Itô’s formula for Walsh’s Brownian motion and applications

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ABSTRACT

We prove an Itô’s formula for Walsh’s Brownian motion in the plane with angles according to a probability measure \( \mu \) on \([0, 2\pi]\). This extends Freidlin–Sheu formula which corresponds to the case where \( \mu \) has finite support. We also give some applications.

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1. Itô’s formula for Walsh’s Brownian motion

Let \( E = \mathbb{R}^2 \); we will use polar co-ordinates \((r, \alpha)\) to denote points in \( E \). We denote by \( C(E) \) the space of all continuous functions on \( E \). For \( f \in C(E) \), we define

\[ f_\alpha(r) = f(r, \alpha), \quad r > 0, \quad \alpha \in [0, 2\pi[. \]

Throughout this paper we fix \( \mu \) a probability measure on \([0, 2\pi[\). We call the following

\[
\bar{f}(r) = \int_0^{2\pi} f(r, \alpha) \mu(\text{d}\alpha), \quad r > 0.
\]

Let \((P^+_t)_{t \geq 0}\) be the semigroup of a reflecting Brownian motion on \([0, \infty[\) and let \((P^0_t)_{t \geq 0}\) be the semigroup of a Brownian motion on \([0, \infty[\) killed at 0. Then for \( t \geq 0 \), define \( P_t \) to act on \( f \in C_0(E) \) as follows:

\[
P_t f(0, \alpha) = P^+_t \bar{f}(0),
\]

\[
P_t f(r, \alpha) = P^+_t \bar{f}(r) + P^0_t (f_\alpha - \bar{f})(r)
\]

for \( r > 0 \) and \( \alpha \in [0, 2\pi[\). We recall the following

Theorem 1.1 (Barlow et al., 1989). \((P_t)_{t \geq 0}\) is a Feller semigroup on \( C_0(E) \), the associated process is by definition the Walsh’s Brownian motion \((Z_t)_{t \geq 0}\) (with angles distributed as \( \mu \)).

We can write \( Z_t = (R_t, \Theta_t) \) where the radial part \( R_t = |Z_t| \) is a reflected Brownian motion and \((\Theta_t)_{t \geq 0}\) is called the angular process of \( Z \). To well define \( \Theta_1 \), when \( Z_t = 0 \), we set by convention \( \Theta_1 = 0 \).

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Notations:
We denote by \( E_\mu \), the state space of Walsh’s Brownian motion, i.e. \( E_\mu = \{(r, \alpha) : r \geq 0, \alpha \in \text{supp}(\mu)\} \). We define the tree-metric \( \rho \) on \( \mathbb{R}^2 \) as follows: for \( z_1 = (r_1, \alpha_1), z_2 = (r_2, \alpha_2) \in \mathbb{R}^2 \),
\[
\rho(z_1, z_2) = (r_1 + r_2)1_{|\alpha_1| \neq |\alpha_2|} + |r_1 - r_2|1_{|\alpha_1| = |\alpha_2|}.
\]
Note that the sample paths of Walsh’s Brownian motion are continuous with respect to the tree-topology, since the process cannot jump from one ray to another one without passing through the origin. These paths are also continuous with respect to the relative topology induced by the Euclidean metric on \( \mathbb{R}^2 \).

Now denote by \( \mathcal{D} \) the set of all functions \( f : E_\mu \to \mathbb{R} \) such that
(i) \( f \) is continuous for the tree-topology.
(ii) For all \( \alpha \in \text{supp}(\mu), f_\alpha \) is \( C^2 \) on \([0, \infty)\) and \( f_\alpha'(0+), f_\alpha''(0+) \) exist and are finite.
(iii) For all \( K > 0 \),
\[
\sup_{0 < r < K, \alpha \in \text{supp}(\mu)} \left( |f_\alpha'(r)| + |f_\alpha''(r)| \right) < \infty.
\]
Note that if \( f \in \mathcal{D} \), then \( \sup_{\alpha \in \text{supp}(\mu)} |f_\alpha'(0+)| < \infty \) and in particular \( \int_{\text{supp}(\mu)} |f_\alpha'(0+)| \mu(d\alpha) \) is finite. From now on, we will extend any function \( g : \text{supp}(\mu) \to \mathbb{R} \) to \([0, 2\pi] \) by setting \( g = 0 \) outside \( \text{supp}(\mu) \). For \( f \in \mathcal{D} \) and \( z = (r, \alpha) \neq 0 \), we set \( f'(z) = f_\alpha'(r) \) and \( f''(z) = f_\alpha''(r) \).

