c = a_1 \otimes a_2 \otimes \ldots \otimes a_{p+q}$ wherein all $a_i$ are supposed themselves homogeneous, we get

$$a \diamond a' = \sum_{\sigma \in Sh(p,q)} F(\sigma, c) sign(\sigma) c_{\sigma^{-1}}$$

and

$$a' \diamond a = \sum_{\sigma \in Sh(q,p)} F(\sigma, c_\tau) sign(\sigma) c_{\tau\sigma^{-1}}$$

where $\tau \in S_{p+q}$ is the permutation

$$\begin{pmatrix} 1 & 2 & \ldots & q & q+1 & \ldots & p+q \\ p+1 & p+2 & \ldots & p+q & 1 & \ldots & p \end{pmatrix}$$

We have to prove now that in both sums the terms are equal and that the commutation factors are equal up to multiplication with the factor prescribed by $(H, \tilde{\chi})$-commutativity. Since multiplication by $\tau$ is a bijective operation on $S_{p+q}$, the former is equivalent to showing that $\sigma\tau \in Sh(q,p)$ for every $\sigma \in Sh(p,q)$. Indeed, let

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & p & p+1 & \ldots & p+q \\ \sigma_1 & \sigma_2 & \ldots & \sigma_p & \sigma_{p+1} & \ldots & \sigma_{p+q} \end{pmatrix} \in Sh(p,q).$$

Then:

$$\sigma\tau = \begin{pmatrix} 1 & 2 & \ldots & q & q+1 & \ldots & p+q \\ \sigma_{p+1} & \sigma_{p+2} & \ldots & \sigma_{p+q} & \sigma_1 & \ldots & \sigma_p \end{pmatrix} \in Sh(q,p)$$

so right multiplication by $\tau$ gives a bijection $Sh(p,q) \to Sh(q,p)$. Using this observation and the product formula, we see now that

$$a' \diamond a = \sum_{\sigma \in Sh(p,q)} F(\sigma\tau, c_\tau) sign(\sigma\tau) c_{\sigma^{-1}} = sign(\tau) F(\tau, c_\tau) a \diamond a'.$$

By inversion counting, we get $sign(\tau) = (-1)^{pq}$. To calculate $F(\tau, c_\tau)$, consider

$$F(\tau, c_\tau) = \prod_{\substack{i<j \\ \tau^{-1}(i) > \tau^{-1}(j)}} \chi(|a_j|, |a_i|) = \chi(\sum_{i=p+1}^{p+q} |a_i|, \sum_{i=1}^{p} |a_i|).$$

Summing up, we have proven now that

$$a' \diamond a = (-1)^{pq} \chi(\sum_{i=p+1}^{p+q} |a_i|, \sum_{i=1}^{p} |a_i|) a \diamond a' = \tilde{\chi}(|a'|, |a|) a \diamond a'$$

as desired.

**Remark 4.1.3** *We would like to add some remarks dealing with the question to what degree it was really necessary to assume that the bicharacter $\chi$ was skew-symmetric. This is a natural question since the relations $\chi(g,h)\chi(h,g)a_g a_h = a_g a_h$ for $a_g \in A_g$ and $a_h \in A_h$ force some degree of skew-symmetry even in the absence of any explicit condition on $\chi$ beyond bilinearity. Going through the above proof, we first notice that without skew-symmetry of the bicharacter the product formula for our signs is no longer true in full generality, since it is no longer so that*

$$\prod_{\substack{i>j \\ \rho^{-1}(i)<\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(j)}|, |a_{(\rho\sigma)^{-1}(i)}) \cdot \prod_{\substack{i>j \\ \rho^{-1}(i)<\rho^{-1}(j) \\ (\rho\sigma)^{-1}(i)>(\rho\sigma)^{-1}(j)}} \chi(|a_{(\rho\sigma)^{-1}(i)}|, |a_{(\rho\sigma)^{-1}(j)}) = 1$$

*automatically for any permutations $\rho, \sigma \in S_n$. However, if $\rho \in Sh(p, q+r), \sigma \in Sh_p(q,r)$, the product formula holds even in this case. This is so because $i > j$ is the same as $\rho(\rho^{-1}(i)) > \rho(\rho^{-1}(j))$, so $i > j$ together with $\rho^{-1}(i) < \rho^{-1}(j)$ means that $(\rho^{-1}(i), \rho^{-1}(j))$ is an inversion pair for $\rho$. Since $\rho \in Sh(p, q+r)$ this implies $\rho^{-1}(i) \leq p$ and $\rho^{-1}(j) > p$. However, since by assumption $\sigma \in Sh_p(q,r)$ we see that $\sigma^{-1}(\rho^{-1}(i)) = \rho^{-1}(i) \leq p < \sigma^{-1}(\rho^{-1}(j))$. From this it follows that both products above are automatically void in this case, yielding the product formula. In the same way, one can also see that without skew-symmetry of $\chi$ the product formula still holds whenever $\rho \in Sh(p+q, r)$ and $\sigma \in Sh(p,q)^r$. This is enough to show associativity of our shuffle product. In fact, as now we did not use anything at all about $\chi$, $\chi$ might up to that point have been just any map sending $G \times G$ into $k$.*
*$(H, \tilde{\chi})$-commutativity of the shuffle product so constructed fails if $\chi$ was merely linear in both components, as can be seen already if one simply calculates according to the definitions $a_1 \diamond a_2$ and $a_2 \diamond a_1$ for $a_i \in A$ arbitrary homogeneous terms.*

**Remark 4.1.4** *Def./Prop. 4.1.2 takes on a particularly beautiful form in case that $A$ was $m$-commutative; in this case, the construction described produces an $m+1$-commutative product on $T(A)$.*

To define Harrison cohomology, we will have to prove that the Hochschild cochain differential is in some sense compatible with the $H$-commutative structure on the $k$-module underlying $T(A)$. The reason why a Harrison-like cohomology theory can be developed only for algebras satisfying some commutation conditions lies in this. In particular, we need the following proposition:

**Proposition 4.1.5** *Let $A$ be an $(G,\chi)$-commutative $k$-algebra. Then, $T(A)$ together with the shuffle product and the bar complex differential and the tensor grading becomes a differential graded algebra, i.e. the bar complex differential $b' : T^p(A) \to T^{p-1}(A)$ satisfies*

$$b'(a \diamond a') = b'(a) \diamond a' + (-1)^p a \diamond b'(a')$$

*for $a$ and $a'$ homogeneous elements of $T(A)$, $a \in T^p(A), a' \in T^q(A)$.*

**Proof** The only thing left to prove is that the differential acts as a graded derivation on $T(A)$. It suffices to check this on generators. Take hence two generators, $a := a_1 \otimes a_2 \otimes \ldots \otimes a_p \in T^p(A)$

and $a' := a_{p+1} \otimes a_{p+2} \otimes \ldots \otimes a_{p+q} \in T^q(A)$ as always with all entries homogeneous. Then, by definition

$$a \diamond a' = \sum_{\sigma \in Sh(p,q)} F(\sigma, c) sign(\sigma) c_{\sigma^{-1}}$$

where $c := a_1 \otimes a_2 \otimes \ldots \otimes a_p \otimes a_{p+1} \otimes \ldots \otimes a_{p+q}$. We have to prove that all terms appearing in the sum $b'(a) \diamond a' + (-1)^p a \diamond b'(a')$ appear also in $b'(a \diamond a')$ and vice versa and that the associated commutation factors are equal on both sides.

To this end, take an arbitrary shuffle $\sigma \in Sh(p,q)$ and consider the term $d_i(c_{\sigma^{-1}})$ appearing in $b(a \diamond a')$. Written out, this is simply

$$d_i(a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \ldots \otimes a_{\sigma^{-1}(i)} \otimes a_{\sigma^{-1}(i+1)} \otimes \ldots \otimes a_{\sigma^{-1}(p)} \otimes a_{\sigma^{-1}(p+1)} \otimes \ldots \otimes a_{\sigma^{-1}(p+q)}) \quad (4.1)$$

so the above term will be what we are searching for in $b'(a) \diamond a' + (-1)^p a \diamond b'(a')$. Looking at the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & p & p+1 & \ldots & p+q \\ \sigma_1 & \sigma_2 & \ldots & \sigma_p & \sigma_{p+1} & \ldots & \sigma_{p+q} \end{pmatrix}$$

we will call the first $p$ entries the left part and the others the right part of the shuffle. Now, we can discard the case where $\sigma^{-1}(i)$ and $\sigma^{-1}(i+1)$ lie in different parts of the permutation. For then, setting $j := \sigma^{-1}(i)$ and $k := \sigma^{-1}(i+1)$ and assuming without loss of generality $j < k$ also

$$\sigma' := \begin{pmatrix} 1 & 2 & \ldots & j & \ldots & p & p+1 & \ldots & k & \ldots & p+q \\ \sigma_1 & \sigma_2 & \ldots & i+1 & \ldots & \sigma_p & \sigma_{p+1} & \ldots & i & \ldots & \sigma_{p+q} \end{pmatrix}$$

is a $(p,q)$-shuffle. The term in $d_i(\ldots)$ associated to $\sigma'$ is

$$d_i(a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(i+1)} \otimes a_{\sigma^{-1}(i)} \otimes \ldots \otimes a_{\sigma^{-1}(p+q)})$$

which is (4.1) multiplied by a factor $\chi(|a_k|, |a_j|)$. Clearly $sign(\sigma) = -sign(\sigma')$. Also, we see

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & \ldots & i & i+1 & \ldots & p+q \\ \sigma_1^{-1} & \sigma_2^{-1} & \ldots & j & k & \ldots & \sigma_{p+q}^{-1} \end{pmatrix}$$

as well as

$$\sigma'^{-1} = \begin{pmatrix} 1 & 2 & \ldots & i & i+1 & \ldots & p+q \\ \sigma_1^{-1} & \sigma_2^{-1} & \ldots & k & j & \ldots & \sigma_{p+q}^{-1} \end{pmatrix}$$

which makes it clear that $F(\sigma', c) = \chi(|a_j|, |a_k|) F(\sigma, c)$. Taking everything together, this gives the equation

$$sign(\sigma') F(\sigma', c) d_i(c_{\sigma'^{-1}}) = -sign(\sigma) F(\sigma, c) d_i(c_{\sigma^{-1}}) \quad (4.2)$$

since $\chi(|a_j|, |a_k|)\chi(|a_k|, |a_j|) = 1$. So the terms of $b(a \diamond a')$ corresponding to the shuffles $\sigma$ and $\sigma'$ cancel out, and we consequently do not need to find them in $b'(a) \diamond a' + (-1)^p a \diamond b'(a')$. For this reason, we will from now on assume that $\sigma^{-1}(i)$ and $\sigma^{-1}(i+1)$ are in the same part of our permutation.

We first consider the case that both are in the left part, i.e. $j := \sigma^{-1}(i) < k := \sigma^{-1}(i+1) \leq p$.

Since the $\sigma_l$ form an ascending sequence for $1 \leq l \leq p$, and because by definition $\sigma_j = i, \sigma_k = i+1$, it is clear that $k = j+1$. Hence, $\sigma$ is of the form

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & j & j+1 & j+2 & \dots & p & p+1 & \dots & p+q \\ \sigma_1 & \sigma_2 & \dots & i & i+1 & \sigma_{j+2} & \dots & \sigma_p & \sigma_{p+1} & \dots & \sigma_{p+q} \end{pmatrix}$$

and so the term we want to recover is

$$a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \dots \otimes a_{\sigma^{-1}(i-1)} \otimes a_j a_{j+1} \otimes a_{\sigma^{-1}(i+2)} \otimes \dots \otimes a_{\sigma^{-1}(p)} \otimes a_{\sigma^{-1}(p+1)} \otimes \dots \otimes a_{\sigma^{-1}(p+q)}$$

Consider now the permutation

$$\tau := \begin{pmatrix} 1 & 2 & \dots & j & j+1 & \dots & p-1 & p & \dots & p+j' & p+j'+1 & \dots & p+q-1 \\ \sigma_1 & \sigma_2 & \dots & i & \sigma_{j+2}-1 & \dots & \sigma_p - 1 & \sigma_{p+1} & \dots & \sigma_{p+j'+1} & \sigma_{p+j'+2}-1 & \dots & \sigma_{p+q}-1 \end{pmatrix}$$

where $j'$ is supposed to be chosen such that $\sigma_{p+j'+1}$ is the last of the $\sigma_l$ which is smaller than $i$. It is easy to verify that this is indeed a permutation, i.e. that all entries appear only once and that the map so defined has the set $\{1, \dots, p+q-1\}$ as range. Furthermore, we have $\sigma_1 < \sigma_2 < \dots < i < \sigma_{i+2} - 1 < \dots < \sigma_p - 1$ and likewise $\sigma_{p+1} < \sigma_{p+2} < \dots < \sigma_{p+j'+1} < \sigma_{p+j'+2} - 1 < \dots < \sigma_{p+q} - 1$ and so $\tau$ is a $(p-1, q)$-shuffle. We will now construct the term of $b'(a) \diamond a'$ associated to $d_j$ and $\tau$. To do this, we first compute the inverse of $\tau$:

$$\tau^{-1}(l) = \begin{cases} \sigma^{-1}(l) & , \ \sigma^{-1}(l) \leq j \\ \sigma^{-1}(l+1) - 1 & , \ j+2 \leq \sigma^{-1}(l+1) \leq p \\ \sigma^{-1}(l) - 1 & , \ p+1 \leq \sigma^{-1}(l) \leq p+j'+1 \\ \sigma^{-1}(l+1) - 1 & , \ \sigma^{-1}(l+1) > p+j'+1. \end{cases}$$

Now set $c_l$ and $c'_l$ in the following way:

$$c_l := \begin{cases} a_l & , \ l < j \\ a_j a_{j+1} & , \ l = j \\ a_{l+1} & , \ l > j \end{cases}$$

and

$$c'_l := \begin{cases} a_{\sigma^{-1}(l)} & , \ l < i \\ a_j a_{j+1} & , \ l = i \\ a_{\sigma^{-1}(l+1)} & , \ l > i. \end{cases}$$

Then, the terms we want to compare are $c_{\tau^{-1}(1)} \otimes c_{\tau^{-1}(2)} \otimes \dots \otimes c_{\tau^{-1}(p+q-1)}$ and $c'_1 \otimes c'_2 \otimes \dots \otimes c'_{p+q-1}$. In other words, we would like to prove that for each $l$ we have $c_{\tau^{-1}(l)} = c'_l$. To this end, we will consider the following cases:

1. *Case:* $l \leq i$
   Then either we have $\sigma^{-1}(l) \leq j$ or $p+1 \leq \sigma^{-1}(l) \leq p+j'+1$. If the former is true, then we get
   $$c_{\tau^{-1}(l)} = c_{\sigma^{-1}(l)} = a_{\sigma^{-1}(l)} = c'_l$$
   as desired. If, on the other hand, we have $p+1 \leq \sigma^{-1}(l) \leq p+j'+1$, then we see
   $$c_{\tau^{-1}(l)} = c_{\sigma^{-1}(l)-1} = a_{\sigma^{-1}(l)}$$
   which is also good. What remains to be done is the other

2. *Case:* $l > i$
   In this case, we get either $j+2 \leq \sigma^{-1}(l+1) \leq p$ or $\sigma^{-1}(l+1) > p+j'+1$. If the first of these, we see immediately that $c_{\tau^{-1}(l)} = c_{\sigma^{-1}(l+1)-1} = a_{\sigma^{-1}(l+1)} = c'_l$. If the second inequality holds, we have $c_{\tau^{-1}(l)} = c_{\sigma^{-1}(l+1)-1} = a_{\sigma^{-1}(l+1)}$ again as desired. So, all cases check out well and we are done.