Now define
\[
B^Z_t = R_t - R_0 - L_t(R)
\]
where \( L_t(R) \) stands for the symmetric local time at zero of \( R \), i.e. \( L_t(R) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{|R_s|<\varepsilon} ds \). Then \( B^Z \) is an \((\mathcal{F}^Z_t)\)-Brownian motion (Lemma 2.2 Barlow et al. (1989)). Our main result is the following

**Theorem 1.2.** Let \((Z_t)_{t \geq 0}\) be a Walsh’s Brownian motion started from \( z \), then for all \( f \in \mathcal{D} \), we have
\[
f(Z_t) = f(z) + \int_0^t 1_{|Z_r| \neq 0} f'(Z_r) dB^Z_r + \frac{1}{2} \int_0^t 1_{|Z_r| \neq 0} f''(Z_r) ds + \left( \int_0^{2\pi} f_\alpha''(0+) \mu(d\alpha) \right) \times L_t(R).
\]
When \( \text{supp}(\mu) \) is finite this formula is due to Freidlin and Sheu (2000) (see also Hajri (2011) for another proof based on the skew Brownian motion). Our proof is based on standard approximations and Riemann–Stieltjes integration.

**Proof.** Note that \( H_t(\omega) = 1_{|Z_t(\omega)| \neq 0} f'(Z_t(\omega)) \) is progressively measurable with respect to the filtration \((\mathcal{F}^Z_t)\) and is locally bounded by the assumption (iii) on \( f \), i.e. a.s. \( \forall t \geq 0, \sup_{|\omega| \leq 2} |H_t| < \infty \). Thus the stochastic integral with respect to \( B^Z \) is well defined. Again by (iii), the integral \( \int_0^t 1_{|Z_r| \neq 0} f''(Z_r) ds \) is well defined.

First, we will consider the case \( z = 0 \). Take \( f \in \mathcal{D}, t > 0 \) and for \( \varepsilon > 0 \), define \( \tau^\varepsilon_0 = 0 \) and for \( n \geq 0 \)
\[
\sigma^\varepsilon_n = \inf\{r \geq \tau^\varepsilon_n : |Z_r| = \varepsilon\}, \quad \tau^\varepsilon_{n+1} = \inf\{r \geq \sigma^\varepsilon_n : Z_r = 0\}.
\]
Set \( g(z) = f(z) - f''_\alpha(0+) |Z_r| 1_{|Z_r| \neq 0} \) where \( \alpha = \arg(z) \). Note that \( g \in \mathcal{D} \) and \( g_\alpha'(0+) = 0 \). We will prove our formula first for \( g \). We have
\[
g(Z_t) - g(0) = \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \left( g(Z_{\sigma^\varepsilon_{n+1} \wedge t}) - g(Z_{\sigma^\varepsilon_n \wedge t}) \right) = \lim_{\varepsilon \to 0} \left( A^\varepsilon_t + D^\varepsilon_t \right)
\]
where
\[
A^\varepsilon_t = \sum_{n=0}^{\infty} \left( g(Z_{\sigma^\varepsilon_{n+1} \wedge t}) - g(Z_{\sigma^\varepsilon_n \wedge t}) \right)
\]
and
\[
D^\varepsilon_t = \sum_{n=0}^{\infty} \left( g(Z_{\sigma^\varepsilon_{n+1} \wedge t}) - g(Z_{\sigma^\varepsilon_n \wedge t}) \right) = \sum_{n=0}^{\infty} \left( g(Z_{\sigma^\varepsilon_{n+1} \wedge t}) - g(Z_{\sigma^\varepsilon_n \wedge t}) \right) 1_{\sigma^\varepsilon_n \leq t}.
\]
We will first identify \( \lim_{\varepsilon \to 0} D^\varepsilon_t \). We need the following

**Lemma 1.3.** Let \( h \in \mathcal{D}, \) then for all \( t \in [\sigma^\varepsilon_n, \tau^\varepsilon_{n+1}] \) we have
\[
h(Z_t) = h(Z_{\sigma^\varepsilon_n}) + \int_{\sigma^\varepsilon_n}^t 1_{|Z_r| \neq 0} h'(Z_r) dB^Z_r + \frac{1}{2} \int_{\sigma^\varepsilon_n}^t h''(Z_r) 1_{|Z_r| \neq 0} ds.
\] (2)

**Proof.** Note that \( B^Z_{\tau^\varepsilon_{n+1}} - B^Z_{\tau^\varepsilon_n} \) is a Brownian motion independent of \( \mathcal{F}^Z_{\sigma^\varepsilon_n} \). Thus by conditioning with respect to \( \mathcal{F}^Z_{\sigma^\varepsilon_n} \), it will be sufficient to prove that a Walsh’s Brownian motion \( Y \) started from a fixed \( y \neq 0 \) satisfies (2) with \( (Y, B^Z_t, 0, \tau^\varepsilon) \) in place

**References**
Applying the previous lemma, we get
\[ D_\varepsilon^t = \sum_{n=0}^{\infty} 1_{[\sigma_n^\varepsilon \leq t]} \int_{\sigma_n^\varepsilon}^{\tau_{n+1}^\varepsilon} \left( g'(Z_s) dB_s^\varepsilon + \frac{1}{2} g''(Z_s) \right) ds \]
\[ = \sum_{n=0}^{\infty} \int_{\sigma_n^\varepsilon}^{\tau_{n+1}^\varepsilon} \left( g'(Z_s) dB_s^\varepsilon + \frac{1}{2} g''(Z_s) \right) ds \]
\[ = \sum_{n=0}^{\infty} \int_0^t 1_{[\sigma_n^\varepsilon, \tau_{n+1}^\varepsilon)}(s) \left( g'(Z_s) dB_s^\varepsilon + \frac{1}{2} g''(Z_s) \right) ds \]
\[ = \int_0^t \sum_{n=0}^{\infty} 1_{[\sigma_n^\varepsilon, \tau_{n+1}^\varepsilon)}(s) \left( g'(Z_s) dB_s^\varepsilon + \frac{1}{2} g''(Z_s) \right) ds \]
where we have used dominated convergence for stochastic integrals (see Revuz and Yor (1999) page 142) in the last line. Using dominated convergence again we see that as \( \varepsilon \to 0 \), \( D_\varepsilon^t \) converges in probability to \( \int_0^t 1_{[0, \varepsilon]}(g'(Z_s) dB_s^\varepsilon + \frac{1}{2} g''(Z_s)) \) ds.

Now \( A_\varepsilon^t = H_\varepsilon^t + K_\varepsilon^t \) with \( H_\varepsilon^t = \sum_{n=0}^{\infty} (g(Z_{\tau_{n+1}^\varepsilon}) - g(0)) 1_{[\tau_{n+1}^\varepsilon \leq t]} \) and \( |K_\varepsilon^t| \leq \int_0^t 1_{[0, \varepsilon]}(g'(Z_s) dB_s^\varepsilon + \frac{1}{2} g''(Z_s)) \) ds.