This means that we have been successful at recovering the term we were looking for in the sum $b'(a) \diamond a'$. What remains to be done is to check that the associated commutation factors are equal. In $b'(a \diamond a')$, the term

$$a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \ldots \otimes a_{\sigma^{-1}(i-1)} \otimes a_j a_{j+1} \otimes a_{\sigma^{-1}(i+2)} \otimes \ldots \otimes a_{\sigma^{-1}(p)} \otimes a_{\sigma^{-1}(p+1)} \otimes \ldots \otimes a_{\sigma^{-1}(p+q)}$$

appears with commutation factor $(-1)^i F(\sigma, c) sign(\sigma)$. In $b'(a)\diamond a'$, on the other hand, we found it multiplied by $(-1)^{\sigma(i)} F(\tau, c') sign(\tau)$ where $c' = a_1 \otimes a_2 \otimes \ldots \otimes a_{j-1} \otimes a_j a_{j+1} \otimes a_{j+2} \otimes \ldots \otimes a_{p+q}$. Consequently, we have to show $sign(\tau) = (-1)^{\sigma(i)-i} sign(\sigma)$ and $F(\tau, c') = F(\sigma, c)$. To show $sign(\tau) = (-1)^{\sigma(i)-i} sign(\sigma)$, we notice that because $\sigma$ is a shuffle, the number $\sigma(l) - l$ gives for any $l$ in the left part of $\sigma$ the number of entries in the right part which are smaller than $\sigma(l)$. In other words, the number $s$ of inversions in $\sigma$ is given by

$$s = \sum_{l=1}^{p} \sigma_l - l$$

Likewise, the number $t$ of inversions in $\tau$ is given by

$$t = \sum_{l=}^{p-1} t_l - l = \sum_{l=1}^{i} \sigma_l - l + \sum_{l=i+1}^{p-1} \sigma_{l+1} - (l+1)$$

Hence we have $s - t = \sigma_{i+1} - (i+1) = \sigma_i - i$ as desired. Now we will prove that also the other contribution to the commutation factors comes out right, i.e. that $F(\tau, c') = F(\sigma, c)$. But, recalling that

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & j & j+1 & j+2 & \ldots & p & p+1 & \ldots & p+q \\ \sigma_1 & \sigma_2 & \ldots & i & i+1 & \sigma_{j+2} & \ldots & \sigma_p & \sigma_{p+1} & \ldots & \sigma_{p+q} \end{pmatrix}$$

and

$$\tau := \begin{pmatrix} 1 & 2 & \ldots & j & j+1 & \ldots & p-1 & p & \ldots & p+j' & p+j'+1 & \ldots & p+q-1 \\ \sigma_1 & \sigma_2 & \ldots & i & \sigma_{j+2}-1 & \ldots & \sigma_p - 1 & \sigma_{p+1} & \ldots & \sigma_{p+j'+1} & \sigma_{p+j'+2}-1 & \ldots & \sigma_{p+q}-1 \end{pmatrix}$$

it is clear that the inversions appearing in the definition of $F(\sigma, c)$ are exactly the same inversions as those appearing in $F(\tau, c')$, with the exception of factors $\chi(|a_l|, |a_i|)$ and $\chi(|a_l|, |a_{i+1}|)$ for $F(\sigma, c)$ and a factor $\chi(|a_l|, |a_i a_{i+1}|)$ for $F(\tau, c')$ respectively. The contributions to the signs coming from such pairs are hence equal on both sides. In other words, we get the desired equation.
We still have to consider the case where both $\sigma^{-1}(i)$ and $\sigma^{-1}(i+1)$ are in the *right* part of

our shuffle. In this case, we must recover the term in (4.1) from $a \diamond b'(a')$ and expect to get an additional sign of $(-1)^p$ in front of it. But, by the arguments just given, our term would appear in $b'(a') \diamond a$ with signs as in $b'(a' \diamond a)$, which because the shuffle product is $(H, \tilde{\chi})$-commutative is simply $\tilde{\chi}(|a'|, |a|)b'(a \diamond a')$. Likewise, we get $b'(a') \diamond a = \tilde{\chi}(|a'| - (1,0), |a|)a \diamond b'(a')$. We used here that $b'$ is a map of degree zero with respect to the grading induced by the grading on $A$ and of degree $-1$ with respect to the tensor grading, which implies that $b'(a')$ is a sum of homogeneous tensor products all of degree $(q, \sum_{i=p+1}^{p+q} |a_i|) - (1,0) = |a'| - (1,0)$. Keeping this in mind, we see

$$\tilde{\chi}(|a'|, |a|) = (-1)^{pq}\chi(\sum_{i=p+1}^{p+q} a_i, \sum_{i=1}^{p} |a_i|)$$

and

$$\tilde{\chi}(|a'| - (1,0), |a|) = (-1)^{(q-1)p}\chi(\sum_{i=p+1}^{p+q} a_i, \sum_{i=1}^{p} |a_i|).$$

Since the difference between these two products is exactly $(-1)^p$, we are done. This concludes the proof that the bar differential endows $(C_*(A), \diamond)$ with a differential graded algebra structure.

**Remark 4.1.6** *If $\chi$ is not supposed skew-symmetric, the terms of $b'(a \diamond a')$ which were shown to cancel out in the proof - i.e., terms containing a product $a_i a_j$ with $i \leq p$ and $j > p$ - still cancel out, because in this case we have $\chi(|a_i|, |a_j|)\chi(|a_j|, |a_i|)a_i a_j = a_i a_j$ and we can use k-linearity of the tensor product to show (eq. 4.2). We can follow the proof verbatim to see that terms on both sides of the 'differential graded' equation that do not vanish in $b'(a \diamond a')$ by the process just described are equal if in the sense of the proof they come from the left side of the shuffle, i.e. appear in some $d_i(a) \diamond a'$). For the terms belonging to $a \diamond d_i(a')$, we can no longer use the argument based on graded commutativity of the shuffle product as with $\chi$ not skew-commutative, we do not have the graded commutativity property proven there. However, if we repeat the procedure of the 'left side proof' for those terms, we see that the only essential change is an additional term of $(-1)^p$. So, the preceding proposition can be adapted to the context of a $\chi$ that lacks skew-symmetry.*

## 4.2 Generalized Harrison cohomology

We now have a sufficient understanding of the shuffle product to define Harrison cohomology. This cohomology will give a nice homological framework for instance for classifying formal deformations which preserve $G$-commutativity of an algebra. Our definition will specialize to the usual definition for commutative algebras if applied to a commutative algebra viewed as an algebra graded over the trivial group. We will in the following continue to presume $A$ $(G, \chi)$-commutative and denote the algebra given by $(T(A), \diamond)$ by $Sh(A)$. When we view $Sh(A)$ as graded by a single index, we shall by default have the tensor grading in mind. After these preliminary remarks, we are ready for the definition/proposition:

**Definition/Proposition 4.2.1** *Let $A$ be an $(G,\chi)$-commutative $k$-algebra and $M$ be an $(G,\chi)$-symmetric $A$-bimodule. Set $I := \sum_{i=1}^{\infty} Sh^i(A)$ and consider the graded $Sh(A)$-module $Q := I/I^2 = \oplus_{i=1}^{\infty} Q^i$. Then, the $k$-modules*

$$\tilde{C}^n(A,M) := \boldsymbol{Hom}_k(Q^n, M)$$

*form a cochain complex if endowed with the differential naturally induced by the ordinary $(G,\chi)$-graded Hochschild cohomology differential. The cohomology of this complex is called the* Harrison cohomology *of $A$ with coefficients in $M$. We will denote the space of Harrison $n$-cocycles of $A$ by $Z^n_{ha}(A,M)$ or $Z^n_{ha}(A)$ if $M = A$. Likewise, the space of Harrison $n$-coboundaries of $A$ with coefficients in $M$ shall be denoted by $B^n_{ha}(A,M)$ or by $B_{ha}(A)$ if $M = A$. We will denote the $n$-th Harrison cohomology group of $A$ with coefficients in $M$ by $H^n_{ha}(A,M)$.*

**Proof** It is clear that $I$ forms with the notations of the previous proposition an $(H,\tilde{\chi})$-graded ideal in $Sh(A)$, so the quotient $Q := I/I^2$ is a graded $Sh(A)$-module in the natural way. Note also that due to $Sh(A)$ being $(H,\tilde{\chi})$-commutative, we know in particular that $I^2$ is as an $Sh(A)$-leftmodule generated by elements of the form $a \diamond a'$ with $a, a' \in I$. As any $r \in Sh(A)$ can be written as $r = r_0 + r_+$ with $r_0 \in k$ and $r_+ \in I$, one then sees that indeed $I^2$ is generated even as a $k$-module by products of the form $a \diamond a'$ with $a, a' \in I$ being homogeneous elements. It is also obvious that if the Hochschild cohomology differential $\beta$ induces a map

$$\tilde{\beta} : \tilde{C}^n(A,M) \to \tilde{C}^{n+1}(A,M)$$

for every $n \in \mathbb{N}$, then the map so defined will satisfy the condition $\tilde{\beta}^2 = 0$ and the groups $\tilde{C}^n(A)$ will hence form a cochain complex. Hence, what needs to be proven is just that the Hochschild differential induces a map on this complex, which in turn is equivalent to the condition

$$\varphi \in C^n(A), \varphi|_{I^2} = 0 \Rightarrow \beta(\varphi)|_{I^2} = 0$$

i.e. to the vanishing on shuffle products of the differentials of Hochschild cochains which vanish on shuffle products. Again, it is sufficient to check this on generators. We quickly recall the definition of the boundary operator of Hochschild cohomology, which is

$$\begin{aligned}&\beta(\varphi)(a_1 \otimes a_2 \otimes \ldots \otimes a_n) := \\ &\chi(|\varphi|,|a_1|)a_1\varphi(a_2 \otimes a_3 \otimes \ldots \otimes a_n) - \varphi(b'(a_1 \otimes \ldots \otimes a_n)) + (-1)^n \varphi(a_1 \otimes a_2 \otimes \ldots \otimes a_{n-1})a_n\end{aligned}$$

with $\varphi \in C^{n-1}(A,M) = \mathbf{Hom}_k(A^{\otimes(n-1)}, M)$ and all the $a_i$ assumed homogeneous. To prove our proposition, we have to assume that $\varphi$ vanishes on shuffle products, and want to show that $\beta(\varphi)$ does so, too. To this end, we take $a \in Sh^p(A)$, $a' \in Sh^q(A)$ as usual and with $n = p+q$ calculate

$$\begin{aligned}&\beta(\varphi)(a \diamond a') = (\sum_{\sigma \in Sh(p,q)} \chi(|\varphi|, |a_{\sigma^{-1}(1)}|)F(\sigma,c)sign(\sigma)a_{\sigma^{-1}(1)}\varphi(a_{\sigma^{-1}(2)} \otimes \ldots \otimes a_{\sigma^{-1}(n)})) \\ - \quad &\varphi(b'(a \diamond a')) \\ + \quad &(-1)^n \sum_{\sigma \in Sh(p,q)} F(\sigma,c)sign(\sigma)\varphi(a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \ldots \otimes a_{\sigma^{-1}(n-1)})a_{\sigma^{-1}(n)}.\end{aligned}$$

In this sum, all three summands are zero. This is easiest to see for the middle term: it is, by Proposition 4.1.5, equal to $\varphi(b'(a)\diamond a' + (-1)^p a \diamond b'(a'))$, which is zero because $\varphi$ was supposed to vanish on $I^2$. To treat the first term, we calculate

$$\sum_{\sigma\in Sh(p,q)} \chi(|\varphi|, |a_{\sigma^{-1}(1)}|)F(\sigma, c)sign(\sigma)a_{\sigma^{-1}(1)}\varphi(a_{\sigma^{-1}(2)} \otimes \ldots \otimes a_{\sigma^{-1}(p+q)}) = \sum_{n\in\{1,p+1\}} \chi(|\varphi|.|a_n|)a_n\varphi(\sum_{\substack{\sigma\in Sh(p,q)\\ \sigma(n)=1}} F(\sigma, c)sign(\sigma)a_{\sigma^{-1}(2)} \otimes \ldots \otimes a_{\sigma^{-1}(p+q)})$$

and for $n = 1$, i.e. the first term in the sum on the right, we obtain, ignoring the multiplicative factor in front:

$$a_1\varphi(\sum_{\substack{\sigma\in Sh(p,q)\\ \sigma(1)=1}} F(\sigma, c)sign(\sigma)a_{\sigma^{-1}(2)}\otimes\ldots\otimes a_{\sigma^{-1}(p+q)}) = a_1\varphi(a_2\otimes a_3\otimes\ldots\otimes a_p\diamond a_{p+1}\otimes\ldots\otimes a_{p+q}) = 0$$

whereas the case $n = p + 1$, i.e. the second term, yields

$$a_{p+1}\varphi(\sum_{\substack{\sigma\in Sh(p,q)\\ \sigma(p+1)=1}} F(\sigma, c)sign(\sigma)a_{\sigma^{-1}(2)} \otimes \ldots \otimes a_{\sigma^{-1}(p)} \otimes a_{\sigma^{-1}(p+1)} \otimes \ldots \otimes a_{\sigma^{-1}(p+q)}) = (-1)^p a_{p+1}\chi(|\sum_{i=1}^{p} |a_i|, |a_{p+1}|)\varphi(a_1 \otimes a_2 \otimes \ldots \otimes a_p \diamond a_{p+2} \otimes \ldots \otimes a_{p+q}) = 0.$$

The vanishing of the third summand follows in exactly the same way, and hence we have completely proven our proposition.

**Remark 4.2.2** *If $\chi$ is not skew-symmetric, the calculations carry over without change, since we used skew-symmetry at no point. Also, $I^2$ viewed as a two-sided $Sh(A)$-ideal can still be seen to be generated as a $k$-module by elements of the form $a \diamond a'$, with $a, a' \in I$, so these calculations suffice to prove that the Hochschild cohomology differential is compatible with quotienting by $I^2$. The conclusion is that skew-symmetry of the bicharacter is not needed for the Harrison construction to work.*

## 4.3 Cohomology of color polynomial rings

We have now proven that a harrison-type cohomology construction works in general for $G$-commutative algebras. In the case of a commutative algebra the cohomology here given becomes classical Harrison cohomology if the algebra is viewed as being concentrated in degree zero, since then the contributions to the shuffle factors coming from the graded structure vanish.
The following results are close analogs of things obtained for the case of commutative algebras in (Harrison [31]). We will assume that $K$ is an arbitrary field and $k$ is a commutative ring.

**Lemma 4.3.1** *Let $A$ be a $(G,\chi)$-commutative $k$-algebra and $M$ a $(G,\chi)$-symmetric $A$-module. Then we have*

$$\begin{aligned} & f(aa'\otimes bb') = \chi(|a'|,|b|)(\chi(|f|,|a|+|b|)abf(a'\otimes b') \\ + \quad & f(ab\otimes a'b') + f(a\otimes b)a'b') - f(a\otimes a')bb' - \chi(|f|,|a|+|a'|)aa'f(b\otimes b') \end{aligned}$$

*for any $a, a', b, b'$ and $f \in Z^2_{ha}(A,M)$ homogeneous.*

**Proof** The proof is based on an elementary calculation using the properties of a second degree Harrison cocycle. There are no differences other than some additional signs between the classical situation and our situation, but the calculation is not explicitly given in (Harrison [31]) - although he does mention that the statement follows from *tedious repeated application of the defining conditions of a cocycle* - and so we give a proof nonetheless. With all elements homogeneous, we first find

$$f(ab\otimes c) = \chi(|f|,|a|)af(b\otimes c) + f(a\otimes bc) - f(a\otimes b)c$$

because of $f$ a Hochschild cocycle. We will now repeatedly use this relation together with $(G,\chi)$-symmetry of a Harrison 2-cochain to obtain our result:

$$\begin{aligned} f(aa',bb') &= \chi(|f|,|a|)af(a',bb') + f(a,a'bb') - f(a,a')bb' \\ &= \chi(|a'|,|b|+|b'|)\chi(|f|,|a|)(\chi(|f|,|b|)abf(b',a') + af(b,b'a') - af(b,b')a') \\ &+ \chi(|a'|,|b|)(f(ab,a'b') + f(a,b)a'b' - \chi(|f|,|a|)af(b,a'b')) - f(a,a')bb' \\ &= \chi(|a'|,|b|)(\chi(|f|,|a|+|b|)abf(a',b') + f(ab,a'b') + f(a,b)a'b') \\ &- \chi(|f|,|a|+|a'|)aa'f(b,b') - f(a,a')bb' \end{aligned}$$

where at the end we see the desired term.

**Remark 4.3.2** *A similar calculation can be carried out also in dimension three. Already in the commutative case, it fills around two printed pages. We will not use it in the sequel, so we omit it.*

We can use the previous lemma to prove the following:

**Proposition 4.3.3** *Let $A$ and $A'$ be $(G,\chi)$-commutative algebras and let $B$ be their $(G,\chi)$-commutative tensor product. Let further $M$ be a $G$-symmetric bimodule over $A$ and $A'$ such that the multiplications with elements of $A$ and $A'$ respectively are compatible, i.e. for $a\in A, a'\in A'$ homogeneous elements we have $a(a'm) = \chi(|a|,|a'|)a'(am)$. Then, $H^i_{ha}(B,M) \cong H^i_{ha}(A,M) \oplus H^i_{ha}(A',M)$ for $i\in\{1,2\}$. The isomorphism is induced by a chain map.*

**Proof** The idea of the proof is to explicitly construct a chain map between the two chain complexes involved and to show by concrete calculation that at least in low dimensions it induces an isomorphism on the level of cohomology spaces. The implementation of this plan

decomposes into several steps:
*Step 0:* First, it is useful to remember that for two graded $k$-modules $U$ and $V$ the direct sum $U \oplus V$ is given a graded structure by setting $(U \oplus V)_g := U_g \oplus V_g$.
Note also that in the course of this proof, there is one possible source of confusion with regards to notation. We will frequently be dealing with functions of types $A^{\otimes n} \to M$, $A'^{\otimes n} \to M$, $(A \otimes A')^{\otimes n} \to M$ and so on. Problems of notation arise here because, while the tensor product itself is associative, the difference e.g. between a function living on $(A \otimes A)^n$ and one living on $A^{2n}$ *is* visible to the Hochschild differential. For reasons of readability, we will therefore use the notation

$$f(a_1 \otimes a'_1, a_2 \otimes a'_2 \dots, a_n \otimes a'_n)$$

when we want to talk about the value a function of type $(A \otimes A')^n \to M$ takes on the specified inputs.
*Step 1:* We start by defining a chain map $\psi : C^*_{ha}(A, M) \oplus C^*_{ha}(A', M) \to C^*_{ha}(A \otimes A', M)$ by setting

$$\begin{aligned} \psi_n(g, g')(a_1 \otimes a'_1, \dots, a_n \otimes a'_n)) &:= F(\sigma, a_1 \otimes a_2 \otimes \dots \otimes a_n \otimes a'_1 \otimes \dots \otimes a'_n) \\ \cdot \quad & (\chi(|g'|, \sum_{i=1}^n |a_i|) a_1 a_2 \cdot \dots \cdot a_n g'(a'_1, a'_2, \dots, a'_n) + g(a_1, a_2, \dots, a_n) a'_1 a'_2 \cdot \dots \cdot a'_n) \end{aligned}$$

with the $a_i$ and $g, g'$ assumed homogeneous as usual and where $\sigma$ denotes the fixed permutation

$$\sigma := \sigma_n := \begin{pmatrix} 1 & 2 & 3 & \dots & n & n+1 & \dots & 2n \\ 1 & 3 & 5 & \dots & 2n-1 & 2 & \dots & 2n \end{pmatrix}.$$

This definition gives a well-defined $k$-linear map $(A \otimes A')^{\otimes n} \to M$ as both summands are induced by natural $n$-linear maps $(A \otimes A')^n \to M$. Also, $\psi_n : C^n(A, M) \oplus C^n(A', M) \to C^n(A \otimes A', M)$ is a map of degree zero since with $g \in C^n(A, M), g' \in C^n(A', M)$ homogeneous elements of the same degree $d$ and $a_i \in A$ homogeneous, the factors $F(\sigma, \dots)$ and $\chi(\dots)$ in the defining formula are clearly degree zero, whereas the factors

$$a_1 a_2 \cdot \dots \cdot a_n g'(a'_1, \dots, a'_n)$$

and

$$g(a_1, a_2, \dots, a_n) a'_1 a'_2 \cdot \dots \cdot a'_n$$

are both degree $\sum_{i=1}^n |a_i| + \sum_{i=1}^n |a'_i| + |(g, g')|$, so $\psi_n(g, g')$ is under these conditions of the same degree as $(g, g')$. Finally, the map $\psi_n$ is easily verified $k$-linear itself.
What remains to be shown to prove well-definedness as a map of type $C^n_{ha}(A, M) \oplus C^n_{ha}(A'M) \to C^n_{ha}(A \otimes A', M)$ is that for all $g \in C^n_{ha}(A, M), g' \in C^n_{ha}(A', M)$ we have $\psi_n(g, g') \in C^n_{ha}(A \otimes A', M)$, i.e. that given the vanishing of $g$ and $g'$ on shuffles in their respective domains we also have vanishing of $\psi_n(g, g')$ on shuffles over $A \otimes A'$. To this end, we calculate for $a_1, \dots, a_{p+q} \in A$

and $a'_1, \ldots, a'_{p+q}$ homogeneous

$$\begin{aligned}
& \psi(g,g')((a_1 \otimes a'_1, a_2 \otimes a'_2, \ldots a_p \otimes a'_p) \diamond (a_{p+1} \otimes a'_{p+1}, \ldots, a_{p+q} \otimes a'_{p+q})) \\
= & \sum_{\tau \in Sh(p,q)} F(\tau, (a_1 \otimes a'_1, \ldots, a_{p+q} \otimes a'_{p+q}))sign(\tau) \\
\cdot & \psi(g,g')(a_{\tau^{-1}(1)} \otimes a'_{\tau^{-1}(1)}, \ldots, a_{\tau^{-1}(p+q)} \otimes a'_{\tau^{-1}(p+q)}) \\
= & \chi(|g'|, \sum_{i=1}^{p+q} |a_i|) a_1 a_2 \cdot \ldots \cdot a_{p+q} \sum_{\tau \in Sh(p,q)} F(\tau) sign(\tau) g'(a'_{\tau^{-1}(1)}, a'_{\tau^{-1}(2)}, \ldots, a'_{\tau^{-1}(p+q)}) + \\
& \left( \sum_{\tau \in Sh(p,q)} G(\tau) sign(\tau) g(a_{\tau^{-1}(1)}, a_{\tau^{-1}(2)}, \ldots, a_{\tau^{-1}(p+q)}) \right) a'_1 a'_2 \cdot \ldots \cdot a'_{p+q}
\end{aligned}$$

wherein

$$\begin{aligned}
& F(\tau) := F(\tau, a_1 \otimes a'_1, \ldots, a_{p+q} \otimes a'_{p+q}) F(\sigma, a_{\tau^{-1}(1)} \otimes \ldots \otimes a_{\tau^{-1}(p+q)} \otimes a'_{\tau^{-1}(1)} \otimes \ldots \otimes a'_{\tau^{-1}(p+q)}) \\
\cdot \; & F(\tau, a_1 \otimes a_2 \otimes \ldots \otimes a_{p+q})^{-1}
\end{aligned}$$

and

$$\begin{aligned}
& G(\tau) := F(\tau, a_1 \otimes a'_1, \ldots, a_{p+q} \otimes a'_{p+q}) F(\sigma, a_{\tau^{-1}(1)} \otimes \ldots \otimes a_{\tau^{-1}(p+q)} \otimes a'_{\tau^{-1}(1)} \otimes \ldots \otimes a'_{\tau^{-1}(p+q)}) \\
\cdot \; & F(\tau, a'_1 \otimes a'_2 \otimes \ldots \otimes a'_{p+q})^{-1}.
\end{aligned}$$

We have to prove that these signs are (essentially) equal to the corresponding signs appearing in the shuffle products

$$a'_1 \otimes a'_2 \otimes \ldots \otimes a'_p \diamond a'_{p+1} \otimes a'_{p+2} \otimes \ldots \otimes a'_{p+q}$$

and

$$a_1 \otimes \ldots \otimes a_p \diamond a_{p+1} \otimes \ldots \otimes a_{p+q}$$

respectively. Once we have done this, we can apply the vanishing of $g, g'$ on shuffle products in their respective domains to conclude that both summands in our term are zero. To get there, we will prove that

$$F(\tau) = F(\sigma, a_1 \otimes \ldots \otimes a_{p+q} \otimes a'_1 \otimes \ldots \otimes a'_{p+q}) F(\tau, a'_1 \otimes a'_2 \otimes \ldots \otimes a'_{p+q})$$

and that similarly

$$G(\tau) = F(\sigma, a_1 \otimes \ldots \otimes a_{p+q} \otimes a'_1 \otimes \ldots \otimes a'_{p+q}) F(\tau, a_1 \otimes a_2 \otimes \ldots \otimes a_{p+q})$$

It will be sufficient to show the former, as the proofs of these two assertions are virtually identical. We start by calculating all the relevant terms:

$$\begin{aligned} & F(\tau,(a_1\otimes a'_1,\dots,a_{p+q}\otimes a'_{p+q})) = F(\tau, a_1\otimes a_2\otimes\dots\otimes a_{p+q}) \\ \cdot & \prod_{\substack{i<j\\ \tau^{-1}(i)>\tau^{-1}(j)}} \chi(|a'_{\tau^{-1}(j)}|,|a_{\tau^{-1}(i)}|)\cdot \prod_{\substack{i<j\\ \tau^{-1}(i)>\tau^{-1}(j)}} \chi(|a_{\tau^{-1}(j)}|,|a'_{\tau^{-1}(i)}|)\cdot F(\tau,a'_1\otimes a'_2\otimes\dots\otimes a'_{p+q}), \\ & F(\sigma, a_{\tau^{-1}(1)}\otimes\dots\otimes a_{\tau^{-1}(p+q)}\otimes a_{\tau^{-1}(1)}\otimes\dots\otimes a_{\tau^{-1}(p+q)}) = \prod_{i<j}\chi(|a'_{\tau^{-1}(i)}|,|a_{\tau^{-1}(j)}|), \\ & F(\sigma, a_1\otimes\dots\otimes a_{p+q}\otimes a'_1\otimes\dots\otimes a'_{p+q}) = \prod_{i<j}\chi(|a'_i|,|a_j|). \end{aligned}$$

Setting

$$D_\tau := F(\sigma, a_{\tau^{-1}(1)}\otimes\dots\otimes a_{\tau^{-1}(p+q)}\otimes a'_{\tau^{-1}(1)}\otimes\dots\otimes a'_{\tau^{-1}(p+q)})F(\sigma,a_1\otimes\dots\otimes a_{p+q}\otimes a'_1\otimes\dots\otimes a'_{p+q})^{-1}$$

we also see that:

$$D_\tau = \prod_{\substack{i<j\\ \tau^{-1}(i)>\tau^{-1}(j)}} \chi(|a'_{\tau^{-1}(i)}|,|a_{\tau^{-1}(j)}|)\cdot \prod_{\substack{i<j\\ \tau^{-1}(i)>\tau^{-1}(j)}} \chi(|a_{\tau^{-1}(i)}|,|a'_{\tau^{-1}(j)}|).$$

Taking these observations together, we obtain

$$\begin{aligned} & F(\tau,(a_1\otimes a'_1,\dots,a_{p+q}\otimes a'_{p+q}))F(\sigma,a_{\tau^{-1}(1)}\otimes\dots\otimes a_{\tau^{-1}(p+q)}\otimes a'_{\tau^{-1}(1)}\otimes\dots\otimes a'_{\tau^{-1}(p+q)}) \\ \cdot\ & F(\tau,a_1\otimes\dots\otimes a_{p+q})^{-1} = F(\tau,a'_1\otimes\dots\otimes a'_{p+q})D_\tau^{-1}F(\sigma,a_{\tau^{-1}}\otimes\dots\otimes a'_{\tau^{-1}(p+q)}) = \\ & F(\tau,a'_1\otimes\dots\otimes a'_{p+q})F(\sigma,a_1\otimes\dots\otimes a_{p+q}\otimes a'_1\otimes\dots\otimes a'_{p+q}) \end{aligned}$$

as desired. We are now done with proving that $\psi$ is well-defined. We will proceed to verify that it is a morphism of chain complexes. To this end, we have to show that for $g\in C^n_{ha}(A,M)$ and $g'\in C^n_{ha}(A',M)$ we always have $\psi_{n+1}(\beta g,\beta g') = \beta(\psi_n(g,g'))$. Set $a := a_1\otimes\dots\otimes a_{n+1}$, $a' := a'_1\otimes\dots a'_{n+1}$, and denote by $aa'$ the tensor concatenation of $a$ and $a'$. Then calculating explicit representations for both sides of the equation to be proven, we see

$$\begin{aligned} & \psi_{n+1}(\beta g,\beta g')(a_1\otimes a'_1,\dots,a_{n+1}\otimes a'_{n+1}) = F(\sigma_{n+1},aa')\cdot \\ \cdot\ & (\chi(|g'|,\sum_{i=1}^{n+1}|a_i|)a_1a_2\cdot\ldots\cdot a_{n+1}(\chi(|g'|,|a'_1|)a'_1g'(a'_2,\dots,a'_{n+1}) \\ +\ & \sum_{i=1}^{n}(-1)^i g'(a'_1,a'_2,\dots,a'_ia'_{i+1},\dots,a'_{n+1}) + (-1)^{n+1}g'(a'_1,a'_2,\dots,a'_n)a'_{n+1}) \\ +\ & (\chi(|g|,|a_1|)a_1g(a_2,a_3,\dots,a_{n+1}) + \sum_{i=1}^{n}(-1)^i g(a_1,a_2,\dots,a_ia_{i+1},\dots,a_{n+1}) \\ & +(-1)^{n+1}g(a_1,\dots,a_n)a_{n+1})a'_1\cdot\ldots\cdot a'_{n+1}) \end{aligned}$$

and

$$\begin{aligned}
& \beta\psi_n(g,g')(a_1\otimes a'_1,\dots,a_{n+1}\otimes a'_{n+1}) = \chi(|g'|,\sum_{i=1}^{n+1}|a_i|)s_0a_1a_2\cdot\ldots a_{n+1}\cdot \\
\cdot\ & \chi(|g'|,|a'_1|)a'_1g'(a'_2,\dots,a'_{n+1}) \\
& +s_0\chi(|g|,|a_1|)a_1g(a_2,\dots,a_{n+1})a'_1a'_2\cdot\ldots\cdot a'_{n+1}+\chi(|g'|,\sum_{j=1}^{n+1}|a_j|)\sum_{i=1}^{n}s_ia_1\cdot\ldots\cdot a_{n+1} \\
& \cdot g'(a'_1,\dots,a'_ia'_{i+1},\dots,a_{n+1})+\sum_{i=1}^{n}s_ig(a_1,\dots,a_ia_{i+1},\dots,a_{n+1})a'_1\cdot\ldots\cdot a'_{n+1} \\
& +\chi(|g'|,\sum_{i=1}^{n+1}|a_i|)s_{n+1}a_1\cdot\ldots\cdot a_{n+1}g'(a'_1,\dots,a'_n)a'_{n+1} \\
+\ & s_{n+1}g(a_1,\dots,a_n)a_{n+1}a'_1\cdot\ldots\cdot a'_{n+1}
\end{aligned}$$

where

$$\begin{aligned}
s_0 &:= F(\sigma_n,a_2\otimes\ldots\otimes a_{n+1}\otimes a'_2\otimes\ldots\otimes a'_{n+1})\prod_{i=2}^{n+1}\chi(|a'_1|,|a_i|) \\
s_i &:= (-1)^i\chi(|a'_i|,|a_{i+1}|)F(\sigma_n,a_1\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_{n+1}\otimes a'_1\otimes\ldots\otimes a'_ia'_{i+1}\otimes\ldots\otimes a'_{n+1}) \\
&\quad (1\le i\le n) \\
s_{n+1} &:= (-1)^{n+1}F(\sigma_n,a_1\otimes\ldots\otimes a_n\otimes a'_1\otimes\ldots\otimes a'_{n+1})\prod_{i=1}^{n}\chi(|a'_i|,|a_{n+1}|).
\end{aligned}$$

Comparing signs of corresponding terms, we see that what is left to prove are basically the following three identities:

1. $F(\sigma_n,a_2\otimes\ldots\otimes a_{n+1}\otimes a'_2\otimes\ldots\otimes a'_{n+1})\prod_{i=2}^{n+1}\chi(|a'_1|,|a_i|)=F(\sigma_{n+1},aa')$
   This is clear from the definitions.

2. $\chi(|a'_i|,|a_{i+1}|)F(\sigma_n,a_1\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_{n+1}\otimes a'_1\otimes\ldots\otimes a'_ia'_{i+1}\otimes\ldots\otimes a'_{n+1})=F(\sigma_{n+1},aa')$
   It is straightforward to verify that

$$\begin{aligned}
& F(\sigma_n,a_1\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_{n+1}\otimes a'_1\otimes\ldots\otimes a'_ia'_{i+1}\otimes\ldots\otimes a'_{n+1})= \\
& \prod_{j<l}\chi(|a'_j|,|a_l|)\cdot\chi(|a_{i+1}|,|a'_i|)=F(\sigma_{n+1},aa')\chi(|a_{i+1}|,|a'_i|)
\end{aligned}$$

   as desired.

3. $F(\sigma_n,a_1\otimes\ldots\otimes a_n\otimes a'_1\otimes\ldots\otimes a'_n)\prod_{i=1}^{n}\chi(|a'_i|,|a_{n+1}|)=F(\sigma_{n+1},aa')$
   This is an easy calculation similar to the first of these identities.

We have now proven that our $\psi$ is indeed a chain map.
*Step 2:* The next thing we will show is that our chain map induces an isomorphism in cohomology at least in dimension one. To this end, first note that $B^1_{ha}(A,M) = 0$ for any $A$ and $M$. To see this, assume $f \in B^1_{ha}(A,M)$ homogeneous. Then by definition, we see $f$ of the form $f(a) = \chi(|m|,|a|)am - ma = 0$ with $m \in M$ some homogeneous element (note that $M$ is $(G,\chi)$-symmetric). To see that $\psi$ induces an injective map in cohomology, let $g \in Z^1_{ha}(A,M), g' \in Z^1_{ha}(A',M)$. Assume $\psi_1(g,g') = 0$. We then have to prove $g = 0, g' = 0$, and to show this we can without loss of generality assume $g, g'$ homogeneous since $\psi_n$ is a map of degree zero, which means that the kernel of $\psi_n$ decomposes along the lines of the grading. Also, as usual it is enough to test $\psi_1(g,g') = 0$ on homogeneous elements. Thus taking arbitrary homogeneous elements $a \in A, a' \in A'$ we have

$$\psi_1(g,g')(a \otimes a') = \chi(|g'|,|a|)ag'(a') + g(a)a' = 0 \tag{4.3}$$

Now $g \in Z^1_{ha}(A,M)$ implies that $\beta g(x,y) = \chi(|g|,|x|)g(y) - g(xy) + g(x)y = 0$ for any homogeneous $x, y \in A$ and in particular $g(1) = 0, g'(1') = 0$. In (Eq. 4.3), this means that for all homogeneous $a \in A$ and all $a' \in A'$ we find $g(a) = 0$ and $g'(a') = 0$ respectively, as desired.
We still have to show surjectivity of the map induced by $\psi_1$ on cohomology. To this end, take any homogeneous $h \in Z^1_{ha}(A \otimes A', M)$. Set now $g := h(\cdot, 1'), g' := h(1, \cdot)$. Then clearly $g$ and $g'$ are $k$-linear functions of degree $|h|$ from $A$ and $A'$ to $M$ respectively and we see

$$\beta g(x,y) = \chi(|h|,|x|)xg(y) - g(xy) + g(x)y = \beta h(x \otimes 1', y \otimes 1') = 0$$

for any homogeneous $x, y \in A$, so $g$ is a cocycle. Virtually the same calculation shows the same for $g'$. However, we also see that

$$\begin{aligned} \beta h(a \otimes 1', 1 \otimes a) &= \chi(|h|,|a|)ah(1 \otimes a') - h(a \otimes a') + h(a \otimes 1)a' = 0 \\ \Rightarrow h(a \otimes a') &= \chi(|h|,|a|)ah(1 \otimes a') + h(a \otimes 1')a' \end{aligned}$$

for all homogeneous $a \in A, a' \in A'$. Using the latter relation, one sees that

$$\psi(g,g')(a \otimes a') = \chi(|h|,|a|)ah(1 \otimes a') + h(a \otimes 1')a' = h(a \otimes a')$$

again for any homogeneous $a \in A, a' \in A'$, and so by extension $\psi(g,g') = h$, so $h \in Im(\psi)$.
*Step 3:* Next on the list is injectivity of $\psi_2$ on the level of cohomology. We first remark that for $g \in Z^2_{ha}(A,M)$ an arbitrary homogeneous element, setting

$$f : A \otimes A \to M, f(x \otimes y) := -\chi(|g|,|x|)xg(1 \otimes y) + g(1 \otimes xy) - g(1 \otimes x)y + g(x \otimes y)$$

for homogeneous elements $x, y \in A$ and extending linearly to nonhomogeneous elements of $A \otimes A$ yields a $k$-linear function $A \otimes A \to M$ with $f - g \in B^2_{ha}(A,M)$ and $f(1 \otimes y) = f(x \otimes 1) = 0$ for all $x, y \in A$. Using the decomposition of an arbitrary $g \in Z^2_{ha}(A,M)$, the same construction can of course be carried out without assuming homogeneity of $g$.
Note in this context that $B^2_{ha}(A,M) = B^2(A,M)$ as dimension one Harrison cochains are just Hochschild cochains, so $f - g$ is indeed automatically a Harrison coboundary.