Note that \( \sigma_n^\varepsilon \) is \( \mathcal{F}^\varepsilon \)-measurable, \( \theta_n = \arg(Z_{\sigma_n^\varepsilon}) \) is independent of \( \mathcal{F}^\varepsilon \) and in particular \( \sigma_n^\varepsilon \) and \( \theta_n \) are independent. Therefore we obtain
\[ \mathbb{E}[|H_\varepsilon^t|] \leq \sum_{n=0}^{\infty} \mathbb{E}[|g(\varepsilon, \theta_{n+1}) - g(0)| 1_{[\sigma_{n+1}^\varepsilon \leq t]}] \leq \sum_{n=0}^{\infty} \mathbb{E}[|g(\varepsilon, \theta_{n+1}) - g(0)|] \mathbb{P}(\sigma_{n+1}^\varepsilon \leq t) \leq \int_0^{2\pi} \frac{|g(\varepsilon, \alpha) - g(0)|}{\varepsilon} \mu(d\alpha) \varepsilon \sum_{n=0}^{\infty} \mathbb{P}(\sigma_{n+1}^\varepsilon \leq t). \]

But \( \varepsilon \sum_{n=0}^{\infty} \mathbb{P}(\sigma_{n+1}^\varepsilon \leq t) \to \mathbb{E}[L_e(R)] \) as \( \varepsilon \to 0 \) and for \( \varepsilon < 1 \), we have
\[ \frac{|g(\varepsilon, \alpha) - g(0)|}{\varepsilon} \leq \sup_{0 < r < 1, \alpha \in \text{supp}(\mu)} |g'_e(r)| < \infty \]
so that by dominated convergence, as \( \varepsilon \to 0 \),
\[ \int_0^{2\pi} \frac{|g(\varepsilon, \alpha) - g(0)|}{\varepsilon} \mu(d\alpha) \to \int_0^{2\pi} |g'_e(0^+)\alpha| \mu(d\alpha) = 0 \]
and finally \( H_\varepsilon^t \to 0 \) in \( L^1 \). Summarizing, we get
\[ g(Z_t) = g(0) + \int_0^t 1_{[Z_s \neq 0]} g'(Z_s) dB_s^\varepsilon + \frac{1}{2} \int_0^t 1_{[Z_s \neq 0]} g''(Z_s) \] ds. \hspace{1cm} (3)

Note that as \( g(Z_t) = f(Z_t) - R_t Y_t \) with \( Y_t = f_{\theta_t}(0^+) 1_{[Z_t \neq 0]} \). Set
\[ C_t = Y_t R_t - \int_0^t Y_t \, dR_s. \]

We will identify \( C_t \) following Prokaj (2009). Put
\[ Y_t^\varepsilon = \sum_{k=0}^{\infty} Y_t 1_{[\sigma_k^\varepsilon, \tau_{k+1}^\varepsilon]}(t). \]

Then \( Y^\varepsilon \) is right-continuous and is constant on every interval \( [\sigma_k^\varepsilon, \tau_{k+1}^\varepsilon] \). Since \( R \) is continuous, then \( R \) is Riemann–Stieltjes integrable with respect to \( Y^\varepsilon \) almost surely and
\[ Y_t^\varepsilon R_t - Y_0^\varepsilon R_0 - \int_0^t Y_s^\varepsilon \, dR_s = \int_0^t R_s \, dY_s^\varepsilon. \hspace{1cm} (4) \]
As $\varepsilon \to 0$, we have $Y^\varepsilon_t \to Y_t$ a.s. Since $|Y^\varepsilon_t| \leq M$ for some $M > 0$, then the dominated convergence for stochastic integrals gives $\int_0^t Y^\varepsilon_s \, dR_s \to \int_0^t Y_s \, dR_s$ in probability. The definition of $Y^\varepsilon$ entails that

$$\int_0^t R_s \, dY^\varepsilon_s = \sum_{k=0}^{\infty} R_{\varepsilon^k} Y_{\varepsilon^k} \mathbf{1}_{[\varepsilon^k \leq t]} = \varepsilon \sum_{k=0}^{\infty} Y_{\varepsilon^k} \mathbf{1}_{[\varepsilon^k \leq t]}.$$