Hence, if necessary we can replace any element of $Z^2_{ha}(A,M)$ or $Z^2_{ha}(A',M)$ by another which is equivalent in cohomology but vanishing on units.
Also, note that with $g \in Z_{ha}(A,M)$, $g' \in Z_{ha}(A',M)$ and $\psi_2(g,g')$ a coboundary, we can assume without loss of generality that $g$ and $g'$ were homogeneous and of the same degree. This is again due to the fact that $\psi_2$ is a degree zero map, meaning that the kernel of $\psi_2$, or more precisely of $\pi\circ\psi_2$ where $\pi$ is projection to cohomology, decomposes into a sum of homogeneous submodules. Suppose now that $g \in Z^2_{ha}(A,M), g' \in Z^2_{ha}(A',M)$ are homogeneous of identical degree and that $h := \psi_2(g,g') \in B^2_{ha}(A\otimes A',M)$. Suppose further that $\tilde{h} \in C^1_{ha}(A\otimes A',M)$ such that $\beta(\tilde{h}) = h$. Since the Hochschild differential $\beta$ is homogeneous of degree zero, and as $h$ is homogeneous of the same degree as $g$, $g'$ and $(g,g')$, we can assume also $\tilde{h}$ homogeneous with $|\tilde{h}| = |g|$. With $x,y \in A$ and $x',y' \in A'$ homogeneous we see

$$\begin{aligned}&\psi(g,g')(x\otimes x',y\otimes y') = \chi(|x'|,|y|)(\chi(|g'|,|x|+|y|)xyg'(x'\otimes y') + g(x\otimes y)x'y') = \\ &\chi(|\tilde{h}|,|x|+|x'|)xx'\tilde{h}(y\otimes y') - \chi(|x'|,|y|)\tilde{h}(xy\otimes x'y') + \tilde{h}(x\otimes x')yy'.\end{aligned}$$

Note that the term $\chi(|x'|,|y|)$ in the middle summand of the last line comes from the definition of multiplication on $A\otimes A'$. Now since $\psi$ is a chain map, we know that changing either $g$ or $g'$ by addition of a coboundary will change also $h$ only by a coboundary. We can for this reason assume without loss of generality that $g(\cdot\otimes 1) = 0, g'(\cdot\otimes 1') = 0$. Setting now $f := \tilde{h}(\cdot\otimes 1'), f' := \tilde{h}(1\otimes\cdot)$ we obtain

$$\psi(g,g')(x\otimes 1',y\otimes 1') = g(x\otimes y) = \chi(|\tilde{h}|,|x|)xf(y) - f(xy) + f(x)y = \beta(f)(x\otimes y).$$

As $|f| = |\tilde{h}|$, this proves $g$ to be a coboundary. That $g'$ is also a coboundary follows in the same way, and hence we are done proving that $\psi_2$ is injective.
*Step 4:* We will now show that $\psi_2$ is also surjective up to cohomology. Suppose that $h \in Z^2_{ha}(A\otimes A',M)$ is a homogeneous Harrison cocycle and assume $x,y \in A$, $x',y' \in A'$ homogeneous. Using Lemma 4.3.1, we see

$$\begin{aligned}&h(x\otimes x',y\otimes y') = h((x\otimes 1')(1\otimes x'),(y\otimes 1')(1\otimes y'))\\ =\ &\chi(|x'|,|y|)(\chi(|h|,|x|+|y|)xyh(1\otimes x',1\otimes y') + h(xy\otimes 1',1\otimes xy) + h(x\otimes 1',y\otimes 1')x'y')\\ -\ &h(x\otimes 1',1\otimes x')yy' - \chi(|h|,|x|+|x'|)xx'h(y\otimes 1',1\otimes y').\end{aligned}$$

Define then $g : A\otimes A \to M, g' : A'\otimes A' \to M$ by setting

$$g(x,y) := h(x\otimes 1',y\otimes 1'), g'(x',y') := h(1\otimes x',1\otimes y')$$

for homogeneous elements $x,y \in A$ and $x',y' \in A'$. This clearly gives well-defined $k$-linear maps of degree $|h|$ and of the desired type. What we have to show is that they are Harrison cocycles and that they satisfy the condition $\psi_2(g,g') = h$ up to cohomology. To see the latter, we calculate

$$\begin{aligned}(\psi_2(g,g')-h)(x\otimes x',y\otimes y') &= h(x\otimes 1',1\otimes x')yy' - \chi(|x'|,|y|)h(xy\otimes 1',1\otimes x'y')\\ &\quad +\chi(|h|,|x|+|x'|)xx'h(y\otimes 1,1\otimes y')\\ &= \beta(\tilde{r})(x\otimes x',y\otimes y')\end{aligned}$$

with $\tilde{r} : A \otimes A' \to M$ defined through $\tilde{r}(x \otimes x') = h(x \otimes 1', 1 \otimes x')$ for $x \in A, x' \in A'$. We will be done if we can show that $g$ and $g'$ are themselves indeed Harrison cocycles. To prove the cocycle condition, we calculate

$$\beta(g)(x_1, x_2, x_3) = \beta(h)(x_1 \otimes 1', x_2 \otimes 1', x_3 \otimes 1') = 0$$

for any $x_1, x_2, x_3 \in A$. Vanishing of $g$ on shuffles follows immediately from the corresponding property of $h$. The corresponding assertions for $g'$ can be seen applying precisely the same methods. This concludes the proof of surjectivity of $\psi_2$.

As a first application, we will derive a characterization of $(G, \chi)$-commutative polynomial rings. As these might have infinitely many variables, we need first a generalization of the previous statement to infinite tensor products:

**Corollary 4.3.4** *Let $\mathcal{A} := \{A_i : i \in I\}$ be a family of $(G, \chi)$ commutative $k$-algebras indexed by a possibly infinite set $I$. Assume that $M$ is a $(G, \chi)$-symmetric bimodule for all $A_i$ and that for all $i, j \in I$ the module structures induced on $M$ by $A_i$ and $A_j$ are compatible in the sense of Prop. 4.3.3. Then, we have*

$$H^i_{ha}(\otimes_{i \in I} A_i, M) \cong \prod_{i \in I} H^i_{ha}(A_i, M)$$

*for $i \in \{1, 2\}$. Here, the infinite product has to be understood as an infinite product in the category of graded $k$-modules. An explicit representation of the direct product in this category was given in Lemma 1.3.12.*

**Proof** First we will show the statement for finite tensor products. In this case it follows immediately from Prop. 4.3.3 and the only thing one needs to check is that if $B_1, B_2, \ldots, B_n, B_{n+1} \in \mathcal{A}$ then the finite subproduct $\otimes_{i=1}^n B_i$ induces on $M$ a bimodule structure which is compatible in the sense of Prop. 4.3.3 with the one induced by $B_{n+1}$. This in turn reduces inductively to the case $n = 2$. Assume hence that $B_1, B_2, B_3$ are $G$-commutative $k$-algebras as indicated. Then for $b_i \in B_i$ homogeneous elements we merely need to verify that

$$(a_1 \otimes a_2)(a_3 m) = \chi(|a_1 \otimes a_2|, |a_3|) a_3((a_1 \otimes a_2) m)$$

which is immediate if one inserts the definitions of these operations. Thus the corollary only remains to be checked for infinite tensor products. To see this, recall first that infinite tensor products were in Example 1.3.6 defined as the direct limit of finite ones. Let $J_1 \subseteq J_2$ be finite subsets of $I$, then the map

$$f_{J_1 J_2} : \bigotimes_{i \in J_1} A_i \to \bigotimes_{i \in J_2} A_i$$

induces maps

$$g^n_{J_1 J_2} : C^n_{ha}(\bigotimes_{i \in J_2} A_i, M) \to C^n_{ha}(\bigotimes_{i \in J_1} A_i, M).$$

Recall in this context that as a consequence of Prop. 1.2.27, we can use any ordering of the indexing sets when defining $f_{J_1 J_2}$. It can be checked by straightforward direct calculation that these maps commute with the Hochschild differential and are degree zero and $k$-linear. Also, for $J_1 \subseteq J_2 \subseteq J_3$ finite subsets of $I$ the compatibility condition

$$g_{J_1 J_2} \circ g_{J_2 J_3} = g_{J_1 J_3}$$

follows easily from the corresponding compatibility condition satisfied by the $f_{J_1 J_2}$, namely

$$f_{J_2 J_3} \circ f_{J_1 J_2} = f_{J_1 J_3}.$$

Finally, the $g_{J_1 J_2}$ are compatible with the modified shuffle product because *any* degree zero $k$-linear map $\varphi : M \to N$ of $G$-graded $k$-modules induces a ring homomorphism $\tilde{\varphi} : Sh(M) \to Sh(N)$ between the corresponding shuffle product algebras by componentwise application. All of this taken together means that the Harrison cochain complexes $C_{ha}(\bigotimes_{i\in J} A_i, M)$, where $J$ runs through all finite subsets of $I$, together with the maps $g_{J_1 J_2}$ form an inverse system of cochain complexes and according to Corollary 1.3.14 the cohomology of the inverse limit of that system will be equal to the inverse limit of the cohomologies. As we have proven the statement already in the finite case, we know that for $J$ a finite subset of $I$

$$H^i_{ha}(\bigotimes_{j\in J} A_j, M) = \bigoplus_{j\in J} H^i_{ha}(A_j, M)$$

for $i \in \{1, 2\}$, and by inspection of the proof of Prop. 4.3.3, or more specifically of step four thereof, we see that the induced maps of the inverse system of cohomology modules are for any pair $J_1 \subseteq J_2$ just the projections

$$\pi_{J_2 J_1} : \bigoplus_{j\in J_2} H^i_{ha}(A_j, M) \to \bigoplus_{j\in J_1} H^i_{ha}(A_j, M).$$

The inverse limit of this system is

$$\prod_{j\in I} H^i_{ha}(A_j, M)$$

where the product is of course the product in the category of $G$-graded $k$-modules as explained in Lemma 1.3.12. This can be seen by the following argument: first, there is for every finite subset $J \subseteq I$ an obvious projection map

$$\pi : \prod_{j\in I} H^i_{ha}(A, M) \to \bigoplus_{j\in J} H^i_{ha}(A_j, M)$$

clearly compatible with the maps defining the inverse system. Second, if $N$ is another module together with projections

$$\rho_J : N \to \bigoplus_{j\in J} H^i_{ha}(A_j, M)$$

which are compatible with the inverse system, then in particular we have for every $j \in I$ a map $\rho_j : N \to H^i_{ha}(A_j, M)$. From this it is easy to see the existence and unicity of a map

$\rho : N \to \prod_{j\in I} H^i_{ha}(A_j, M)$ commuting with all the projections on $N$ and the product. One needs for this only verify that

$$\rho(v) := (\rho_j(v))_{j\in I}$$

is well-defined, i.e. that the righthand side is always a sum of a finite number of homogeneous elements of $\prod_{j\in I} H^i_{ha}(A_j, M)$. One verifies this by considering that the $\rho_j$ are all degree zero and that every $v \in N$ can be written as a finite sum of homogeneous elements of $N$.
What remains to be proven is that the limit complex of the inverse system $(C^i_{ha}(A_J, M), g_{J_1J_2})$ is the Harrison complex of the limit algebra of the direct system $(A_J, f_{J_1J_2})$.
To this end, set first

$$A := \lim_{\rightarrow} A_J = \bigotimes_{i\in I} A_i.$$

By definition of the direct limit, there is for every finite $J \subseteq I$ a map $f_J : A_J \to A$ compatible with the other maps of the direct system and $A$ is in some sense the smallest object such that all of these exist. We can use these maps to obtain maps $g_J : C^i_{ha}(A, M) \to C^i_{ha}(A_J, M)$ and by the same arguments that were used in the construction of the inverse system of Harrison cochain complexes, it turns out that these are compatible with the Harrison construction. Recall also that by Lemma 1.3.12, Lemma 1.3.13 a cochain of the limit complex is of the form $(\varphi_J)_{J\in\mathcal{P}_{fin}(I)}$, where the $\varphi_J$ are Harrison cochains over $A_J$ with coefficients in $M$ which are all compatible with each other in the sense of the inverse system, i.e. for any $J_1 \subseteq J_2$ finite we have $\varphi_{J_2} \circ f_{J_1J_2} = \varphi_{J_1}$. We can associate to such a cochain a cochain in $C^n_{ha}(A, M)$ in the following way: for $a = a_1 \otimes a_2 \otimes \ldots \otimes a_n \in A^{\otimes n}$ we find a finite $J \subseteq I$ such that all the $a_i$ can be identified with elements of $A_J$. Then we define $\varphi(a) := \varphi_J(a)$. We will have to show that this gives a chain map

$$\psi' : \lim_{\leftarrow} C_{ha}(A_J, M) \to C_{ha}(A, M)$$

and that this chain map is the inverse of the map

$$\psi : C_{ha}(A, M) \to \lim_{\leftarrow} C_{ha}(A_J, M)$$

which is given by setting for a $\varphi \in C^n_{ha}(A, M)$

$$\psi(\varphi) := (\varphi \circ f_J)_{J\in\mathcal{P}_{fin}}.$$

Well-definedness of $\psi'$ follows easily from the compatibility conditions on the $\varphi_J$. Linearity of $\varphi = \psi'((\varphi_J)_{J\in\mathcal{P}_{fin}(I)})$ follows from linearity of all the $\varphi_J$ together with the fact that for any pair of elements $a_1, a_2 \in A^{\otimes n}$ there is some finite $J \subseteq I$ such that $\varphi(a_1 + a_2) = \varphi_J(a_1 + a_2)$. Vanishing of $\varphi$ on arbitrary graded shuffles can be seen in the same way. Compatibility with the limit complex differential and the Hochschild differential can then be checked by remembering that the differential of the limit complex acts on a cochain

$$(\varphi_J)_{J\in\mathcal{P}_{fin}(I)}$$

component-wise, so that for any particular $a \in A^{\otimes n}$ and $(\varphi_J)_{J\in\mathcal{P}_{fin}(I)}$ we see

$$\beta(\psi'(\varphi_J)_{J\in\mathcal{P}_{fin}(I)})(a) = \beta(\varphi_{J'})(a) = \psi'(\beta((\varphi_J)_{J\in\mathcal{P}_{fin}}))(a)$$

for some $J' \subseteq I$ and so $\psi' \circ \beta = \beta \circ \psi'$ as desired. Hence, our $\psi'$ is indeed a chain map. $\psi \circ \psi' = id$ is then a consequence of $\psi \circ \psi'$ being a chain map of type

$$\lim_{\leftarrow} C_{ha}(A_J, M) \to \lim_{\leftarrow} C_{ha}(A_J, M)$$

and hence by virtue of the universal property of the inverse limit the identity. To see $\psi' \circ \psi = id_{C_{ha}(A,M)}$ as well, recall that $\psi$ was for $\varphi \in C_{ha}(A, M)$ given by

$$\psi(\varphi)_J := \varphi \circ f_J.$$

Inserting the definition of $\psi'$ we see for an arbitrary $a \in A^{\otimes n}$ and $J \subset I$ finite such that $a \in A_J$ (up to equivalence in $A$) that

$$\psi'(\psi(\varphi))(a) = \psi(\varphi)_J(a) = (\varphi \circ f_J)(a) = \varphi(a)$$

and so $\psi \circ \psi' = id$ as desired. This means that $C_{ha}(A, M)$ is as a cochain complex isomorphic to the inverse limit of the complexes $C_{ha}(A_J, M)$. The claim about the cohomology groups in low dimension follows easily.

Another tool that we will need is the following

**Lemma 4.3.5** *Let $A$ be a $(G, \chi)$-commutative algebra over a field $K$ and let $\mathfrak{a} \subseteq A$ be a graded ideal in $A$. Assume that $M$ is a $(G, \chi)$-symmetric $A$-module annihilated by $\mathfrak{a}$. Then we have an exact sequence*

$$0 \to H^1_{ha}(A/\mathfrak{a}, M) \to^{p_1} H^1_{ha}(A, M) \to^{q_1} \boldsymbol{Hom}_A(\mathfrak{a}, M) \to^{\delta_1} H^2_{ha}(A/\mathfrak{a}, M) \to^{p_2} H^2_{ha}(A, M).$$

**Proof** The proof works basically like the proof of the existence of a long exact sequence in cohomology coming from an exact sequence of chain complexes, except that this exact sequence is not strictly speaking derived from an exact sequence of chain complexes, and the construction cannot be carried out in arbitrary dimension. Also, the graded situation needs more careful arguing in some places than the ordinary commutative case of the present lemma, mainly because we allowed cochains to be of nonzero degree or even nonhomogeneous. Finally, (Harrison [31]) does not explicitly supply a proof in the commutative case, so it makes sense in any case to supply one here.
Define first $p : A \to A/\mathfrak{a}$ in the natural way and $\tilde{p}_1 : C^1_{ha}(A/\mathfrak{a}, M) \to C^1_{ha}(A, M)$ by $\tilde{p}_1(\varphi) := \varphi \circ p$. Due to $(G, \chi)$-symmetry of $M$ there are no 1-Hochschild coboundaries, we can show compatibility with taking cohomology simply by showing that this maps cocycles to cocycles. It suffices to prove this, in turn, for homogeneous cocycles. Let therefore $\varphi \in Z^1_{ha}(A/\mathfrak{a}, M)$ be a homogeneous cocycle. Then with $x, y \in A$ homogeneous elements and using the notation $\overline{a} := p(a)$ for any $a \in A$ we see

$$\begin{aligned} & \beta(\tilde{p}_1(\varphi))(x \otimes y) = \chi(|\varphi|, |x|)x\tilde{p}_1(\varphi)(y) - \tilde{p}_1(\varphi)(xy) + \tilde{p}_1(\varphi)(x)y \\ = \; & \chi(|\varphi|, |x|)x\varphi(\overline{y}) - \varphi(\overline{xy}) + \varphi(\overline{x})y = \overline{x}\varphi(\overline{y}) - \varphi(\overline{xy}) + \varphi(\overline{x})\overline{y} \\ = \; & \beta(\varphi)(\overline{x} \otimes \overline{y}) = 0 \end{aligned}$$

where $\mathfrak{a}M = 0$ is used in the second line. As usual, vanishing of $\beta(\tilde{p_1}(\varphi))$ on arbitrary homogeneous generators implies vanishing on arbitrary elements.
After having thereby verified that $\tilde{p_1}$ induces a map in cohomology, define $p_1$ to be the induced map $p_1 : H^1_{ha}(A/\mathfrak{a}, M) \to H^1_{ha}(A, M)$. It is clear that $p_1$ is injective. Next, we notice that for $\varphi \in C^1_{ha}(A, M)$ a homogeneous Hochschild cocycle and homogeneous $r \in A$ and $a \in A$ we have

$$\varphi(ra) = \chi(|\varphi|, |r|)r\varphi(a)$$

and

$$\varphi(ar) = \varphi(a)r$$

because of $\mathfrak{a}M = M\mathfrak{a} = 0$. This means that the map given by restriction to $\mathfrak{a}$,

$$\tilde{q_1} : Z^1_{ha}(A, M) \to \mathbf{Hom}_A(\mathfrak{a}, M), \tilde{q_1}(\varphi) := \varphi|_{\mathfrak{a}}$$

is well-defined. It has as kernel the cocycles vanishing on $\mathfrak{a}$, which are precisely those in $Im(p_1)$. Note that here we need to use $K$ a field.
Due to absence of coboundaries at this level, $Z^1_{ha}(A, M) \cong H^1_{ha}(A, M)$, and the corresponding map $H^1_{ha}(A, M)$ we denote by $q_1$.
Next, we have to define $\delta_1$. This will use the same machinery as the construction of the connecting homomorphism in the construction of a long exact cohomology sequence coming from a short exact sequence of chain complexes . This is to say that for $\varphi \in \mathbf{Hom}_A(\mathfrak{a}, M)$ a homogeneous map, we set

$$\delta_1(\varphi) := \tilde{p_2}^{-1}(\beta(\tilde{q_1}^{-1}(\varphi)).$$

Some remarks on how to understand this definition are in order. $\tilde{q_1}^{-1}$ is meant to associate to $\varphi$ an arbitrarily chosen homogeneous $K$-linear extension to all of $A$. Hence this extension is a Harrison cochain but not necessarily a cocycle. Also, $\tilde{p_2} : C^2_{ha}(A/\mathfrak{a}, M) \to C^2_{ha}(A, M)$ and $p_2 : H^2_{ha}(A/\mathfrak{a}, M) \to H^2_{ha}(A, M)$ here are the maps induced by $p : A \to A/\mathfrak{a}$ or more precisely $\tilde{p} : A \otimes A \to A/\mathfrak{a} \otimes A/\mathfrak{a}$. One has to check here that $p_2$ is well-defined, which means that $\tilde{p_2}$ should map cocycles to cocycles, coboundaries to coboundaries, and preserve $(G, \chi)$-symmetry of Harrison cochains. All of these verifications are easy.
We will now prove that also $\delta_1$ is well-defined. First, this means showing independence of all the choices inherent in the definition of $\delta_1$, i.e. that the definition indeed associates to any $\varphi \in \mathbf{Hom}_A(\mathfrak{a}, M)$ a cochain unique up to equivalence in $C^2_{ha}(A/\mathfrak{a}, M)$. Second, one must also show that the cochain so produced is always a cocycle.
To see the former, assume $\varphi$ a homogeneous element of $\mathbf{Hom}_A(\mathfrak{a}, M)$, then $\varphi_1, \psi_1 \in \mathbf{Hom}_K(A, M)$ homogeneous maps which restrict to $\varphi$, $\varphi_2 := \beta(\varphi_1), \psi_2 := \beta(\psi_1)$, and $\varphi_3, \psi_3 \in \mathbf{Hom}_K(A/\mathfrak{a} \otimes A/\mathfrak{a}, M)$ homogeneous maps satisfying

$$\varphi_3 \circ \tilde{p_2} = \varphi_2, \psi_3 \circ \tilde{p_2} = \psi_2.$$

For a start, one should quickly check that, starting with some homogeneous $\varphi \in \mathbf{Hom}_A(\mathfrak{a}, M)$, it is in fact possible to make all the choices with homogeneous results. To construct from $\varphi$ a suitable $\varphi_1$, one just chooses a homogeneous $K$-basis of $\mathfrak{a}$ and extends to a homogeneous $K$-basis of $A$. Then one sets $\varphi_1(b) = 0$ for all basis elements outside $\mathfrak{a}$. To construct $\varphi_3$ from

$\varphi_2$, one similarly uses the fact that $U \oplus \mathfrak{a} = A$ for some graded $K$-subspace $U \subseteq A$. With this summand, $A/\mathfrak{a} \cong U$ as graded $K$-vectorspaces, and with this identification a suitable $\varphi_3$ can be constructed by restriction of $\varphi_2$ to $U$. To prove that this construction works as desired, one then needs to show that $\varphi_2$ vanishes on $A \otimes \mathfrak{a}$ and $\mathfrak{a} \otimes A$. As $\varphi_2$ is automatically $(G, \chi)$-symmetric, it is sufficient to verify that for $a \in \mathfrak{a}$ and $a \in A$ homogeneous elements we have $\varphi_2(a \otimes a_2) = 0$. To this end, we calculate

$$\varphi_2(a \otimes a_2) = \chi(|\varphi|, |a|)a\varphi_1(a_2) - \varphi_1(aa_2) + \varphi_1(a)a_2 = -\varphi(aa_2) + \varphi(a)a_2 = 0,$$

where the second-to-last transformation is justified by $\mathfrak{a}M = 0$ whereas the last one uses $\varphi \in \mathbf{Hom}_A(\mathfrak{a}, M)$.
We still need to see independence up to addition of a coboundary of $\varphi_3$ of the choice of $\varphi_1$, which is the same as $\varphi_3 - \psi_3 \in B^2_{ha}(A/\mathfrak{a}, M)$. Additionally, we still need to see $\varphi_3 \in Z^2_{ha}(A/\mathfrak{a}, M)$. It is equivalent to the former statement that applied to the case $\varphi = 0$, the construction presented yields a coboundary; we will therefore now assume $\varphi = 0$. We define then

$$\tilde{\varphi_1} \in Hom_K(A/\mathfrak{a}, M), \tilde{\varphi_1}(\overline{a}) := \varphi_1(a)$$

which is a well-formed definition as by assumption $\varphi_1$ vanishes on $\mathfrak{a}$. It then follows that

$$\begin{aligned}\beta(\tilde{\varphi_1})(\overline{a_1} \otimes \overline{a_2}) &= \chi(|\tilde{\varphi_1}|, |\overline{a_1}|)\overline{a_1}\tilde{\varphi_1}(\overline{a_2}) - \tilde{\varphi_1}(\overline{a_1 a_2}) + \tilde{\varphi_1}(\overline{a_1})\overline{a_2} \\ &= \chi(|\varphi_1|, |a_1|)a_1\varphi_1(a_2) - \varphi_1(a_1a_2) + \varphi_1(a_1)a_2 = \varphi_3(\overline{a_1} \otimes \overline{a_2})\end{aligned}$$

for all homogeneous $a_1, a_2 \in A$, and so $\varphi_3$ a coboundary as desired.
It is clear that our construction returns a $(G, \chi)$-symmetric cochain, so to show that $\varphi_3$ is a Harrison cocycle one only needs to check that $\beta(\varphi_3) = 0$. It suffices to check this on generators, where we see

$$\begin{aligned}\beta(\delta_1\varphi) &= \chi(|\varphi_3|, |\overline{a_1}|)\overline{a_1}\varphi_3(\overline{a_2} \otimes \overline{a_3}) - \varphi_3(\overline{a_1 a_2} \otimes \overline{a_3}) + \varphi_3(\overline{a_1} \otimes \overline{a_2 a_3}) - \varphi_3(\overline{a_1} \otimes \overline{a_2})\overline{a_3} \\ &= \beta\varphi_2(a_1 \otimes a_2 \otimes a_3) = \beta^2\varphi_1(a_1 \otimes a_2 \otimes a_3) = 0.\end{aligned}$$

For exactness we now have to show $Ke(\delta_1) = Im(q_1)$ and $Ke(p_2) = Im(\delta_1)$.
*$Ke(\delta_1) = Im(q_1)$:* we have $\delta_1 q_1 = 0$, because for $\varphi \in \mathbf{Hom}_A(\mathfrak{a}, M)$ a quasihomomorphism which is the restriction of a Harrison cocycle $\tilde{\varphi} \in Hom_K(A, M)$ we have by definition $\beta\tilde{\varphi} = 0$ and so $\delta_1(\varphi) = 0$. On the other hand, we have also $Ke(\delta_1) \subseteq Im(q_1)$ because $\delta_1(\varphi) = 0$ implies with notations as in the proof of the well-definedness of $\delta_1$ that $\varphi_3$ is a coboundary, say $\varphi_3 = \beta(\tilde{\varphi_3})$. Since $\beta \circ \tilde{p_1} = \tilde{p_2} \circ \beta$ and because by definition, we have $\beta(\varphi_1) = \tilde{p_2}(\varphi_3)$ as well as $\beta(\tilde{\varphi_3}) = \varphi_3$, we realize that $\rho := \varphi_1 - \tilde{p_1}(\tilde{\varphi_3})$ is a cocycle, i.e. $\rho \in Z^1_{ha}(A, M)$. Since $\tilde{p_1}(\tilde{\varphi_3})$ is zero restricted to $\mathfrak{a}$, this implies that $q_1(\rho) = \varphi$, meaning $\varphi \in Im(q_1)$.
*$Ke(p_2) = Bi(\delta_1)$:* Using again the same notations as in the proof of well-definedness of $\delta_1$, we see $\tilde{p_2}(\varphi_3) = \varphi_2 = \beta(\varphi_1)$, so $p_2 \circ \delta_1 = 0$. Likewise, by inspection of the construction of $\delta_1$ we see that if $\varphi \in Ke(p_2)$, then $\tilde{p_2}(\varphi) = \beta(\tilde{\varphi})$ for some suitable $\tilde{\varphi}$, and with $\psi = \tilde{q_1}(\tilde{\varphi})$ we see $\delta_1(\psi) = \varphi$.

**Lemma 4.3.6** *Let $k$ be a commutative ring and let $A := k[X^g]$ be the $(G,\chi)$-commutative polynomial algebra in one variable of degree $g \in G$. Assume $\varphi \in Z^2_{ha}(A,M)$ is homogeneous, where $M$ is a $(G,\chi)$-symmetric $A$-module and where we assume[1] $\varphi(1,x) = 0$ for all $x \in A$. Then we see for all $m,n \in \mathbb{N}$ that*

$$\varphi(X^m, X^n) = -\chi(|\psi|,|X|)^m X^m \psi(X^n) + \psi(X^{m+n}) - \psi(X^m)X^n \tag{4.4}$$

*where*

$$\psi(X^m) = \sum_{i=0}^{m-1} \chi(|\varphi|,|X|)^i X^i \varphi(X, X^{m-1-i})$$

*In other words, any $\varphi \in Z^2_{ha}(A,M)$ is in this situation automatically a coboundary, so $H^2_{ha}(A,M) = 0$.*

**Proof** In the classical situation the definition of $\psi$ is unproblematic since the polynomial algebra $k[X]$ of a commutative ring $k$ is a free $k$-module, but in our case one needs to check that $\psi$ is well-defined since there may be relations among the powers of $X$. The rest of the proof works as in the commutative case, except that one has to be a bit more careful insofar as one cannot as easily commute variables.
So, we first check well-definedness of $\psi$. As the $X^i$ form a system of $k$-module generators of $k[X]$, this is the same as verifying that for

$$\sum \mu_i X^i = 0$$

a relation in $A$ we have $\sum \mu_i \psi_i = 0$, where

$$\psi_n = \sum_{i=0}^{n-1} \chi(|\varphi|,|X|)^i X^i \varphi(X, X^{n-1-i})$$

as in the lemma. We will show first that these relations are effectively the same as in a $(G,\chi)$-graded power series ring over $k$ of degree $g$. One sees without difficulty that the relations forced on $A$ by the $(G,\chi)$-commutative structure are $k$-linearly generated by

$$(1-\lambda^p)X^n X^m = 0$$

where $p = n'm'$ with $1 \leq n' \leq n$, $1 \leq m' \leq m$ and where $\lambda = \chi(g,g)$. From the definition of a bicharacter it follows then that $\lambda^2 = 1$, which means that the above family of relations reduces to those of the type

$$(1-\lambda)X^n = 0, n \geq 2.$$

It suffices to show, therefore, that under the definitions given, $(1-\lambda)\psi(X^n) = 0$ for any $n \geq 2$. We calculate that for $n \geq 2$

$$(1-\lambda)\sum_{i=0}^{n-1} \chi(|\varphi|,|X|)^i X^i \varphi(X, X^{n-1-i}) = (1-\lambda)\varphi(X, X^{n-1}) + (1-\lambda)\chi(|\varphi|,|X|)X\varphi(X, X^{n-2}).$$

[1]recall that we can do so without loss of generality when the final target is to compute cohomology, since we have already shown that every 2-cocycle is equivalent to one such cocycle

The first term here is zero because $\varphi$ is bilinear and $(1-\lambda)X^2 = 0$ if $n \geq 3$ and because $(1-\lambda)\varphi(X,X) = 0$ due to $(G,\chi)$-symmetry of $\varphi$ if $n = 2$. The second term is zero for $n \geq 4$ since $(1-\lambda)X^2 = 0$, for $n = 3$ because $(1-\lambda)\varphi(X,X) = 0$ and for $n = 2$ because $\varphi(X,1) = 0$. Also, we see without difficulty that $|\psi| = |\varphi|$, i.e. the degrees of these two functions are interchangeable in sign calculations.
The assertion now follows through induction on $m$. The case $m = 0$ yields zero on both sides of (Eq. 4.4).
The case $m = 1$ may be treated by calculating

$$\begin{aligned}
& -\chi(|\varphi|,|X|)X\psi(X^n) + \psi(X^{n+1}) - \psi(X)X^n \\
= \ & -\chi(|\varphi|,|X|)X\sum_{i=0}^{n-1}\chi(|\varphi|,|X|)^i X^i\varphi(X,X^{n-1-i}) + \sum_{i=0}^{n}\chi(|\varphi|,|X|)^i\varphi(X,X^{n-i}) - \varphi(X,1)X^n \\
= \ & -\sum_{i=0}^{n-1}\chi(|\varphi|,|X|)^{i+1}X^{i+1}\varphi(|X|,|X|^{n-1-i}) + \sum_{i=0}^{n}\chi(|\varphi|,|X|)^i X^i\varphi(X,X^{n-i}) \\
= \ & -\sum_{i=1}^{n}\chi(|\varphi|,|X|)^i X^i\varphi(X,X^{n-i}) + \sum_{i=0}^{n}\chi(|\varphi|,|X|)^i X^i\varphi(X,X^{n-i}) = \varphi(X,X^n).
\end{aligned}$$

For the induction step, calculate

$$\begin{aligned}
& \varphi(X^{m+1},X^n) = \chi(|\varphi|,|X|)X\varphi(X^m,X^n) + \varphi(X,X^{m+n}) - \varphi(X,X^m)X^n \\
= \ & \chi(|\varphi|,|X|)X(-\chi(|\psi|,|X|)^m X^m\psi(X^n) + \psi(X^{m+n} - \psi(X^m)X^n) \\
+ \ & (-\chi(|\psi|,|X|)X\psi(X^{m+n}) + \psi(X^{m+n+1}) - \psi(X)X^{m+n}) \\
- \ & (-\chi(|\psi|,|X|)X\psi(X^m) + \psi(X^{m+1}) - \psi(X)X^m)X^n \\
= \ & -\chi(|\psi|,|X|)^{m+1}X^{m+1}\psi(X^n) + \psi(X^{m+n+1}) - \psi(X^{m+1})X^n
\end{aligned}$$

as desired.

In order to be be able to state the next result, the main proposition of this section, in a concise way, we set the following:

**Definition 4.3.7** *Suppose that $G$ and $H$ are abelian groups and that $A$ is a $k$-module which is graded over both $G$ and $H$. Then the gradings over $G$ and $H$ are called* compatible *if a grading of $M$ over $G \times H$ is induced by setting*

$$A_{(g,h)} := A_{(g,\bullet)} \cap A_{(\bullet,h)}$$

*for any $g \in G$ and $h \in H$, where $A_{(g,\bullet)}$ denotes the component of $A$ of degree $g \in G$ under the $G$-grading and $A_{(\bullet,h)}$ denotes the component of degree $h \in H$ under the $H$-grading. When it is clear which of the two groups an indexing element $g$ or $h$ belongs to, we will also write $A_g$ instead of $A_{(g,\bullet)}$ and $A_h$ instead of $A_{(\bullet,h)}$ respectively.*

Now, with all the necessary machinery in place, the proof of the following proposition is exactly as in the classical commutative case. We explain it only to keep the thesis self-contained:

**Proposition 4.3.8** *If $A$ is a $(G,\chi)$-commutative polynomial ring over a commutative ring $k$, then*

$$H^2_{ha}(A,M)=0$$

*for any $A$-module $M$.*
*Suppose on the other hand that $K$ is a field and $A$ a $(G,\chi)$-graded algebra carrying in addition a nonnegative $\mathbb{Z}$-grading compatible to the $G$-grading. If $A_0=K$ and $A_n=(A_1)^n$ for all $n\geq 1$ and $H^2_{ha}(A,K')=0$, then $A$ is isomorphic to a $(G,\chi)$-commutative polynomial ring over $K$.*

**Proof** Using Example 1.3.7, any $(G,\chi)$-commutative polynomial ring over a commutative ring $k$ can be written as an infinite tensor product of univariate polynomial rings. By Corollary 4.3.4, color Harrison cohomology of this infinite tensor product is the infinite graded direct product of the Harrison cohomologies of each factor. By Lemma 4.3.6 each of those has zero second Harrison cohomology, so we get the first part of the proposition.
On the other hand, suppose that $K$ is a field and that $A$ is a $(G,\chi)$-commutative $K$-algebra, endowed in addition with a nonnegative compatible $\mathbb{Z}$-grading, such that $A_n=(A_1)^n$ for any $n\in\mathbb{N}$ and $A_0=:K$. Choose now a homogeneous $K$-basis $X$ of $A_1$ and consider the polynomial ring $K[X]$. Write $X=\{x_i : i\in I\}$ with $I$ a suitable indexing set. The formal variable coming from $x_i\in X$ will be denoted by $X_i$. We can now construct a graded ring epimorphism $\varphi : K[X]\to A$ by multiplicative extension of $\varphi(X_i):=x_i$ for any $X_i\in X$, because by the construction of $K[X]$, relations among monomials in $K[X]$ are mirrored by relations among the corresponding products of basis elements in $A_1$. Let $\mathfrak{b}$ be the kernel of that epimorphism, then we see that $f|_{\mathfrak{b}}=0$ for any $f\in Z^2_{ha}(A,K)$. This is due to $\mathfrak{b}\subseteq\mathfrak{a}^2$ and $f|_{\mathfrak{a}^2}=0$ by direct calculation. Using Lemma 4.3.5, we have an exact sequence

$$0\to H^1_{ha}(A,K)\to H^1_{ha}(K[X],K)\to \mathbf{Hom}_{K[X]}(\mathfrak{b},K)\to 0.$$