Let $N(t, \varepsilon)$ be the number of upcrossings of the interval $[0, t]$ by $R$. So

$$\int_0^t R_s \, dY^\varepsilon_s = \varepsilon N(t, \varepsilon) \sum_{j=0}^{N(t, \varepsilon)} f_{\varepsilon^j}(0+).$$

Since $(f_{\varepsilon^j}(0+))_{j \geq 0}$ is a sequence of independent random variables having the same law $\mu$, the Bernoulli law of large numbers yields

$$\int_0^t R_s \, dY^\varepsilon_s = \varepsilon N(t, \varepsilon) \sum_{j=0}^{N(t, \varepsilon)} f_{\varepsilon^j}(0+) \to L_t(R) \, \mathbb{E}[f_{\varepsilon^0}(0+)].$$

But $\mathbb{E}[f_{\varepsilon^0}(0+)] = \int_0^{2\pi} f_0(0+) \, \mu(\mathrm{d}x)$ and so $\int_0^t R_s \, dY^\varepsilon_s \to [\int_0^{2\pi} f_0(0+) \, \mu(\mathrm{d}x)] L_t(R)$ as $\varepsilon \to 0$. Letting $\varepsilon \to 0$ in (4), we obtain

$$C_t = \left( \int_0^{2\pi} f_0(0+) \, \mu(\mathrm{d}x) \right) L_t(R).$$

Using the fact that $1_{\{R_s > 0\}} dR_s = 1_{\{R_s > 0\}} dB^2_s$, it follows that

$$\int_0^t Y_s \, dR_s = \int_0^t Y_s \, dB^2_s = \int_0^t 1_{\{Z_s \neq 0\}} f_{\varepsilon^0}(0+) \, dB^2_s.$$

Now for $z = 0$, our formula follows from (3). The case $z \neq 0$ holds by discussing $t \leq \tau$ and $t > \tau$ where $\tau$ is the first hitting time of $0$ by $Z$ and using the Itô’s formula satisfied by $Z_{\tau+}$ which is a Walsh’s Brownian motion started from $0$.  

2. Some applications

2.1. Walsh’s Brownian motion on graphs

The result of this section can be deduced from Hajri and Raimond (in press) but it has not been announced in that paper. Our purpose is to give an Itô’s formula for Walsh’s Brownian motion with the more general space state of a graph. This diffusion is defined by means of its infinitesimal generator in Freidlin and Sheu (2000) (see also Chapter 4 in Jehring (2009)). Along an edge it is like a Brownian motion and around a vertex it behaves like a Walsh’s Brownian motion.

Let $G$ be a metric graph equipped with the shortest path distance, denote by $V$, the set of its vertices, and by $\{E_i; \ i \in I\}$ the set of its edges where $I$ is a finite or countable set. To each edge $E_i$, we associate an isometry $e_i : f_i = [0, L_i]$ when $L_i < \infty$ and $f_i = \{0, \infty\}$ or $f_i = (-\infty, 0]$ when $L_i = \infty$. When $L_i < \infty$, denote $[g_i, d_i] = (e_i(0), e_i(L_i))$. For $L_i = \infty$, denote $[g_i, d_i] = (e_i(0), \infty)$ when $f_i = \{0, \infty\}$ and $[g_i, d_i] = (\infty, e_i(0))$ when $f_i = (-\infty, 0]$. For all $v \in V$, denote $I^+ = \{i \in I; \ g_i = v\}$ and $I^- = \{i \in I; \ d_i = v\}$ and $I^*_v = I^+_v \cup I^-_v$. We assume that Card $I^*_v < \infty$ for all $v$ and that $\inf_{i \in I} L_i > 0$. To each $v \in V$ and $i \in I_v$, we associate a parameter $p_{i,v} \in [0, 1]$ such that $\sum_{i \in I_v} p_{i,v} = 1$. Let $p = (p_{i,v}, v \in V, i \in I_v)$ and denote by $\mathcal{D}_p^G$ the set of all continuous functions $f : G \to \mathbb{R}$ such that for all $i \in I, f \circ e_i$ is $C^2$ on the interior of $f_i$ with bounded first and second derivatives both extendable by continuity to $f_i$ and such that for all $v \in V$