Since $f|_{\mathfrak{b}}=0$, the map $H^1_{ha}(K[X],K)\to\mathbf{Hom}_{K[X]}(\mathfrak{b},K)$ therein is zero. As the sequence is exact, this means that $\mathbf{Hom}_{K[X]}(\mathfrak{b},K)=0$. Now for any $f\in\mathfrak{b}$ which in the sense of the $G$-grading on $K[X]$ is homogeneous and for any $\varphi\in\mathbf{Hom}_{K[X]}(\mathfrak{b},K)$ and any $X_i\in X$ we have

$$\varphi(X_if)=\chi(|\varphi|,|X_i|)X_i\varphi(f)=0$$

because of the definition of $\mathbf{Hom}_{K[X]}(\mathfrak{b},K)$ and because $\mathfrak{a}K=0$. This means that any such $\varphi$ is zero on $\mathfrak{ab}$, so we have a degree zero bijective $K$-linear map $\mathbf{Hom}_{K[X]}(\mathfrak{b}/(\mathfrak{ab}),K)\to\mathbf{Hom}_{K[X]}(\mathfrak{b}/(\mathfrak{ab}),K)$. As a $K[X]$-quasihomomorphism is automatically a $K$-quasihomomorphism, this extends to an degree zero linear bijective map

$$\Psi:\mathbf{Hom}_{K[X]}(\mathfrak{b},K)\to\mathbf{Hom}_K(\mathfrak{b}/(\mathfrak{ab}),K).$$

However $\mathbf{Hom}_K(\mathfrak{b}/(\mathfrak{ab}),K)=0$ means $\mathfrak{b}=\mathfrak{ab}$, because otherwise we could take a nonzero homogeneous element in $\mathfrak{b}/(\mathfrak{ab})$, extend to a homogeneous $K$-basis of $\mathfrak{b}/(\mathfrak{ab})$, and define on that basis a nonzero quasihomomorphism to $K$.
Finally, from $\mathfrak{b}=\mathfrak{ab}$ follows $\mathfrak{b}=0$ because $\cap_{i=1}^{\infty}\mathfrak{a}^i=0$. So the epimorphism $K[X]\to A$ is injective, which concludes the proof.

## 4.4 Deformation theory of color-commutative algebras

In this section, as an application of the cohomology theory just developed, we will study formal deformation problems in the category of color-commutative algebras. The goal is to prove, in analogy to the classical commutative case, that infinitesimal *color-commutative* deformations of a color-commutative algebra $A$ are classified by second Harrison cohomology and that the obstructions to extension of such deformations to higher order correspond to third Harrison cochains.
Let hence $A$ be a $(G, \chi)$-commutative $k$-algebra. We will assume as in the chapter on deformation theory of $(G, \chi)$-graded associative algebras that 2 is invertible in $k$, that $g \in G$ is a fixed element, and that $\lambda := \chi(g, g)$. The problem we are interested in is then the following:

*Among the degree $g$ formal deformations of $A$, which ones remain color-commutative?*

Then from the treatment of $(G, \chi)$-graded deformation theory in chapter three, we know already that associative color deformations of $A$ are classified by $HH^2(A)$. Second coboundaries are the same in Hochschild and Harrison cohomology, which corresponds to the observation that equivalences of $(G, \chi)$-graded associative deformations preserve color-commutation. Also, it is clear that a second Hochschild cocycle $\varphi \in Z^2(A)$ corresponds to a color-commutative deformation exactly if it is $(G, \chi)$-symmetric. This yields

**Proposition 4.4.1** *The infinitesimal color-commutative deformations of $A$ are classified up to equivalence by $H^2_{ha}(A)$.*

We now turn to the question of obstructions to extension of deformations to higher order. Recall for this purpose with the notations of the relevant section of chapter three (Eq. 3.8):

$$e_\lambda \beta(\mu_{N+1})(a, b, c) = e_\lambda \sum_{\substack{i+j=N+1 \\ i \leq N, j \leq N}} (\mu_j(\mu_i(a, b), c) - \chi(|a|, |X|)^i \mu_j(a, \mu_i(b, c)))$$

for all homogeneous $a, b, c \in A$, where $e_\lambda := \frac{1+\lambda}{2}$, and our formal deformation was given by an infinite formal sum $\sum_{i \geq 0} X^i \mu_i$, with $\mu_i \in \mathbf{Hom}_k(A \otimes A, A)$. We already know from chapter three that the righthand side of this is a third Hochschild cocycle. The $\mu_i$ are $(G, \chi)$-commutative up to multiplication with $e_\lambda$, because our deformation is supposed to be a $(G, \chi)$-commutative deformation. Setting

$$\Psi(a, b, c) = e_\lambda \sum_{\substack{i+j=N+1 \\ i \leq N, j \leq N}} (\mu_j(\mu_i(a, b), c) - \chi(|a|, |X|)^i \mu_j(a, \mu_i(b, c)))$$

for homogeneous $a, b, c \in A$ and extending linearly, we have to prove then that $\Psi$ vanishes on shuffle-products in $A^{\otimes 3}$. This needs to be tested only on generators and relations and is therefore equivalent to the two equations

$$\Psi(a, b, c) - \chi(|a|, |b|)\Psi(b, a, c) + \chi(|a|, |b| + |c|)\Psi(b, c, a) = 0 \qquad (4.5)$$

and

$$\Psi(a,b,c) - \chi(|b|,|c|)\Psi(a,c,b) + \chi(|a|+|b|,|c|)\Psi(c,a,b) = 0 \qquad (4.6)$$

for all homogeneous $a, b, c \in A$. We verify first (Eq. 4.5). We calculate

$$\begin{aligned} & \Psi(a,b,c) - \chi(|a|,|b|)\Psi(b,a,c) + \chi(|a|,|b|+|c|)\Psi(b,c,a) \\ = \ & e_\lambda \sum_{i+j=N+1} (\mu_j(\mu_i(a,b),c) - \chi(|a|,|X|)^i \mu_j(a,\mu_i(b,c))) \\ - \ & e_\lambda \chi(|a|,|b|) \sum_{i+j=N+1} (\mu_j(\mu_i(b,a),c) - \chi(|a|,|X|)^i \mu_j(b,\mu_i(a,c))) \\ + \ & e_\lambda \chi(|a|,|b|+|c|) \sum_{i+j=N+1} (\mu_j(\mu_i(b,c),a) - \chi(|a|,|X|)^i \mu_j(b,\mu_i(c,a))). \end{aligned}$$

Here everything cancels out, because

$$\begin{aligned} & e_\lambda \mu_j(\mu_i(a,b),c) = e_\lambda \chi(|a|,|b|)\mu_j(\mu_i(b,a),c) \text{ and} \\ & \chi(|a|,|b|+|c|)e_\lambda \mu_j(\mu_i(b,c),a) = e_\lambda \chi(|a|,|b|+|c|)\chi(|\mu_i|+|b|+|c|,|a|)\mu_j(a,\mu_i(b,c)) \\ = \ & e_\lambda \chi(|X|,|a|)^{-i}\mu_j(a,\mu_i(b,c)) = e_\lambda \chi(|a|,|X|)\mu_j(a,\mu_i(b,c)), \text{ and finally} \\ & e_\lambda \chi(|a|,|b|)\mu_j(b,\mu_i(a,c)) = e_\lambda \chi(|a|,|b|+|c|)\mu_j(b,\mu_i(c,a)). \end{aligned}$$

(Eq. 4.6) is shown similarly. We therefore obtain by the same additional arguments as were used in chapter three in the determination of a homological description of obstructions to extension of associative deformations

**Proposition 4.4.2** *Obstructions to extension of a graded formal $(G,\chi)$-commutative deformation of a $(G,\chi)$-commutative k-algebra $A$ are controlled by $H^3_{ha}(e_\lambda A, e_\lambda A)$.*

**Final remarks**

We close the chapter with a final remark on color-commutative deformation of color-commutative algebras which is beyond the scope of the cohomological tools just described.
Suppose we are given an associative algebra $A$ without additional structure and suppose that $A$ can be endowed with a color-commutation rule. Then it is natural to interpret the question which color-commutative deformations it admits as including deformations of *all* admissible color-commutative structures on $A$.
In this case, one encounters two distinct subproblems. The first one is to determine all color-commutative structures on $A$. The second one is to determine graded formal color-commutative deformations in each case. The cohomological tools presented here help only with the second problem, but not with the first. We do not have a good method for carrying out that first step, but we study a simple example in Rem. 2.2.7 and a more complicated example in Appendix A.

# Chapter 5

# Open questions

In this chapter, we will outline some possible targets and questions work on which would in our opinion be interesting in the framework of this thesis.

### Color-commutative algebra

**1.**It would in our view be interesting to start a more systematic effort towards transplanting techniques and results about commutative algebras to the color-commutative context. An attractive intermediate target in this direction would be to obtain a color-commutative version of the theorem about the existence of coefficient fields for regular local commutative rings. The first reason is that this would require reconstruction of a number of fundamental objects and constructions from commutative algebra in the color-commutative context, which would probably yield interesting results regardless of the final outcome. Secondly, the existence of coefficient fields is one of the primary ingredients in the proof of characterization of regular local rings by classical Harrison cohomology in (Harrison [31]), so success would be very encouraging in terms of obtaining an analog also of this theorem.

**2.** All algebras appearing in this thesis were algebras over some commutative base ring. While this is reasonable insofar as the degree zero component of any color-commutative algebra is commutative, it would still be interesting to see to what extent we can remove the requirement of having a commutative base ring. The idea would be to replace it with e.g. a color-commutative base. There are two approaches to this. The less ambitious one would consist in allowing color-commutative base rings, all the while still demanding that the bicharacter prescribing any commutation rules should have as target the set of invertible elements of degree zero of $k$. The more ambitious approach would consist in removing also this commutation condition from the bicharacter. This would require admitting gradings over noncommutative groups, which was our original motivation when we carried out the investigations that led us to Rem. 1.2.16.

**3.** It would be desirable to have better methods for classifying compatible color-commutative structures given an ungraded associative $k$-algebra $A$. The subproblem of determining all

gradings which can be installed on a given $A$, even those induced by abelian groups, seems to be nontrivial even in very restricted special cases, see e.g. (Boboc, Dascalescu [10]), (Caenepeel, Dascalescu, Nastasescu [12]), (Chun, Lee [15]), (Dascalescu, Ion, Nastasescu, Rios Montes [19]) for various special cases of the problem of determining the group gradings of full matrix algebras over fields. Classification of bicharacters compatible with a given grading is an additional problem that one would need to solve to obtain a generally satisfying solution. Although the need for color-commutativity rules out many *a priori* possible gradings and thereby makes the combined task simpler than the sum of both parts of the problem, this is at least partially offset by the fact that the color-commutative structure reduces the usefulness of classification results on some special kinds of gradings: for instance, gradings over cyclic groups cannot in the case of $A$ an algebra over a field $K$ give rise to other color-commutative structures than commutative or supercommutative ones.
While for the reasons outlined above it seems unlikely that one can obtain a solution to this problem which is applicable to a wide class of algebras, it would in our opinion be a productive effort to try solving special cases of this, chosen such that there is some interest in the result. At the very least, one might obtain interesting examples of color-commutative algebras in this way.
We treat this problem for one interesting class of algebras in the appendix.
Of course, similar questions can be posed also in the generalized color-commutative context.

### Hochschild and Harrison cohomology

**1.** It would be useful to adapt other cohomology theories for commutative algebras, e.g. André-Quillen cohomology, to the color-commutative context. In particular, it might be interesting to see if results analogous to the commutative situation can be obtained in the characteristic zero case, where the different approaches that exist to commutative algebra cohomology coincide if the base ring $k$ contains $\mathbb{Q}$ and if furthermore the $k$-algebra $A$ is flat as $k$-module.

**2.** Also if $\mathbb{Q} \subseteq k$, there exists an approach to Harrison cohomology through a decomposition of Hochschild cohomology. This construction can be carried out using combinatorial tools quite similar to those used in the construction of Harrison cohomology, see e.g. (Gerstenhaber, Schack [28]), (Loday [38]). One could try to see if it generalizes to our version of Hochschild cohomology and if so, if it can be similarly used to recover generalized Harrison cohomology in characteristic zero.

**3.** Suppose that $K$ is a field and that $A$ is a finite-dimensional (as $K$-vectorspace) $K$-algebra. Assume further that $A$ has infinite global dimension, i.e. that there exists some finitely generated $A$-module without finite projective resolution. Happel asked in [30] if Hochschild dimension would in this case also necessarily be infinite, i.e. if there exist arbitrarily large $n \in \mathbb{N}$ such that $HH^n(A) \neq 0$. This question was affirmatively answered by (Avramov, Iyengar [3]) in the case of commutative algebras, but in the general associative case, (Buchweitz, Green, Madsen, Solberg [11]) produced a counterexample showing finite Hochschild cohomological dimension and infinite global dimension. Now, their counterexamples - the algebras $A_q = K <$

$X, Y > /(X^2, XY + qXY, Y^2)$ with $q \in K$ and $K$ a field - happen to be color-commutative if equipped with the right grading and bicharacter. It is natural to ask if this class of algebras still contains counterexamples to Happel's conjecture if instead of computing ordinary Hochschild cohomology, we decided to view these algebras as color-commutative algebras and applied the modified Hochschild cohomology explained in chapter two. Again, a full treatment of this problem decomposes into two steps, namely first a classification of color-commutative structures admitted by these algebras and second calculation of $(G, \chi)$-graded Hochschild cohomology in each of these cases. If at the end of this process nothing contradicting the conjecture of Happel are found among the examples of color-commutative algebras so obtained, an investigation into the possibility of generalizing the proof in the commutative case to the color-commutative situation would seem to be in order.
We execute the first step of this program in appendix A.

### Deformation theory

In the course of construction a deformation theory for $(G, \chi)$-graded associative algebras in chapter three, we found that there was, in this category, more than one reasonable choice of trivial deformation ring of a $(G, \chi)$-graded $k$-algebra $A$: we could choose the degree of the formal deformation parameter $X$ and we could choose between a deformation theory arising from taking as trivial deformation the power series ring $A[[X]]$ or the alternative power series ring $A[[\tilde{X}]]$. We believe it could make sense to transport this observation back into the classical setting of ungraded algebras and consider deformation theories arising from alternatives to taking as trivial deformation something else than the ordinary power series ring $A[[X]]$. Ideas which come to mind immediately would be to consider sub- or superalgebras of $A[[X]]$. Here, at least special cases of the idea of using subalgebras of the full power series ring have of course already been heavily researched, because this direction of generalization includes cases such as rings of (globally) *convergent* power series when $A$ is endowed with some topology. Purely algebraic alternatives to the ordinary power series ring however may not have received very much attention yet in the context of deformation theory. There are many possibilities one could try. If for instance one wants a larger trivial deformation ring, one possibility that comes to mind immediately would be to use the incidence algebra over $A$ of the natural numbers with their natural ordering, or some subalgebra thereof different from the power series ring in one variable over $A$. We would be very interested in seeing previous work in this direction.

### Other directions of generalization

In this thesis, we have extended and adapted the Hochschild and Harrison cohomological theories as well as some related results from deformation theory to the context of $(G, \chi)$-graded and $(G, \chi)$-commutative algebras respectively. This means in comparison to the classical situation, in the case of Harrison cohomology, that we relaxed the assumption of commutativity. One natural other direction in which to extend this theory is to investigate the consequences of weakening the associative condition. Recently, we have become interested in this context in a particular weakened version of associativity known as *hom-associativity*. In a hom-associative ring $R$, the ordinary condition of associativity is changed to $\alpha(x) * (y * z) = (x * y) * \alpha(z)$, where $\alpha : R \to R$ is an abelian group homomorphism, not necessarily in any way respecting the multiplicative structure. In the case of a hom-associative $k$-algebra $A$, $\alpha$ is supposed to be a morphism. The beginnings of a deformation theory and a deformation cohomology for hom-associative algebras were developed in (Makhlouf, Silvestrov [40]). In terms of deformation theory, the differences between unital and non-unital associative algebras are not very big: given a unital algebra, it is not much harder to find unital deformations than arbitrary ones. In joint work with Yael Fregier (Fregier, Gohr [22]), we have recently proven a number of results concerning the structure of unital hom-associative algebras which suggest that this will be quite different for hom-associative algebras, because unitality implies many identities involving arbitrary elements which do not hold for a nonunital hom-algebra. For instance, in unital hom-associative rings, $\alpha$ injective implies that the ring in question is already associative. Also, the map $\alpha$ can be shown to be of a special form in this case, being multiplication with $\alpha(1)$, which turns out to be in the center of the algebra and to generate a hom-algebra ideal consisting of elements which associate with all elements of the algebra. In the deformation framework of (Makhlouf, Silvestrov [40]), the associativity twisting map $\alpha$ is in the deformed algebra replaced by a formal series of the type

$$\tilde{\alpha} := \sum_{i=0}^{\infty} \alpha_i$$

with $\alpha_0 = \alpha$ in the usual manner. Because this deformation must in the case of unital deformations, unlike in the general case of an arbitrary hom-associative deformation, respect the restrictions imposed by general identities true on any unital hom-associative algebra, and because in particular it is in this case strongly linked to the multiplication by being defined by multiplication with $\tilde{\alpha}(1)$, we believe that developing a deformation and cohomology theory especially for unital hom-associative algebras could be both interesting from the point of view of advancing the understanding of both topics for hom-associative algebras in general and from the point of view of developing a theory that may have interesting properties in its own right. We also think that construction of such a theory may shed light also on *associative* deformation problems. To illustrate this point, we point out that for instance in the case of a unital hom-algebra with injective $\alpha$, the form of the deformed twisting map $\tilde{\alpha}$ above means that it will be injective, too, hence forcing associativity on $A$ if we are searching for unital deformations. However, the deformations so obtained would not be *arbitrary* associative deformations, because an associative algebra which is hom-associative is hom-associative only with respect to some but not all possible twisting maps. Indeed, it can be shown that the set of twisting maps is for a unital hom-associative algebra $A$ in one-one correspondence with the center of $A$. This means, in terms of deformation theory, that if we fix a twisting map on the deformed algebra, this is the same as keeping certain elements in the center of the algebra.