$$\sum_{i \in I_v} p_{i,v} \lim_{r \to 0^+} (f \circ e_i)'(r) = \sum_{i \in I_v} p_{i,v} \left( \lim_{r \to L_i^-} (f \circ e_i)'(r) 1_{[t_i < \infty]} + \lim_{r \to 0^-} (f \circ e_i)'(r) 1_{[t_i = \infty]} \right).$$

For $f \in \mathcal{D}_p^G$ and $x = e_i(r) \in G \setminus V$, set $f'(x) = (f \circ e_i)'(r)$, $f''(x) = (f \circ e_i)''(r)$ and take the following conventions for all $v \in V, f'(v) = f''(v) = 0$.

Let $Z$ be a Walsh’s Brownian motion on $G$ associated to the family $p$ and started from $z$, i.e. if $E_i$ is an adjacent edge to $v$, then from $v, Z$ “jumps” to $E_i$ with probability $p_{i,v}$. We have the following

**Proposition 2.1.** There exists a Brownian motion $W$ such that for all $f \in \mathcal{D}_p^G$, we have

$$f(Z_t) = f(z) + \int_0^t f'(Z_u) \, dW_u + \frac{1}{2} \int_0^t f''(Z_u) \, du.$$

**Proof.** We take $z = v$ a vertex point. We will define $W$ until $Z$ hits another vertex $v'$. The definition of $W$ should be clear then using the strong Markov property and the successive hitting times of $V$ by $Z$. Thus we can also assume that $G$ is a star graph (has only one vertex and eventually an infinite number of edges) which we embed in the complex plane just to
there exists a probability space $(\Omega, \mathcal{F}, P)$ and measurable functions $\phi_1, \phi_2 \colon [0, 2\pi] \to \mathbb{R}$ such that $P(\{\phi_1(\omega) = \phi_2(\omega)\}) = 1$ for all $\omega \in \Omega$. Then the map $\omega \mapsto (\phi_1(\omega), \phi_2(\omega))$ is an $\mathcal{F}$-measurable map from $\Omega$ to $\mathbb{R}^2$. 

2.2. Harmonic functions on Walsh’s Brownian motion

In Jehring (2009), Harmonic functions on Walsh’s Brownian motion are studied. The following lemma (Lemma 3.2 in Jehring (2009)) is the most technical point in proving Theorem 3.2 in Jehring (2009). Its proof uses the Excursion theory and non trivial calculations (see Appendix A in Jehring (2009)). We will derive it in few lines from Theorem 1.2.

Lemma 2.2. Let $h : E \to \mathbb{R}$ be of the form

$$h(\pi, \alpha) = m_h(\alpha) \cdot s + h(0)$$

for $\alpha \in E$, where the function $m_h : [0, 2\pi] \to \mathbb{R}$ is an integrable function with respect to $\mu$ and satisfies $\int_0^{2\pi} m_h(\alpha) \mu(d\alpha) = 0$. Then $h(Z_t, t \geq 0)$ is a martingale with respect to $(\mathcal{F}_s)_{s \geq 0}$ for every starting point $z \in E$.