Finally, there exist different notions of deformation in the hom-algebra category and at least some of them can also be linked to unitality conditions. For instance, we show in (Fregier, Gohr [22]) that hom-associative algebras with bijective twisting map satisfying a condition we call weak unitality can always be viewed as deformations of associative algebras.

A deeper analysis of these issues would in our view be desirable.

# Appendix A

# Color-commutative structures on $A_q$

The goal of this appendix is to provide a case study in the classification of color-commutative structures compatible with a class of algebras which we find interesting. Let $K$ be a field and $q \in K$, then the algebra we are interested in is

$$A_q := K < X, Y > /(X^2, Y^2, XY + qYX).$$

These algebras have been studied by other researchers mainly for their ability to generate pathologies in homological algebra, as in (Bergh [8]), (Buchweitz, Green, Madsen, Solberg [11]). We are interested in them because of their role in (Buchweitz, Green, Madsen, Solberg [11]), where they are used as a counterexample to a conjecture of (Happel [30]). The conjecture relates global dimension and Hochschild dimension of a finite-dimensional algebra over a field, claiming that finite Hochschild dimension implies finite global dimension. As Happel's conjecture is true for commutative algebras and as the counterexample algebras support color-commutative structures, it becomes natural to ask whether his conjecture holds true for color-commutative algebras. A natural attempt to answer this question is to check the counterexample algebras supplied in the associative case whether they can supply a counterexample also in the colored setting. The first step in this direction is the classification of color-commutative structures on the algebras $A_q$. To do this is the purpose of this appendix.
It is clear that $A_q$ supports (for all $q \neq 0$) some color-commutative structures and it is in fact not too difficult to construct all of them. It is more difficult to show that the list so obtained is really exhaustive and the subsequent calculation of graded Hochschild cohomology.
Hence, our goal is to identify all gradings of $A_q$ such that arbitrary homogeneous elements $x, y \in A_q$ satisfy a commutation rule of the form $xy = \chi(|x|, |y|)yx$ for some bicharacter $\chi : G \times G \to K^*$ and to determine all bicharacters that might be compatible with $A_q$. In doing so, we have to distinguish between one generic case and three degenerate cases for the value of $q$.
When one talks about a classification of a certain kind of mathematical structure, one usually has to specify some kind of equivalence up to which the proposed classification classifies the objects in question. In our case, the notion of isomorphy of graded algebras is too narrow to serve our needs, as the notion of graded algebra isomorphism distinguishes between graded algebras which differ only by a relabeling of the graded components given by a group automorphism.

We do not want to make this distinction and therefore introduce the following definition:

**Definition A.0.3** *Let $G$ be an abelian group and let $A$ and $A'$ be associative algebras graded over $G$. Suppose that there exists a group automorphism $\varphi \in Aut(G)$ such that the graded algebra given by*

$$B := \bigoplus_{g \in G} A'_{\varphi(g)}$$

*is isomorphic as a graded algebra to $A$. Then we call $A$ and $A'$* equivalent *graded algebras.*

### A.0.1 Generic case

We will first treat the generic case $q \notin \{-1, 0, 1\}$. For this case, we prove the following claim:

**Proposition A.0.4** *Let $G$ be an abelian group and $A_q = \oplus_{g \in G}(A_q)_g$ be a decomposition of $A_q$ into $K$-vector spaces which induces a grading on $A_q$. Suppose that $G$ is generated by $Supp(A_q) = \{g \in G : (A_q)_g \neq 0\}$ and that for every pair of nonzero homogeneous elements $a, b \in A_q$ we have $ab = \chi(|a|, |b|)ba$ where $\chi$ is supposed some bicharacter. Then the given grading on $A_q$ is equivalent to one where $X$ and $Y$ are supposed homogeneous. $G$ is a noncyclic epimorphic image of $\mathbb{Z}^2$. The possible choices for the degrees of $X$ and $Y$ are classified here by the orbits of the action of $Aut(G)$ on the set*

$$M := \{(g_1, g_2) : g_i \in G \text{ with } Ke(f_{(g_1,g_2)}) \subseteq Ke(\tilde{\chi}_{x,y}) \forall x, y \in \mathbb{Z} \text{ and } < g_1, g_2 >= G\}$$

*where for $g_1, g_2 \in G$ the function $f_{(g_1,g_2)} : \mathbb{Z}^2 \to G$ is given through*

$$f_{(g_1,g_2)}(n, m) := ng_1 + mg_2$$

*and where $\tilde{\chi}_{x,y} : \mathbb{Z}^2 \to K^*$ is given by*

$$\tilde{\chi}_{x,y}(n, m) := \xi_X^{nx} \xi_Y^{my} p^{ny-mx}$$

*with $\xi_X := \chi(|X|, |X|)$ and $\xi_Y := \chi(|Y|, |Y|)$.*

**Proof** First, we prove that under the conditions given there can be no appropriate grading of $A_q$ by a cyclic group. Suppose that $G$ is cyclic and that $g$ is a generator of $G$. Because $K$ is a field, we have $\chi(g, g) = \xi \in \{-1, 1\}$ and consequently $\chi(mg, ng) = \xi^{nm}$. The map $\varphi : G \to \mathbb{Z}_2$ given by $\varphi(h) := \chi(g, h)$ is then a group homomorphism into $\mathbb{Z}_2$ which is compatible with commutation conditions and so what we need to show is just that $A_q$ cannot be endowed with a supercommutative structure ($\xi = -1$ since $A_q$ is not commutative). Also, noncommutativity of $A_q$ means that in this case, we have $char(K) \neq 2$. Now if there was a way to make $A_q$ supercommutative, then again as $A_q$ is not commutative we could assume that there are two odd elements $a, b \in A_q$ such that $ab \neq ba$, in particular $ab \neq 0$. We see $a = \lambda_1 X + \lambda_2 Y + \lambda_3 XY$ and likewise $b = \mu_1 X + \mu_2 Y + \mu_3 XY$ because elements of the form $1 + \epsilon$ with $\epsilon$ an element of the maximal ideal of $A_q$ are invertible - and hence not odd - as $\epsilon$ is nilpotent in this case. Computing $ab$ and $ba$ and using the assumed anticommutation property of $a$ and $b$, we see

$(*)\lambda_1\mu_2 = -\lambda_2\mu_1$, and as $char(K) \neq 2$ we have as well $a^2 = 0, b^2 = 0$, so $\lambda_1\lambda_2 = 0$ which together with $(*)$ implies $0 = \lambda_1\mu_2 = \lambda_2\mu_1$. But this means $ab = ba$, leading to a contradiction. So $G$ is not cyclic and we can choose $a, b \in A_q$ homogeneous such that $|a|$ and $|b|$ span a subgroup of $G$ strictly larger than the subgroup generated by either element. Again we can without loss of generality assume $ab \neq 0$, and setting $\lambda := \chi(|a|, |b|)$ we have $ab = \lambda ba$ and $\lambda \neq 1$. One sees from these conditions easily that $\{1, a, b, ab\}$ is a homogeneous $K$-basis of $A_q$ since the degrees of these elements are mutually different and $dim_K(A_q) = 4$. We will now show that $a$ and $b$ must both be noninvertible. Setting $a = \lambda_0 + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$, $b = \mu_0 + \mu_1 X + \mu_2 Y + \mu_3 XY$ and $p := (-q)^{-1}$ this is an easy calculation: we get

$$ab - ba = (\lambda - 1)ba = (1-p)(\lambda_1\mu_2 - \lambda_2\mu_1)XY$$

and

$$ba = \mu_0\lambda_0 + (\mu_0\lambda_1 + \mu_1\lambda_0)X + (\mu_0\lambda_2 + \mu_2\lambda_0)Y + (\mu_0\lambda_3 + \mu_1\lambda_2 + p\mu_2\lambda_1 + \mu_3\lambda_0)XY$$

so the condition $ab = \lambda ba$ with $\lambda \neq 1$ by comparison of coefficients implies the following equations:

$$\begin{aligned} \lambda_0\mu_0 &= 0 \\ \mu_0\lambda_1 + \mu_1\lambda_0 &= 0 \\ \mu_0\lambda_2 + \mu_2\lambda_0 &= 0 \\ \mu_0\lambda_3 + \mu_1\lambda_2 + p\mu_2\lambda_1 + \mu_3\lambda_0 &= \xi \end{aligned}$$

where $\xi := (1-p)(\lambda_1\mu_2 - \lambda_2\mu_1)(\lambda - 1)^{-1}$. Taking without loss of generality $\lambda_0 = 0$ we see that this implies either $\lambda_1, \lambda_2, \lambda_3 = 0$ in contradiction to the assumption $a \neq 0$ or $\mu_0 = 0$, i.e. also $b$ not invertible. Given that, the conditions on the grading force $a^2 = 0$ and $b^2 = 0$ since if nonzero these would have to be homogeneous elements of degree zero, and so nonzero elements of the base field $K$. We have $a^2 = (1-p)\lambda_1\lambda_2 XY$, so we have $\lambda_1 = 0$ or $\lambda_2 = 0$ and likewise $\mu_1 = 0$ or $\mu_2 = 0$. Since $ab \neq 0$ we cannot have the same case on both sides, so without loss of generality $\lambda_1 \neq 0, \lambda_2 = 0, \mu_1 = 0$ and $\mu_2 \neq 0$. But in this case $ab + qba = 0$, so $a$ and $b$ satisfy exactly the same relations as $X$ and $Y$ and it is easy to construct an isomorphism of $(G, \chi)$-commutative algebras taking $a$ to $X$ and $b$ to $Y$, where of course $X$ and $Y$ are assigned the same degrees as $a$ and $b$ respectively.
We will from now on always assume $X$ and $Y$ homogeneous and turn our attention to the statement about the admissible degrees of $X$ and $Y$. Suppose for this purpose that $\chi : G \times G \to K^*$ is a bicharacter encoding the commutation relations on $A_q$. Then, we first see that

$$\chi(n_1|X| + m_1|Y|, n_2|X| + m_2|Y|) = \xi_X^{n_1 n_2}\xi_Y^{m_1 m_2} p^{n_1 m_2 - m_1 n_2}$$

and this is well-defined if and only if with the definitions of the proposition and $g_1 := |X|, g_2 := |Y|$ we have

$$n_1 g_1 + m_1 g_2 = 0 \Rightarrow \tilde{\chi}_{n_2, m_2}(n_1, m_1) = 1$$

for all $n_2, m_2 \in \mathbb{Z}$, or in other words $Ke(f_{(g_1,g_2)}) \subseteq Ke(\tilde{\chi}_{n_2,m_2})$ for the same. By definition, this condition is just $(g_1, g_2) \in M$ as $g_1$ and $g_2$ are generating for $G$ by assumption anyway. What remains to be proven is just that $Aut(G)$ induces a group action on $M$. We show only stability of $M$ under automorphisms of $G$. Let $\varphi \in Aut(G)$ and $(g_1, g_2) \in M$. Then $\varphi(g_1)$ and $\varphi(g_2)$ satisfy the same relations as $g_1$ and $g_2$, so in particular $Ke(f_{(g_1,g_2)}) = Ke(f_{(\varphi(g_1),\varphi(g_2))})$. This proves the claim.

### $p$ of infinite order

To further constrain the degrees and the underlying group, we have to consider various subcases for the value of $p$. Curiously, these different cases to be considered coincide [1] with the cases that appear in the treatment of Hochschild cohomology of the ungraded versions of these algebras in (Buchweitz, Green, Madsen, Solberg [11]). If $p$ is of infinite order, then setting $g_X := deg(X)$, $g_Y := deg(Y)$ and $H_X :=< g_X >, H_Y :=< g_Y >$ we see that $\chi$ induces nonzero homomorphisms from $H_X$ and $H_Y$ respectively to $\mathbb{Z}$, implying that the degrees of $X$ and $Y$ must be of infinite order as well. This means that either $G = \mathbb{Z} \times \mathbb{Z}$ or that $G = \mathbb{Z} \times \mathbb{Z}_n$ for some $n \in \mathbb{N}$.
Now the second case is impossible in our situation. The reason for this is very simple: assume that $G = \mathbb{Z} \times \mathbb{Z}_n$ and that $|X|$ and $|Y|$ generate $G$. Then there exists some $\mathbb{Z}$-linear combination of $|X|$ and $|Y|$ yielding an element $g$ of order $n$ of $\mathbb{Z} \times \mathbb{Z}_n$. Applying $\chi$, we would get

$$\chi(|X|, g) = \chi(|X|, n_X|X| + n_Y|Y|) = \xi_X^{n_X} p^{n_Y}$$

with $\xi_X$ as before. Raising this to the $n$-th power would then yield

$$\chi(|X|, g)^n = \chi(|X|, 0) = 1 \text{ on the one hand and } \chi(|X|, g)^n = \xi_X^{nn_X} p^{nn_Y}$$

on the other hand. Since $\xi_X^2 = 1$ and $p$ was supposed to be not of finite order, this cannot be unless $n_Y = 0$. This in turn is incompatible with the assumptions that $|X|$ was an element of $G$ of infinite order and that $g$ is of finite order.
In the case $G = \mathbb{Z} \times \mathbb{Z}$, we note that $Aut(G)$ acts transitively on the set of two-element group generating sets of $G$, implying that we can then assume without loss of generality that

$$deg(X) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, deg(Y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So in the case at hand, there is essentially only one grading to be considered on $A_q$. We now need to determine the bicharacters compatible to it. We will show in the following remark that there are four choices for this, two of them equivalent:

**Remark A.0.5** *Suppose that $p$ is of infinite order in $K^*$ and that $A_q$ is endowed with a graduation over $G = \mathbb{Z}^2$ in the way just described, i.e. with $X$ and $Y$ homogeneous and of degrees given by the standard basis of $\mathbb{Z}^2$. Suppose that $\chi : G \times G \to K^*$ is a bicharacter compatible with $A_q$. Set $\xi_X := \chi(|X|, |X|)$ and $\xi_Y := \chi(|Y|, |Y|)$. With $v, w \in \mathbb{Z}^2$ we calculate*

$$\chi(v, w) = \xi_X^{v_1 w_1} \xi_Y^{v_2 w_2} p^{v^t S w}$$

[1]We do not think that this is indeed more than coincidence

*where $S$ is the matrix*

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

*where for $\xi_X$ and $\xi_Y$ we have a choice between the values 1 and $-1$. Note that if we set $\xi_X = \xi_Y = -1$, we obtain a $(G,\chi)$-commutative structure on $A_q$ which makes it into a $(G,\chi)$-commutative polynomial ring. In particular, the results of the fourth chapter imply that generalized Harrison cohomology of $A_q$ vanishes in dimension two in this case.*

### $p$ a root of unity

We now consider the case that $p$ is an $n$-th root of unity. Recall that $ord(p)$ and $ord(q)$ are (outside of characteristic two) related through

$$ord(p) = \begin{cases} ord(q) & \text{for } ord(q) \equiv 0 \bmod 4 \\ 2ord(q) & \text{for } ord(q) \equiv 1,3 \bmod 4 \\ \frac{ord(q)}{2} & \text{for } ord(q) \equiv 2 \bmod 4 \end{cases}$$

By Proposition A.0.4 we can under the conditions spelled out there continue to assume that $G$ is generated by exactly two elements and that $X$ and $Y$ are homogeneous elements with degrees that form a system of generators of $G$. If $G = \mathbb{Z} \times \mathbb{Z}$, the arguments given carry over without change. If $G$ is not free as $\mathbb{Z}$-module, the situation gets more complicated. Indeed we do not have a nice classification of the orbits on $M$ in this case. We will content ourselves with showing that in general there is more than one orbit:

**Remark A.0.6** *Suppose that $p$ is a root of unity of order $r \in \mathbb{P}$ a prime number. Assume further that $G = \mathbb{Z} \times \mathbb{Z}_r$ and that $\xi_X = \xi_Y = 1$. Then, setting*

$$deg(X) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, deg(Y) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

*is an admissible choice of degrees for $X$ and $Y$ since in that case we have*

$$Ke(f_{|X|,|Y|}) \quad = \quad < \begin{pmatrix} 0 \\ r \end{pmatrix} > \quad \subseteq \quad Ke(\tilde{\chi}_{n,m})$$

*for all $n, m$, but another choice is also given by setting*

$$deg(X) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, deg(Y) := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

*since in that case we find*

$$Ke(f_{|X|,|Y|}) \quad = \quad < \begin{pmatrix} r \\ r \end{pmatrix} >$$

*which is a subset of all the $Ke(\tilde{\chi}_{n,m})$ as well. The two choices are not transformed by any automorphism of $G$ into each other for reasons of incompatible element orders.*

### A.0.2 The case $q = -1$

The next case that has to be considered is $q = -1$, i.e. the case where $X$ and $Y$ commute. In this case, $A_q$ is the commutative algebra

$$A_{-1} = K[X,Y]/(X^2, Y^2)$$

and any grading on this algebra is compatible with at least the trivial color-commutative structure but the grading together with possibly a bicharacter of nontrivial signature can introduce additional structure different from the standard case (recall that the signature of a bicharacter was defined to be the function given by $\chi \circ \Delta$ where $\Delta : G \to G \times G$ is the diagonal map). As the algebraic structure places much less restrictions on the grading than in the previous case, we cannot hope for better results for the gradings over groups with torsion than we got there. For this reason, we will only consider gradings over powers of $\mathbb{Z}$ and of course assume as usual that $Supp(A_{-1})$ generates the grading group $G$. With these provisions, we see

**Proposition A.0.7** *The compatible gradings of $A_{-1}$ over free abelian groups are up to equivalence given by the following cases:*

1. *$G = 0$ with trivial $\chi$.*

2. *$G = \mathbb{Z}$ and $char(K) = 2$, then up to equivalence $X$ and $Y$ homogeneous of coprime degree. Of course in this case the bicharacter must be trivial.*

3. *$G = \mathbb{Z}$ and $char(K) \neq 2$, then up to equivalence $X$ and $Y$ homogeneous of coprime degree. In this case, the admissible bicharacters depend on the degrees of $X$ and $Y$, with a supercommuting structure compatible with the underlying algebraic structure if exactly one of $deg(X)$ and $deg(Y)$ is even and the other is odd.*

4. *$G = \mathbb{Z} \oplus \mathbb{Z}$, then up to equivalence $X$ and $Y$ homogeneous with*

$$deg(X) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, deg(Y) := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

*and $\xi_X, \xi_Y$ may be freely chosen from $\{-1, 1\}$ in this case.*

**Proof** The first case is trivial. To deal with the other cases, we make some simple preliminary remarks:

- If $char(K) = 2$, then all non-invertible elements of $A_{-1}$ are nilpotent of order two.