Proof. By Theorem 1.2, using the function $h^N(\pi, \alpha) = (m_h(\alpha) \cdot N) \cdot s + h(0), N \geq 0$, we see that

$$h^N(Z_t) = h^N(z) + \int_0^t (m_h(\Theta_u) \cdot N) 1_{[\pi, \pi]} dB_u + \int_0^t (m_h(\alpha) \cdot N) \mu(d\alpha) \times L_t(R).$$

Thus for all $s \leq t$,

$$E[h^N(Z_t) | \mathcal{F}_s] = h^N(z) + \int_0^s (m_h(\Theta_u) \cdot N) 1_{[\pi, \pi]} dB_u + \int_0^s (m_h(\alpha) \cdot N) \mu(d\alpha) \times L_t(R)$$

$$h^N(Z_s) = \int_0^s (m_h(\alpha) \cdot N) \mu(d\alpha) \times L_t(R) + \int_0^s (m_h(\alpha) \cdot N) \mu(d\alpha) \times E[L_t(R) | \mathcal{F}_s].$$

Letting $N \to \infty$ and using dominated convergence (note that $E[|m_h(\Theta_t)| | Z_t] < \infty$ since $m_h(\Theta_t)$ and $Z_t$ are independent), we get $E[h(Z_t) | \mathcal{F}_s] = h(Z_s)$.

2.3. Stochastic flows in the plane

Let $Z$ be a Walsh’s Brownian motion started from 0 and define $W = B^2$. Then, we have $R_t = W_t - \min_{0 \leq u \leq t} W_u$ by Skorokhod Lemma (see Revuz and Yor (1999) page 239), $\sigma(R) = \sigma(W)$ and $\Theta_t$ is independent of $W$ for all $t > 0$. This situation is close to Tanaka’s equation: If $X$ satisfies

$$X_t = \int_0^t \text{sgn}(X_s) dW_s,$$

then $\sigma(|X|) = \sigma(W)$ and $\text{sgn}(X_t)$ is independent of $W$ for all $t > 0$. Theorem 2 allows thus to define the analogue of Tanaka’s equation in the plane. This problem has been considered from the stochastic flows view point in Hajiri (2011) for $\text{supp}(\mu)$ finite.

We say that $(\psi_{t, z}(z))_{0 \leq s \leq t, z \in E}$ is a stochastic flow of mappings (SFM) on $E$ as soon as $\psi_{t, z}$ is measurable with respect to $(\pi, \omega)$, $\psi_{t, z}$ and $\psi_{0, z}$ are equal in law, for any sequence $\{[t_i, t_i]\} \subseteq [0, 1]$, $1 \leq i \leq n$ of non-overlapping intervals the mappings $\psi_{t_i, t_i}$ are independent, and we have the flow property: for all $s \leq t \leq u$ and $z \in E$, a.s. $\psi_{s, u}(z) = \psi_{t, u} \circ \psi_{t, s}(z)$.

Definition 2.3. Let $\varphi$ be a SFM and $W$ be a Brownian motion. We say that $(\varphi, W)$ solves Tanaka’s equation with angle $\mu$ if for all $0 \leq s \leq t$, $z \in E$ and all $f \in D$ such that $\int_0^{2\pi} f'_u(0) \mu(d\alpha) = 0$, a.s.

$$\int_0^s (|z_{\pi, u}(z)| \varphi_{s, u}(z)) dB_u + \int_0^s (|z_{\pi, u}(z)| \varphi_{s, u}(z)) du = \frac{1}{2} \int_0^s (|z_{\pi, u}(z)| \varphi_{s, u}(z)) du$$

Following Le Jan and Raimond (2006), we will construct a SFM $\varphi$ solving this equation: by Kolmogorov extension theorem, there exists a probability space $(\Omega, \mathcal{F}, P)$ on which one can construct a process $(\Theta_{t, z}, W_{s, t})_{0 \leq s \leq t < \infty}$ taking values in $[0, 2\pi] \times \mathbb{R}$ such that (i)-(iv) below are satisfied

(i) $W_{s, t} := W_t - W_s$ for all $s \leq t$ and $W$ is a Brownian motion.

(ii) For fixed $s < t$, $\Theta_{s, t}$ and $W$ are independent and the law of $\Theta_{s, t}$ is $\mu$.
(iii) Define \( \min_{s,t} = \inf\{W_u; u \in [s, t]\} \). Then for all \( s < t \) and \( u < v \)
\[
\mathbb{P}(\Theta_{s,t} = \Theta_{u,v}|\min_{s,t} = \min_{u,v}) = 1.
\]
(iv) For all \( s < t \) and \( \{s_i, t_i\}; 1 \leq i \leq n \) with \( s_i < t_i \), the law of \( \Theta_{s,t} \) knowing \( (\Theta_{h_i,t_i})_{1 \leq i \leq n} \) and \( W \) is given by \( \mu \) on the event \( \min_{s,t} \notin \{\min_{h_i,t_i}; 1 \leq i \leq n\} \) and is given by \( \delta_{\Theta_{h_i,t_i}} \) on the event \( \{\min_{s,t} = \min_{h_i,t_i}\} \) with \( 1 \leq i \leq n \).

Note that (i)–(iv) uniquely define the law of \( (\Theta_{s_1,t_1}, \ldots, \Theta_{s_n,t_n}, W) \) for all \( s_i < t_i, 1 \leq i \leq n \). This family of laws is consistent by construction.

For \( 0 \leq s \leq t, z \in E \), define
\[
\tau_z(u) = \inf\{r \geq s: W_{\alpha,r} = -|z|\}
\]
and
\[
\varphi_z(u) = (|z| + W_{\alpha,t}, \arg(z))1_{[t \leq \tau_z(u)]} + (W_t - \min_{s,t}, \Theta_{s,t})1_{[t > \tau_z(u)]}.
\]

Following Le Jan and Raimond (2006), one can prove that \( \varphi \) is a SFM such that \( (\varphi, W) \) satisfies (5). It is also possible to extend (5) to stochastic flows of kernels as in Le Jan and Raimond (2006). A stochastic flow of kernels (SFK) \( K = (K_{s,t}; s \leq t) \) is the same as a SFM, but the mappings are replaced by kernels, and the flow property being now that for all \( z \in E, s \leq t \leq u, a.s. \)
\[
K_{s,u}(x) = K_{s,t}K_{t,u}(x).
\]

For example
\[
K_{s,t}^W(z) = E[\delta_{\Theta_{s,t}(z)}|W] = \delta(|z| + W_{\alpha,t}, \arg(z))1_{[t \leq \tau_z(u)]} + \int_0^{2\pi} \delta(W_t - \min_{s,t}, \alpha)\mu(d\alpha)1_{[t > \tau_z(u)]}
\]
is a SFK. By conditioning with respect to \( W \) in (5), we see that \( (K^W, W) \) is solution of the following equation.

**Definition 2.4.** Let \( K \) be a SFK and \( W \) be a Brownian motion. We say that \( (K, W) \) solves Tanaka’s equation with angle \( \mu \) if for all \( 0 \leq s \leq t, z \in E \) and all \( f \in \mathcal{D} \) such that \( \int_0^{2\pi} f_t'(0+)\mu(d\alpha) = 0 \), a.s.
\[
K_{s,t}(f) = f(z) + \int_s^t K_{s,u}(f'1_{\mathbb{R}^2\setminus\{0\}})(z)dW_u + \frac{1}{2} \int_s^t K_{s,u}(f''1_{\mathbb{R}^2\setminus\{0\}})(z)du.
\]

Note that, now we have two solutions to (6): \((\delta_{\varphi}, W)\) and \((K^W, W)\). In Hajri (2011), when \( \text{supp}(\mu) \) is finite a complete description of the laws of all SFK’s \( K \) solutions of (6) is given under some further assumptions on stochastic flows (see the definitions in Le Jan and Raimond (2004)). Under these assumptions, it should be possible to prove in our setting that \( \varphi \) is the law-unique SFM solution of (5), \( K^W \) is the unique Wiener (i.e. \( \sigma(W) \)-measurable) flow solution of (6) and to give a complete classification of solutions to (6).

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**References**