- Invertible elements in $A_q$ must be of order zero, since with $G$ being a free abelian group we know that an invertible element of nonzero degree would make an infinite number of graded components of $A_{-1}$ nonzero.

- If $\{1, a, b, c\}$ is a homogeneous basis of $A_{-1}$, we can without loss of generality assume $a, b, c$ not invertible since if any of them were degree zero and invertible, we could replace them by a homogeneous noninvertible element of zero degree by adding a suitable multiple of the unit.

- If $\{1, a, b, c\}$ is a homogeneous basis of $A_{-1}$, we can assume without loss of generality that $ab \neq 0$ since $a, b, c$ form a homogeneous basis of the maximal ideal $\mathfrak{m} \subseteq A_{-1}$ of noninvertible elements and because there exist noninvertible elements which do not multiply to zero. We can also assume in this situation that $a^2 = c$ or $a^2 = 0$, since $a^2$ must be homogeneous or zero and $a^3 = 0$.
- Suppose that $\{1, a, b, c\}$ is a homogeneous basis of $A_{-1}$ such that $a^2 = 0, b^2 = 0$ and $0 \neq ab \in < c >$. This is for instance necessarily true when $a$ and $b$ are elements of nonzero degree with $ab \neq 0$ and which square to zero. Then there is an isomorphism of graded algebras from $A_{-1}$ equipped with the given grading to $A_{-1}$ with the grading induced by taking $X$ and $Y$ as homogeneous elements and giving them the same degrees as $a$ and $b$.

We start now with the case that $char(K) = 2$. Then per the above, with $\{1, a, b, c\}$ a homogeneous basis of $A_{-1}$ we have $a^2 = b^2 = c^2 = 0$ and can without loss of generality assume $ab \neq 0$. If we can show that indeed $ab = c$ up to multiplication with a scalar, we will be done. Indeed, $ab \neq 0$ implies that neither $a$ nor $b$ is in $\mathfrak{m}^2$, but $ab$ is. It follows from this that $ab$ is not a scalar multiple of either $a$ or $b$ and as it is certainly not invertible, the only remaining option is that it be in $< c >$. We can hence from now on assume without loss of generality that $a = X, b = Y$ and $c = XY$. The rest of the statement for $G = \mathbb{Z}$ and $char(K) = 2$ is now immediate. For $G = \mathbb{Z} \oplus \mathbb{Z}$, arguments of the usual type about the automorphism group of $G$ yield the rest of the claim now.
Suppose on the other hand that $char(K) \neq 2$. As before, we find a homogeneous basis $\{1, a, b, c\}$ of $A_{-1}$ with $a, b, c \in \mathfrak{m}$ and $ab \neq 0$. Also, we see from $ab \neq 0$ that neither $a$ nor $b$ are in $\mathfrak{m}^2$ but $ab, a^2$ and $b^2$ are. Consequently, we have $ab \in < c >$ and if either of $a^2$ and $b^2$ are nonzero, they must be in $< c >$ as well. Suppose now $G = \mathbb{Z}$. If $a^2 = b^2 = 0$, we can as before see that up to equivalence $a, b, c$ may be assumed to be $X, Y, XY$ and get also the statements about the degrees and the bicharacters. If not, we can on the other hand assume $a^2 \neq 0$ without loss of generality. In this case, $a^2$ and $ab$ are both in $\mathfrak{m}^2 =< XY >=< c >$, from which $deg(a) = deg(b)$ follows. Since $deg(a)$ and $deg(b)$ are supposed to generate $\mathbb{Z}$, there is up to equivalence only the possibility $deg(a) = deg(b) = 1$. Now as $a, b, c$ form a basis of $\mathfrak{m}$ and as $c \in \mathfrak{m}^2$ it is clear that the projections $a'$ and $b'$ of $a$ and $b$ to the subspace spanned by $X$ and $Y$ must form a basis of $< X, Y >$. Hence, there exist linear combinations of $a$ and $b$ of form $X + \lambda XY$ and $Y + \mu XY$ respectively. Since these linear combinations are linear combinations of homogeneous elements of degree one, the elements so produced are of degree one again and we find ourselves again in the situation that up to equivalence $X$ and $Y$ are homogeneous of degree one. Since $X$ and $Y$ commute with each other, the only admissible bicharacter in this situation would be the trivial one.
What remains to be done is the case $G = \mathbb{Z} \oplus \mathbb{Z}$. As before, we find a homogeneous basis of $A_{-1}$ of form $\{1, a, b, c\}$ with $a, b, c \in \mathfrak{m}$ and $ab \neq 0$ and can by the usual arguments about automorphisms of $\mathbb{Z}^2$ assume without loss of generality that

$$deg(a) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, deg(b) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

By dimension counting we see immediately that in this situation we must have $a^2 = 0, b^2 = 0$

and $ab = c$. This concludes the proof.

### A.0.3 The case $q = 0$

**Remark A.0.8** *As one would intuitively expect, there is no way to define a color-commutative structure on $A_0$.*

**Proof** Assume that $\{1, a, b, c\}$ is a homogeneous basis of $A_0$. We can without loss of generality assume that $a, b, c$ are not invertible. Since $A_0$ is a noncommutative algebra with sole maximal ideal $\mathfrak{m} =< a, b, c >$ and multiplication not identically zero on $\mathfrak{m}$, we know that if there is to be a not strictly commutative color-commuting structure on $A_0$ we can assume $0 \neq ab = \lambda ba$ with some $\lambda \neq 0, 1$. By direct calculation we see for $a = \lambda_1 X + \lambda_2 Y + \lambda_3 YX$ and $b = \mu_1 X + \mu_2 Y + \mu_3 YX$ that $ab = \lambda_2\mu_1 YX$ and $ba = \lambda_1\mu_2 YX$. Given the relation prescribed between $ab$ and $ba$, this implies that all of $\lambda_1, \lambda_2, \mu_1, \mu_2$ must be nonzero. In this case, however, we see $XY \in< a^2 >=< b^2 >=< ab >$ and so $deg(a) = deg(b)$. But, on the other hand we also see from $ab = \lambda ba$ with a $\lambda \neq 0, 1$ that with the same notations $\lambda_1\mu_2 - \lambda_2\mu_1 \neq 0$. Using again projection to $< X, Y >$ we can then conclude that in the same graded component as $a$ and $b$ we have also elements of the form $X + \rho_1 XY$, $Y + \rho_2 XY$ and we can therefore without loss of generality assume $a$ and $b$ to be of this form. This gives a contradiction to $ab = \lambda ba$, concluding the proof of this case.

### A.0.4 The case $q = 1$

In this case, we will again treat the cases only that arise over free abelian groups. We summarize the situation with the following remark:

**Remark A.0.9** *Up to equivalence, we can assume $X$ and $Y$ to be homogeneous. If $G = \mathbb{Z}$, we can further assume their weights to be of odd coprime order. If $G = \mathbb{Z} \oplus \mathbb{Z}$ we can without loss of generality assume again that $deg(X)$ and $deg(Y)$ coincide with the canonical basis of $\mathbb{Z}^2$. If $G = \mathbb{Z}$, the only admissible bicharacter is given by the standard supercommutative one, i.e. by $\chi(n, m) = (-1)^{nm}$.*

**Proof** Again we can assume that invertible homogeneous elements are of degree zero because the index groups of the grading are assumed free abelian and because the algebras in question are finite dimensional. Also as before we can with $\{1, a, b, c\}$ a homogeneous basis of $A_1$ assume without loss of generality that $ab \neq 0$ and that $a, b, c \in \mathfrak{m}$. In addition, we have $ab \in \mathfrak{m}^2$, $a \notin \mathfrak{m}^2, b \notin \mathfrak{m}^2$ and $deg(a) \neq 0$. By homogeneity of $ab$ it follows that $\mathfrak{m}^2 =< XY >=< c >$. If we have $a^2 \neq 0$, we see by the same argument that $deg(a) = deg(b)$. Since $ab$ depends only on the projections of $a$ and $b$ to $< X, Y >$, and because $ab \neq ba$, we know that in this case these projections must be linearly independent. Given this, we can without loss of generality assume $a = X + \rho_1 XY$ and $b = Y + \rho_2 XY$. Hence without loss of generality $a^2 = b^2 = 0$ and $a = X, b = Y, c = XY$.
Now we consider the case $G = \mathbb{Z}$. Since $A_1$ is not a commutative algebra, it is clear that the bicharacter given on any $G$-commutative structure on $A_1$ must be given by $\chi(n, m) := (-1)^{nm}$.

The assumption that $G$ should be generated by elements in the support of $A_1$ implies that $X$ and $Y$ should be of coprime degree. Finally, the fact that they anticommute means that their weights should both be odd.

# Bibliography

[1] Michael Atiyah and Ian MacDonald. *Introduction to commutative algebra*. Perseus Books, 1994.

[2] Luchezar Avramov, Vesselin Gasharov, and Irena Peeva. Complete intersection dimension. *Publications IHES*, 86:67–114, 1997.

[3] Luchezar Avramov and Srikanth Iyengar. Gaps in Hochschild cohomology imply smoothness for commutative algebras. *Mathematical Research Letters*, 12, no. 5-6:789–804, 2005.

[4] Luchezar Avramov and Micheline Vigue-Poirrier. Hochschild homology criteria for smoothness. *Duke Mathematical Journal*, 65:17–25, 1992.

[5] John Baez. Hochschild homology in a braided tensor category. *Transactions of the American Mathematical Society*, 344, 1994.

[6] F. Bayen, M. Flato, A. Fronsdal, C. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization I. Deformations of symplectic structures. *Ann. Phys.*, 111 no. 1:61–110, 1978.

[7] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization $II$: Physical applications. *Ann. Phys.*, 111 no. 1:111–151, 1978.

[8] Petter Andreas Bergh. On the Hochschild cohomology of quantum exterior algebras. *Communications in Algebra*, 35(11):3440–3450, 2007.

[9] Petter Andreas Bergh and Steffen Oppermann. Cohomology of twisted tensor products. *Journal of Algebra*, 320, no. 8:3327–3338, 2008.

[10] C. Boboc and S. Dascalescu. Gradings of matrix algebras by cyclic groups. *Communications in Algebra*, 29(11):5013–5021, 2001.

[11] Ragnar-Olaf Buchweitz, Edward L. Green, Dag Madsen, and Oyvind Solberg. Finite Hochschild cohomology without finite global dimension. *Mathematical Research Letters*, 12:805–816, 2005.

[12] S. Caenepeel, S. Dascalescu, and C. Nastasescu. On gradings of matrix algebras and descent theory. *Communications in Algebra*, 30(12):5901–5920, 2002.

[13] Alberto Cattaneo and Davide Indelicato. Formality and star products. *London Mathematical Society Lecture Note Series. Poisson-Geometry, deformation quantisation and group representations.*, 323:79–144, 2005.

[14] Xiao-Wu Chen, Freddy van Oystaeyen, and Toukaiddine Petit. Note on the cohomology of Color Hopf and Lie algebras. *Journal of Algebra*, 299:1:419–442, 2006.

[15] Jang-Ho Chun and Jaeun Lee. Abelian group gradings on full matrix rings. *Communications in Algebra*, 35(10):3095–3102, 2007.

[16] Albert Crumeyrolle. *Orthogonal and symplectic Clifford algebras.* Kluwer Academic Publishers, 1990.

[17] Joachim Cuntz and Daniel Quillen. Algebra extensions and nonsingularity. *Journal of the American Mathematical Society*, Vol. 8, No. 2:251–289, 1995.

[18] Joachim Cuntz and Daniel Quillen. Cyclic homology and nonsingularity. *Journal of the American Mathematical Society*, 8:373–442, 1995.

[19] S. Dascalescu, B. Ion, C. Nastasescu, and J Rios Montes. Group gradings on full matrix rings. *Journal of Algebra*, 220(2):709–728, 1999.

[20] Deligne. Notes on spinors. In Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison, and Edward Witten, editors, *Quantum fields and strings: a course for mathematicians 1.* American Mathematical Society, 1996.

[21] RM Fossum. The structure of indecomposable injective modules. *Math. Scand*, 36:291–312, 1975.

[22] Yael Fregier and Aron Gohr. On Hom-type algebras. *arxiv:0903.3393*, 2009.

[23] Christian Fronsdal. Harrison Cohomology and Abelian Deformation Quantization on Algebraic Varieties. In Gilles Halbout, editor, *Deformation quantization*, pages 149–161. De Gruyter Berlin, 2002.

[24] Christian Fronsdal. Deformation Quantization on singular coadjoint orbits. *arxiv: math.RT/0510580*, 2005.

[25] Christian Fronsdal and Maxim Kontsevich. Quantization on Curves. *Letters in Mathematical Physics*, 79:109–129, 2007.

[26] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008.

[27] Murray Gerstenhaber. The cohomology structure of an associative ring. *Annals of Mathematics*, 78, 1963.

[28] Murray Gerstenhaber and S.D. Schack. A hodge-type decomposition for commutative algebra cohomology. *Journal of Pure and Applied Algebra*, 48:229–247, 1987.

[29] Victor Ginzburg. Double derivations and cyclic homology. *http://arxiv.org/abs/math/0505236v3*, na, 2005.

[30] D. Happel. Self-orthogonal modules. In *Abelian groups and modules (Padova, 1994)*, 1995.

[31] D.K. Harrison. Commutative algebras and cohomology. *Transactions of the American Mathematical Society*, 104:191–204, 1962.

[32] Robin Hartshorne. *Algebraic Geometry*. Springer Verlag, 1977.

[33] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[34] Tornike Kadeishvili. Structure of $A(\infty)$-algebra and Hochschild and Harrison cohomology. *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR*, 91:19–27, 1989.

[35] Maxim Kontsevich and Alexander Rosenberg. Noncommutative smooth spaces. In *The Gelfand mathematical seminars 1996-1999*. Birkhäuser Boston, 2000.

[36] P.S. Landweber. Unique Factorization in Graded Power Series Rings. *Proceedings of the American Mathematical Society*, pages 73–76, 1974.

[37] Viktor Levandovskyy. *Non-commutative Computer Algebra for polynomial algebras: Gröbner bases, application and implementation*. PhD thesis, Universität Kaiserslautern, 2005.

[38] Jean-Louis Loday. *Cyclic homology*. Springer Verlag, Berlin, 1992.

[39] V.S. Lychagin. Colour calculus and colour quantizations. *Acta Appl. Math.*, 41:193–226, 1995.

[40] Abdenacer Makhlouf and Sergei Silvestrov. Notes on formal deformations of hom-associative and hom-lie algebras. *arxiv:math.RA/0712.3130v1*, 2007.

[41] Martin Markl, Steve Shnider, and Jim Stasheff. *Operads in Algebra, Topology and Physics*. American Mathematical Society, 2002.

[42] Sophie Morier-Genoud and Valentin Ovsienko. Well, Papa, can you multiply triplets? *arxiv:0810.5562*, 2008.

[43] Francois Nadaud. Generalised deformations, Koszul resolutions, Moyal products. *Rev. Math. Phys.*, 10, no. 5:685–704, 1998.

[44] Francois Nadaud. Generalized Deformations and Hochschild Cohomology. *Letters in Mathematical Physics*, 58:41–55, 2001.

[45] Cyrille Ospelt. Cohomological properties of the quantum shuffle product and application to the construction of quasi-Hopf algebras. *Journal of pure and applied Algebra*, 173, no.2:315–337, 2002.

[46] Donald Passman. Simple Lie Color algebras of Witt type. *Journal of Algebra*, 208:698–721, 1998.

[47] Georges Pinczon. Noncommutative Deformation Theory. *Letters in Mathematical Physics*, 41:101–117, 1997.

[48] Daniel Quillen. On the (co-) homology of commutative rings. *Proc. Symp. Pure Math.*, 17:65–87, 1970.

[49] V. Rittenberg and D. Wyler. Generalized superalgebras. *Nuclear Physics B*, 139(3):189–203, 1978.

[50] M. Rosso. Quantum groups and quantum shuffles. *Invent. Math.*, 2:399–416, 1998.

[51] Manfred Scheunert. Generalized Lie algebras. *Journal of Mathematical Physics*, 20, no.4:712–720, 1979.

[52] Manfred Scheunert. Introduction to the cohomology of Lie superalgebras and some applications. In *Recent Advances in Lie Theory*, 2002.

[53] Yucai Su, Kaiming Zhao, and Linsheng Zhu. Classification of derivation-simple color algebras related to locally finite derivations. *Journal of Mathematical Physics*, 45:525–536, 2004.

[54] Veeravalli Varadarajan. *Supersymmetry for mathematicians.* American Mathematical Society, 2004.

[55] Stefan Waldmann. *Poisson-Geometrie und Deformationsquantisierung: Eine Einführung.* Springer Verlag, 2007.

# Declaration of independent completion of this dissertation

I hereby declare that this dissertation is my own work and that no assistance or literature was used without proper credit.
Luxembourg, the
Signature of the author: