An inverse to the antisymmetrization map of Cartan & Eilenberg

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Abstract

The first section of this paper gives the construction of an explicit contracting homotopy of the Chevalley-Eilenberg resolution of any Lie algebra $\mathfrak{g}$ over a ring containing the field of rational numbers. In the second section, this homotopy is used to define a functorial quasi-inverse to the antisymmetrization map of Cartan and Eilenberg. More precisely, it is shown that the Chevalley-Eilenberg (co)chain complex of $\mathfrak{g}$ is a deformation retract of the Hochschild (co)chain complex of its universal enveloping algebra.

Notations

1 A contracting homotopy for the Chevalley-Eilenberg resolution

1.1 Convolution, Cofree coalgebras and coderivations

Definition 1.1.1. Let $(A,\mu)$ be an graded algebra and $(C,\Delta)$ be a graded coalgebra in the category of graded modules over some commutative graded ring $U$ equipped with the graded tensor product $\otimes_U$. Then the graded module $\text{Hom}_U,gr(C,A)$ can be endowed with a graded associative composition product $\star$, called convolution product, defined by $f \star g := \mu \circ (f \otimes g) \circ \Delta$ for all $f$ and $g$ in $\text{Hom}_U(C,A)$.

Let $R$ be a commutative ring, $(C,\epsilon)$ be a cocommutative counital coalgebra in the category of graded $R$-modules, and $V$ a graded $R$-submodule of $C$. Denote by $\Delta : C \rightarrow C \otimes_R C$ the coproduct of $C$, $\epsilon : C \rightarrow R$ its counit.

Definition 1.1.2. \cite{Qui69} $C$ is said to be connected if there exists morphism of coalgebras $\eta : R \rightarrow C$ such that $\epsilon \eta = \text{Id}_R$ and $\overline{C} := C/\text{Im} \eta$ is a conilpotent coalgebra.

Assume that $C$ is connected. $C$ is said to be cofreely generated by $V$ if there exists a morphism of graded $R$-modules $p : C \rightarrow V$ such that for every connected graded $R$-coalgebra $D$ and every morphism of graded $R$-modules $\overline{f} : D \rightarrow V$, there exists a unique morphism of coalgebras $f : D \rightarrow C$ such that the following diagramm

$$
\begin{array}{ccc}
D & \rightarrow & C \\
\downarrow f & & \downarrow p \\
 & V & \\
\end{array}
$$

commutes.

Denote by $|x|$ the degree of an homogenous element in $V$.

Proposition 1.1.3. \cite{Qui69}, appendix B. Let $V$ be a graded $R$-module.

- Two cofree connected cocommutative coalgebras cogenerated by $V$ are isomorphic.

- Moreover, one of them is given by the connected cocommutative coalgebra $\text{proj} : S,V \rightarrow V$, where $S,V$ is the quotient of the (graded) tensor algebra $T,V := \bigoplus_{n \geq 0} V \otimes_R V$ by the ideal generated by relations of the form $x \otimes y - (\epsilon (x) \otimes y) - (x \otimes \epsilon (y))$. $S,V$ is a graded commutative algebra and can be equipped with a graded cocommutative coproduct $\Delta : S,V \rightarrow S,V \otimes_R S,V$ turning it into a Hopf algebra such that every element in $V \subset S,V$ is primitive. The projection morphism $\text{proj} : S,V \rightarrow V$ is induced by the canonical projection of $TV$ on its length 1 term.

- In particular, the unique morphism of coalgebras $f : D \rightarrow C$ lifting a given linear map $\overline{f} : D \rightarrow V$, where $D$ is any connected cocommutative coalgebra, can be defined thanks to convolution in $\text{Hom}_R(D,S,V)$ (see 1.1.4) via $f := \exp_1(\overline{f}) := \sum_{n \geq 0} \frac{1}{n!} \overline{f}^n$ with $\overline{f}^0 := \eta \epsilon$.


Definition 1.1.4. Let \((C, \Delta)\) be a graded coalgebra, and \(\phi : C \to C\) be an endomorphism of coalgebra. A **coderivation of \(C\) along \(\phi\)** is a morphism \(d : C \to C\) such that

\[
\Delta \circ d = (\phi \otimes d + d \otimes \phi) \circ \Delta
\]

When \(\phi = \text{Id}_C\), we simply say that \(d\) is a coderivation.

**Proposition 1.1.5.** [(Qui69], appendix B.] Let \(\bar{d} : S_* V \to V\) be a graded R-linear map. Then

- There exists a unique coderivation \(d : S_* V \to S_* V\) along \(\phi\) such that \(\bar{d} = \text{proj} \circ d\).
- \(d\) is given by \(d := \bar{d} \ast \phi\).

**Proposition 1.1.6.** Let \(\phi : C \to C\) and \(\psi : C \to C\) be two coalgebra endomorphisms of a given graded R-coalgebra \(C\), and \(d\) (resp. \(D\)) be a coderivation of \(C\) along \(\phi\) (resp. along \(\psi\)). Then

- \(\psi \circ d = D\) is a coderivation of \(C\) along \(\psi \circ \phi\).
- Suppose that \(\phi \circ \psi = \psi \circ \phi\). Then the graded bracket

\[
[d, D] := d \circ D - (-1)^{|d||D|} D \circ d
\]

is a coderivation of \(C\) along \(\phi \circ \psi\).

### 1.2 The Chevalley-Eilenberg resolution

Let \(L\) be a Lie algebra over some commutative ring \(R\) of characteristic 0 with Lie bracket \([- , -] : L \otimes_R L \to L\) (Here \(\Lambda_R\) stands for the exterior product of \(R\)-modules). Denote by \(UL\) its universal enveloping algebra, that is the algebra obtained by quotienting the tensor algebra \(TL := \oplus_{n \geq 0} L \otimes_R L^n\) by the ideal generated by relations of the form \(g \otimes g' - g' \otimes g - [g, g']\) when \(g\) and \(g'\) run over \(L\). The product of two elements \(x\) and \(y\) of \(UL\) will be written \(xy\). Recall that \(UL\) can be endowed with

- a comultiplication \(\Delta : UL \to UL \otimes_R UL\) determined by saying that every element of \(L \subset UL\) is primitive,
- a counit \(\epsilon : UL \to R\) and a unit \(\eta : R \to UL\), both induced by the canonical ones of \(TL\),
- an antipode \(S : UL \to UL\) which is the only algebra antimorphism such that \(S(g) = -g\) for all \(g\) in \(L\),

turning it into Hopf algebra (for a brief account on the Hopf algebra structure on \(UL\), one can for instance consult [Kas95]). Following [Lod94], this Hopf algebra structure gives rise to a convolution product \(*\otimes\) on \(End_R(UL)\), the \(R\)-module of linear endomorphism of \(UL\), such that

\[
f * h := \mu(f \otimes h)\Delta
\]

for all \(f\) and \(h\) in \(End_R(UL)\), where \(\mu\) denotes the associative product of \(UL\).

**Definition 1.2.1.** [Lod94, Lod98, Ren93, Lod98] The **first eulerian idempotent** of \(L\) is the \(R\)-linear endomorphism \(pr : UL \to UL\) defined by

\[
pr := \sum_{i \geq 0} \frac{(-1)^i}{i + 1} (\text{Id} - \eta)^i + 1
\]

**Theorem 1.2.2.** [Poincaré-Birkhoff-Witt] The first eulerian idempotent \(pr\) takes its values in \(L\). Moreover, \((UL, pr : UL \to UL)\) is a cofree connected cocommutative coalgebra cogenerated by the \(R\)-module \(L\).

**Proof.** The fact that \(pr\) takes its values in \(L\) is proved for instance in [Ren93]. To our knowledge, Quillen was the first to notice in [Qui69] that the symmetrization map

\[
sym : SL \to UL
\]

sending a monomial \(g_1 \cdots g_n\) in \(S^n L\) to its symmetrization \(\sum_{\sigma \in \Sigma_n} g_{\sigma(1)} \cdots g_{\sigma(n)}\) in \(UL\) is an isomorphism of cocommutative coalgebras, which has for immediate consequence that \(UL\) is cofree cogenerated by \(L\). The fact that the projection of \(UL\) on its cogenerators is given by the first eulerian idempotent \(pr\), defined this time as the multilinear part of the BCH formula, is established in [Ren93] in the case when \(L\) is a free Lie algebra (which implies the general case) and an explicit formula for it is given. More general formulas are given in [Hei89] and the definition of \(pr\) in terms of convolution seems to appear in [Lod94] for the first time (see also [Lod98]). A more general formulation of the universality of the eulerian idempotent in the framework of triple of operads is developed in [Lod08].

We give here a self-contained proof of theorem 1.2.2 mainly based on ideas present in [Hei89] and [Lod94].

Suppose that \(L = \text{Lie}(V)\) is the free Lie algebra generated by a \(K\)-module \(V\). Then, using the universal property characterizing \(UL\), one sees that \(UL = TV\), the free associative algebra generated by \(V\), which is endowed with the shuffle coproduct \(\Delta : TV \to TV \otimes TV\) turning it into a Hopf algebra.

Clearly, the Lie subalgebra \(\text{Prim}(TV)\) of primitive elements of \(TV\) satisfies

\[
\text{Lie}(V) \subset \text{Prim}(TV)
\]
The reverse inclusion also holds: this is Friedrichs’ theorem, a short proof of which can be found in [Wig89]. Thus, the inclusion of \( \text{Lie}(V) \) in \( TV \) factors through an isomorphism on \( \text{Prim}(TV) \):

\[
\text{Lie}(V) = \text{Prim}(TV) \subset TV
\]

But clearly

\[
(\text{Id} - \eta \epsilon)^k(x) = 0 \quad k \geq 2
\]

for any primitive element \( x \) in \( TV \), which implies that \( pr \) is the identity on \( \text{Lie}(V) \). Moreover, \( pr \) is a coderivation along \( \eta \epsilon \), i.e.

\[
\Delta \circ pr = (pr \otimes \eta \epsilon + \eta \epsilon \otimes pr) \circ \Delta
\]

which shows that

\[ \square \]

**Proposition 1.2.3.** [Eulerian idempotents] For all \( k \) and \( l \) in \( \mathbb{N} \)

\[
\frac{1}{k!l!} p^*r^k \circ pr^l = \begin{cases} 
\frac{1}{i!} p^*r^i & \text{if } k = l \\
0 & \text{if } k \neq l
\end{cases}
\]

Notice that \( L \) can be seen as a graded \( R \)-module concentrated in degree 0. When \( V = \{V_i\}_{i \geq 0} \) is a graded module, denote by \( V[1] \) the shifted module whose degree \( i \) component is \( V[1]_i := V_{i-1} \).

**Definition 1.2.4.** The Chevalley-Eilenberg resolution of \( L \) is the chain complex of \( R \)-modules \( C_*(L) := UL \otimes_R SL[1] \) with differential \( d : C_*(L) \to C_{*-1}(L) \) of degree \(-1\) defined by

\[
d(x \otimes g_1 \wedge \ldots \wedge g_n) := x \sum_{i=1}^n (-1)^{i+1} g_i \otimes g_1 \wedge \ldots \wedge \hat{g}_i \wedge \ldots \wedge g_n \\
+ \sum_{i < j} (-1)^{i+j} x \otimes g_i \wedge \ldots \wedge [g_i, g_j] \wedge \ldots \wedge g_n
\]

for all \( x \) in \( UL \) and \( g_1, \ldots, g_n \) in \( L \), where \( \hat{g}_i \) means that \( g_i \) has been omitted.

**Remark 1.2.5.** When \( V = \{V_n\}_{n \geq 0} \) is a graded module concentrated in degree \( 1 \), we will always identify \( S_n V \) with the \( n \)-th exterior power \( \Lambda^n V_1 \).

**Proposition 1.2.6.** Define \( \text{PR} : C_*(L) \to L \otimes_R R \oplus R \otimes_R L[1] \cong L \oplus L[1] \) by

\[
\text{PR} := pr \otimes \epsilon + \epsilon \otimes \text{proj}
\]

Then \( (C_*(L), \text{PR}) \) is a cofree cocommutative connected (graded) coalgebra generated by \( L \oplus L[1] \). Moreover, the differential \( d \) is the unique coderivation generated by

\[
d : C_*(L) \to L \oplus L[1] \\
x \otimes y \mapsto pr(x \text{proj}y) + \epsilon(x)B(y)
\]

for all \( x \) in \( UL \) and \( y \) in \( SL[1] \), where \( B : SL[1] \to L[1] \) coincides with the Lie bracket in degree \( 2 \) and is zero elsewhere.

Let \( g \) be a Lie algebra over a commutative ring \( K \) containing \( \mathbb{Q} \), and denote by \( g[t] \) the \( K[t] \)-Lie algebra \( g \otimes_K K[t] \). An element of \( g[t] \) is just a polynomial expression in \( t \) with coefficients in \( g \). We have obvious isomorphisms

\[
U(g[t]) \cong U\ell g[t] := Ug \otimes_K K[t]
\]

and

\[
C_*(g[t]) \cong C_*(g)[t] := C_*(g) \otimes_K K[t]
\]

Moreover, “formal integration on \([0, 1]\)” gives a \( K \)-linear map \( I_{[0,1]} : K[t] \to K \), sending each \( t^n \) to \( \frac{1}{n+1} \), providing a morphism of chain complexes

\[
I := \text{Id} \otimes I_{[0,1]} : C_*(g)[t] \to C_*(g)
\]

which behaves with respect to “formal derivation” \( \frac{d}{dt} : t^n \mapsto nt^{n-1} \) as in the usual real case. The inclusion \( K \subset K[t] \) induces an inclusion of chain complexes

\[
C_*(g) \hookrightarrow C_*(g)[t]
\]

Given a \( K \)-module \( V \), \( V[t] \) will always denote the \( K[t] \)-module \( V \otimes K[t] \), and \( K[t] \)-linear morphism from \( V[t] \) to some other \( K[t] \)-module will always be defined on \( V \) and extended to \( V[t] \) by linearity. Note that all previous considerations can be easily generalized to the case when one replaces \( K[t] \) by \( K[t_1, t_2, \ldots, t_n] \), the algebra of polynomials in \( n \) indeterminates \( t_1, t_2, \ldots, t_n \). From sequel, \( \otimes \) will always mean \( \otimes_K \).
Notation 1.2.7. We’ll make an intensive use of Sweedler’s notation to write iterated comultiplications in cocommutative coalgebras:

\[ \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes \cdots \otimes x^{(n)} \]

will stand for

\[ (\Delta \otimes \text{Id}^{\otimes (n-2)}) \circ (\Delta \otimes \text{Id}^{\otimes (n-3)}) \circ \cdots \circ (\Delta \otimes \text{Id}) \circ \Delta (x) \]

Definition 1.2.8. Define two \( \mathbb{K}[t] \)-linear maps \( \phi_t : U[\mathfrak{g}] \to U[\mathfrak{g}] \) and \( A_t : U \otimes [\mathfrak{g}] \to U[\mathfrak{g}] \) by

\[ \phi_t := \sum_{k \geq 0} \frac{t^k}{k!} \text{pr}^k \]

and

\[ A_t(x, g) := A_t(x \otimes g) := \sum_{(x)} \phi_{-1}(x^{(1)}) \phi_t(x^{(2)} g) \]

for all \( x \) in \( U[\mathfrak{g}] \) and \( g \) in \( \mathfrak{g} \).

Proposition 1.2.9. • As endomorphisms of \( U[\mathfrak{g}][t_1, t_2] := U[\mathfrak{g}] \otimes_{\mathbb{K}[t_1, t_2]} \mathbb{K}[t_1, t_2] \):

\[ \phi_{t_1} \circ \phi_{t_2} = \phi_{t_1 t_2} \]

and

\[ \phi_{t_1} \ast \phi_{t_2} = \phi_{t_1 + t_2} \]

• \( A_t \) takes its values in \( \mathfrak{g}[t] \) i.e.

\[ A_t(x, g) \in \mathfrak{g}[t] \]

for all \( x \) in \( U[\mathfrak{g}] \) and \( g \) in \( \mathfrak{g} \).

• \( \frac{d}{dt} = \phi_t \ast \text{pr} \) as a \( \mathbb{K}[t] \)-linear endomorphism of \( U[\mathfrak{g}][t] \).

Definition 1.2.10. Define \( \mathbb{K}[t] \)-linear morphisms of graded modules \( a_t : C^*_s(\mathfrak{g})[t] \to C^*_s(\mathfrak{g})[t] \) and \( b_t : C^*_s(\mathfrak{g})[t] \to C^*_{s+1}(\mathfrak{g})[t] \) by

\[ a_t(x \otimes g_1 \wedge \cdots \wedge g_n) := \sum_{(x)} \phi_t(x^{(1)}) \otimes A_t(x^{(2)} \wedge g_1) \wedge \cdots \wedge A_t(x^{(n+1)}, g_n) \]

and

\[ b_t(x \otimes g_1 \wedge \cdots \wedge g_n) := \sum_{(x)} \phi_t(x^{(1)}) \otimes \text{pr}(x^{(2)}) \wedge A_t(x^{(3)} \wedge g_1) \wedge \cdots \wedge A_t(x^{(n+2)}, g_n) \]

Proposition 1.2.11. \( a_t \) is an endomorphism of coalgebra and \( b_t \) is a degree 1 coderivation of \( C^*_s(\mathfrak{g})[t] \) along \( a_t \).

The following theorem implies that the Chevalley-Eilenberg resolution is indeed a resolution:

Theorem 1.2.12. The degree 1 \( \mathbb{K} \)-linear map \( s : C^*_s(\mathfrak{g}) \to C^*_{s+1}(\mathfrak{g}) \) defined by

\[ s := I \circ b_t \]

is a contracting homotopy of the chain complex \((C^*_s(\mathfrak{g}), d)\).

Proof. The theorem is a direct consequence of the three following facts:

• \( \frac{d}{dt} a_t \) is a coderivation along \( a_t \): Proposition 1.2.11 asserts that \( a_t \) is a coalgebra endomorphism i.e.

\[ \Delta a_t = (a_t \otimes a_t) \Delta \]

Thus

\[ \Delta \frac{d}{dt} a_t = \frac{d}{dt} \Delta a_t = \frac{d}{dt} (a_t \otimes a_t) \Delta = \left( \frac{d}{dt} a_t \otimes a_t + a_t \otimes \frac{d}{dt} a_t \right) \Delta \]

which exactly means that \( \frac{d}{dt} a_t \) is a coderivation along \( a_t \).

• Proposition 1.2.11 (resp. 1.2.6) tells us that \( b_t \) (resp. \( d \)) is a coderivation along \( a_t \) (resp. the identity map of \( C^*_s(\mathfrak{g})[t] \)). By proposition 1.1.6, since the identity map obviously commutes with \( a_t \), the graded bracket \([d, b_t] = db_t + b_t d\) is a coderivation along \( a_t \).

• The two preceding coderivations are equal:

\[ db_t + b_t d = \frac{d}{dt} a_t \]

As both sides of this equation are coderivations along \( a_t \), propositions 1.1.5 and 1.2.6 imply that all we need to check is whether their postcompositions by PR are equal. Since PR vanishes on \( U[\mathfrak{g}] \otimes S_{\geq 2}[\mathfrak{g}][1] \), we can restrict to length lower than 2. Let \( x \) be an element of \( U[\mathfrak{g}] \) and \( g \) be in \( \mathfrak{g} \):

\[ (db_t + b_t d)(x) = \sum_{(x)} \phi_t(x^{(1)}) \text{pr}(x^{(2)}) = \phi_t \ast \text{pr}(x) \]
But the last point of proposition 1.2.9 tells us that \( \frac{d}{dt} \phi_t = \phi_t \ast \text{pr} \) so that

\[
\text{PR}(d_b + b_t d) = \text{pr}(\frac{d}{dt} \phi_t) = \text{PR} \frac{d}{dt} a_t(x)
\]

which proves that 1 holds in length 0. For length 1, we have, thanks to the cocommutativity of the coproduct and the properties of \( \phi_t \) listed in proposition 1.2.9:

\[
(d_b + b_t d)(x \otimes g) = \sum_{(x)} \phi_t(x^{(1)}) \text{pr}(x^{(2)}) \otimes A_t(x^{(3)}, g) - \phi_t(x^{(1)}) A_t(x^{(2)}, g) \otimes \text{pr}(x^{(3)})
\]

\[
- \phi_t(x^{(1)}) \otimes [\text{pr}(x^{(2)}), A_t(x^{(3)}, g)] + \sum_{(xg)} \phi_t((xg)^{(1)}) \otimes \text{pr}((xg)^{(2)})
\]

\[
= \sum_{(x)} \frac{d}{dt} \phi_t(x^{(1)}) \otimes A_t(x^{(2)}, g) + \phi_t(x^{(1)}) \otimes \text{pr}(x^{(2)}) - \sum_{(x)} \phi_t(x^{(1)}) \otimes [\text{pr}(x^{(2)}), A_t(x^{(3)}, g)]
\]

But for any \( y \) in \( Ug \)

\[
\frac{d}{dt} A_t(y, g) = - \sum_{(y)} \text{pr}(y^{(1)}) A_t(y^{(2)}, g) + \sum_{(yt)} \phi_t(y^{(1)}) \phi_t((yg)^{(2)}) \text{pr}((yg)^{(3)})
\]

\[
= - \sum_{(y)} [\text{pr}(y^{(1)}), A_t(y^{(2)}, g)] + \text{pr}(yg)
\]

Thus

\[
(d_b + b_t d)(x \otimes g) = \sum_{(x)} \frac{d}{dt} \phi_t(x^{(1)}) \otimes A_t(x^{(2)}, g) + \phi_t(x^{(1)}) \otimes \frac{d}{dt} A_t(x^{(2)}, g)
\]

\[
= \frac{d}{dt} a_t(x \otimes g)
\]

which obviously implies the desired equality by applying PR.

Finally, we have

\[
sd + ds = I(b_t d + db_t) = I \frac{d}{dt} a_t = a_1 - a_0 = \text{Id}_{C_*(g)}
\]

on \( C_*(g) \subset C_*(g)[t] \).

1.3 The Koszul resolution

The Chevalley-Eilenberg resolution of \( Ug \) enables one to build a new chain-complex, this time consisting of \( Ug \)-bimodules:

**Definition 1.3.1.** The **Koszul resolution** of \( Ug \) is the complex of \( Ug \)-bimodules \( CK_*(g) \) defined by

\[
CK_*(g) := Ug \otimes S_*(g) [1] \otimes Ug
\]

with differential \( d^K : CK_*(g) \rightarrow CK_{*-1}(g) \) defined by

\[
d^K(1 \otimes g_1 \wedge \cdots \wedge g_n \otimes 1) := \sum_{i=1}^n (-1)^{i+1} (g_i \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_n \otimes 1 - 1 \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_n \otimes g_i)
\]

\[
+ \sum_{1 \leq i < j \leq n} (-1)^{i+j+1} 1 \otimes g_1 \wedge \cdots \wedge [g_i, g_j] \wedge \cdots \wedge g_n \otimes 1
\]

for all \( g_1, g_2, \ldots, g_n \) in \( g \).

**Proposition 1.3.2.** The degree +1 map \( h : CK_*(g) \rightarrow CK_{*+1}(g) \) defined in degree \( n \) by

\[
h(x \otimes g_1 \wedge \cdots \wedge g_n \otimes y) := \sum_{(x)} \int_0^1 dt \phi_t(x^{(1)}) \otimes \text{pr}(x^{(2)}) \wedge A_t(x^{(3)}, g_1) \wedge \cdots \wedge A_t(x^{(n+2)}, g_n) \otimes \phi_{t-1}(x^{(n+3)})y
\]

for all \( x, y \) in \( Ug \) and \( g_1, g_2, \ldots, g_n \) in \( g \), is a contracting homotopy.

As a corollary, we recover the well known following fact (at least when \( g \) is free over \( K \)):

**Corollary 1.3.3.** If \( g \) is projective over \( K \), the Koszul resolution of \( Ug \) is a projective resolution of the \( Ug \)-bimodule \( Ug \) via the product map

\[
CK_0(g) = Ug^{\otimes 2} \rightarrow Ug \quad x \otimes y \mapsto xy
\]
2 An inverse to the antisymmetrisation map

In this section, we drop the symbol \( \sum \) in Sweedler’s notation of iterated coproducts so that
\[
x^{(1)} \otimes \cdots \otimes x^{(n)}
\]
will stand for
\[
\sum_x x^{(1)} \otimes \cdots \otimes x^{(n)}
\]

2.1 The antisymmetrisation morphism \( F_* \)

**Definition 2.1.1.** The bar resolution of \( U\mathfrak{g} \) is the complex of \( U\mathfrak{g} \)-bimodules \( B_*(U\mathfrak{g}) \) defined in degree \( n \) by
\[
B_n(U\mathfrak{g}) := U\mathfrak{g} \otimes U\mathfrak{g}^\otimes n \otimes U\mathfrak{g}
\]
with differential \( d^B : B_*(U\mathfrak{g}) \to B_{*+1}(U\mathfrak{g}) \) defined by
\[
d^B(a < x_1 | \cdots | x_n > b) := a x_1 < x_2 | \cdots | x_n > b + \sum_{i=1}^{n-1} (-1)^i a < x_1 | \cdots | x_ix_{i+1} | \cdots | x_n > b
\]
for all \( a, b, x_1, \ldots, x_n \) in \( U\mathfrak{g} \). The notation \( a < x_1 | \cdots | x_n > b \) stands for the element \( a \otimes x_1 \otimes \cdots \otimes x_n \otimes b \) in \( B_n(U\mathfrak{g}) = U\mathfrak{g}^\otimes (n+2) \) and \( 1 < x_1 | \cdots | x_n > 1 \) will be abbreviated in \( < x_1 | \cdots | x_n > \) in the sequel.

**Proposition 2.1.2.** If \( \mathfrak{g} \) is projective over \( \mathbb{K} \), the bar resolution defined above is a projective resolution of the \( U\mathfrak{g} \)-bimodule \( U\mathfrak{g} \) via the same map as \( CK_*(\mathfrak{g}) \).

**Definition 2.1.3.** The antisymmetrisation map \( F_* : CK_*(\mathfrak{g}) \to B_*(U\mathfrak{g}) \) is the morphism of graded \( U\mathfrak{g} \)-bimodules defined in degree \( n \) by
\[
F_n(1 \otimes g_1 \wedge \cdots \wedge g_n \otimes 1) := \sum_{\sigma \in S_n} \text{sgn}(\sigma) < g_{\sigma(1)} | \cdots | g_{\sigma(n)} >
\]
for all \( g_1, \ldots, g_n \) in \( \mathfrak{g} \), where \( S_n \) denotes the \( n \)-th symmetric group and \( \text{sgn}(\sigma) \) stands for the signature of a permutation \( \sigma \).

**Theorem 2.1.4.** [Cartan-Eilenberg] Suppose that \( \mathfrak{g} \) is projective over \( \mathbb{K} \). Then, the antisymmetrisation map \( F_* : CK_*(\mathfrak{g}) \to B_*(U\mathfrak{g}) \) defined above is a morphism of projective resolutions of the \( U\mathfrak{g} \)-bimodule \( U\mathfrak{g} \) over the identity map \( \text{Id}_{U\mathfrak{g}} : U\mathfrak{g} \to U\mathfrak{g} \).

Denoting by \( U\mathfrak{g}^{op} \) the opposite algebra of \( U\mathfrak{g} \), this implies that

**Corollary 2.1.5.** For any \( U\mathfrak{g} \)-bimodule \( M \), the map \( \text{Id}_M \otimes F_* : M \otimes t\mathfrak{g} \otimes t\mathfrak{g}^{op} \to CK_*(\mathfrak{g}) \to M \otimes t\mathfrak{g} \otimes t\mathfrak{g}^{op} B_*(U\mathfrak{g}) \) is an homotopy equivalence of chain complexes.

2.2 Building a quasi-inverse to \( F_* \)

**Definition 2.2.1.** Let \( G_* : B_*(U\mathfrak{g}) \to CK_*(\mathfrak{g}) \) be the unique morphism of \( U\mathfrak{g} \)-bimodules defined by induction on the homological degree via
\[
G_0 := \text{Id} : B_0(U\mathfrak{g}) = U\mathfrak{g}^\otimes 2 \to CK_0(\mathfrak{g}) = U\mathfrak{g}^\otimes 2
\]
and
\[
G_n(1 < x_1 | \cdots | x_n > 1) := hG_{n-1}d^B(1 < x_1 | \cdots | x_n > 1) , \quad n > 0
\]
for all \( x_1, \ldots, x_n \) in \( U\mathfrak{g} \).

**Proposition 2.2.2.** The map \( G_* : B_*(U\mathfrak{g}) \to CK_*(\mathfrak{g}) \) defined above is a morphism of resolutions of \( U\mathfrak{g} \) over the identity map \( \text{Id}_{U\mathfrak{g}} : U\mathfrak{g} \to U\mathfrak{g} \).

Thanks to the explicit formula defining \( h \), one can get rid of the induction in definition 2.2.1.

**Theorem 2.2.3.** The morphism of resolutions \( G_* : B_*(U\mathfrak{g}) \to CK_*(\mathfrak{g}) \) defined above satisfies
\[
G_n(< x_1 | \cdots | x_n >) = \int_{[0,1]^n} dt_1 \cdots dt_n \sum \Gamma_n(x^{(1)}_1, \ldots, x^{(1)}_n) \otimes B^1_{n}(x^{(2)}_1, \ldots, x^{(2)}_n) \wedge \cdots \wedge B^n_{n}(x^{(n+1)}_1, x^{(n+1)}_n) \otimes \text{ST}_{n}(x^{(n+2)}_1, x^{(n+2)}_n) x^{(n+3)}_1 \cdots x^{(n+3)}_n
\]
for all \( x_1, \ldots, x_n \) in \( U\mathfrak{g} \). Here, \( \Gamma_n : U\mathfrak{g}^\otimes n \to U\mathfrak{g}[t_1, \ldots, t_n] \) and \( B^i_{n} : U\mathfrak{g}^\otimes n \to U\mathfrak{g}[t_1, \ldots, t_n] \), \( 1 \leq i \leq n \) are the operators defined by
\[
\Gamma_n(y_1, \ldots, y_n) := \phi_{t_1}(y_1) \phi_{t_2}(y_2) \phi_{t_3}(y_3) \cdots \phi_{t_n}(y_n) \cdots)
\]
and
\[ B_n^i := \Gamma_n(y_1^{(1)}, \ldots, y_n^{(1)}) \frac{d\Gamma_n}{dt_i}(y_1^{(2)}, \ldots, y_n^{(2)}) \]
for all \( y_1, \ldots, y_n \) in \( U_g \).

**Proof.** Define \( \hat{G}_n \) to be the \( \mathbb{K}[t_1, \ldots, t_n] \)-linear map equal to the integrand under \( \int_{[0,1]} dt_1 \cdots dt_n \) of the right-hand side of (3). We have to prove that for all \( n, \) \( G_n = \int_{[0,1]} dt_1 \cdots dt_n \hat{G}_n. \) Since both are bimodule maps that coincide in degree zero, we only have to check that the \( \int_{[0,1]} dt_1 \cdots dt_n \hat{G}_n \)'s satisfy the induction relation (2) on elementary tensors of the form \( < x_1 | \cdots | x_n >. \)

**Lemma 2.2.4.** For every \( n \geq 0 \) and \( y_1, \ldots, y_n \) in \( U_g \),
\[ h \hat{G}_n(y_1, \ldots, y_n) = 0 \]
where it is understood that \( h \) has been extended to \( C_\infty(g)[t_1, \ldots, t_n] \) by \( \mathbb{K}[t_1, \ldots, t_n] \)-linearity.

**Proof of lemma 2.2.4.** Let \( h_t : C_\infty(g)[t_1, \ldots, t_n] \to C_\infty(g)[t, t_1, \ldots, t_n] \) be the \( \mathbb{K}[t_1, \ldots, t_n] \) linear map defined by
\[ h_t(a \otimes g_1 \wedge \cdots \wedge g_n \otimes b) := \phi_t(a^{(1)}) \otimes \Gamma_t(\phi_t(a^{(2)})) \wedge \cdots \wedge \Gamma_t(\phi_t(a^{(n+2)})) \otimes \phi_t^{-1}(\phi_t(a^{(n+3)}))b \]
for all \( a, b \) in \( U_g \) and \( g_1, \ldots, g_n \) in \( g \), so that
\[ h \hat{G}_n = \int_0^1 dt h_t \hat{G}_n \]
on \( C_\infty(g)[t_1, \ldots, t_n] \). We have
\[ h_t \hat{G}_n(< y_1 | \cdots | y_n >) = h_t \left( \Gamma_n(y_1^{(1)}, \ldots, y_n^{(1)}) \otimes B_n(y_1^{(2)}, \ldots, y_n^{(2)}) \wedge \cdots \wedge B_n(y_1^{(n+1)}, y_n^{(n+1)}) \otimes ST_n(y_1^{(n+2)}, y_n^{(n+2)}) \right) \]
\[ = \phi_t(\Gamma_n(y_1^{(1)}, \ldots, y_n^{(1)})) \otimes \phi_t(\Gamma_t(y_1^{(2)})) \wedge \cdots \wedge \phi_t(\Gamma_t(y_1^{(n+2)})) \otimes \phi_t^{-1}(\phi_t(a^{(n+3)}))ST_n(y_1^{(n+4)}, y_n^{(n+4)}) \]
But for all \( z_1, \ldots, z_n \) in \( U_g \), the identities of proposition 1.2.3 imply that
\[ \phi_t((\Gamma_n(z_1^{(1)}, \ldots, z_n^{(1)}), B_n(z_1^{(2)}, \ldots, z_n^{(2)})) = h_t \left( \Gamma_n(z_1^{(1)}, \ldots, z_n^{(1)}) \otimes B_n(z_1^{(2)}, \ldots, z_n^{(2)}) \right) \]
Thus, we see that by cocommutativity of the coproduct of \( U_q \), \( h_t \hat{G}_n(< y_1 | \cdots | y_n >) \) is invariant under the transposition that exchanges its first and second wedge factors, which implies that it must be zero.

We are now ready to prove that the \( \int_{[0,1]} dt_1 \cdots dt_n \hat{G}_n \)'s satisfy the induction relation (2). Indeed, the preceding lemma implies that all terms but the first of \( d^\delta (< x_1, \ldots, x_n >) = x_1 < x_2 | \cdots | x_n > + \cdots \) are sent to zero under \( h_t \hat{G}_n \). Writing \( \hat{G}_n, \Gamma_n \) and \( B_n \) for the operators \( \hat{G}_n, \Gamma_n \) and \( B_n \) where the variables \( t_1, \ldots, t_n \) have been changed to \( t_2, \ldots, t_n, t_1 \), one gets
\[ h_t \hat{G}_n^{-1}d^\delta(< x_1 | \cdots | x_n >) = h_t \hat{G}_n^{-1}(x_1 < x_2 | \cdots | x_n >) \]
\[ = \phi_t((x_1^{(1)}, \Gamma_n^{-1}(x_2^{(1)}, \ldots, x_n^{(1)})) \otimes (x_2^{(2)}, \ldots, x_n^{(2)}) \wedge \cdots \wedge (x_1^{(n+2)}, \Gamma_n^{-1}(x_2^{(n+2)}, \ldots, x_n^{(n+2)})) \otimes (x_2^{(n+3)}, \ldots, x_n^{(n+3)}))ST_n(x_2^{(n+4)}, x_n^{(n+4)})x_2 \]
\[ = \phi_t((x_1^{(1)}, \Gamma_n^{-1}(x_2^{(1)}, \ldots, x_n^{(1)}))ST_n(x_2^{(1)}, x_n^{(1)})) \]
Since, for all \( z_1, \ldots, z_n \) in \( U_g \) and \( i \) in \( \{ 1, \ldots, n-1 \} \) the following identities hold
- \( \phi_t(z_1 \Gamma_n^{-1}(z_2, \ldots, z_n)) = \Gamma_n(z_1, \ldots, z_n), \)
- \( \phi_t(z_1 \Gamma_n^{-1}(z_2, \ldots, z_n)) = B_n(z_1, \ldots, z_n), \)
- \( \phi_t(z_1 \Gamma_n^{-1}(z_2, \ldots, z_n)) = \Gamma_n(z_1, \ldots, z_n), \)
- \( \phi_t^{-1}(z_1 \Gamma_n^{-1}(z_2, \ldots, z_n))ST_n(x_2^{(1)}, x_n^{(1)}))ST_n(x_2^{(1)}, x_n^{(1)})) = \Gamma_n(z_1, \ldots, z_n), \)
- \( \phi_t^{-1}(z_1 \Gamma_n^{-1}(z_2, \ldots, z_n))ST_n(x_2^{(1)}, x_n^{(1)}))ST_n(x_2^{(1)}, x_n^{(1)})) = \Gamma_n(z_1, \ldots, z_n), \)
- \( \phi_t^{-1}(z_1 \Gamma_n^{-1}(z_2, \ldots, z_n))ST_n(x_2^{(1)}, x_n^{(1)}))ST_n(x_2^{(1)}, x_n^{(1)})) = \Gamma_n(z_1, \ldots, z_n), \)

- \( \phi_t^{-1}(z_1 \Gamma_n^{-1}(z_2, \ldots, z_n))ST_n(x_2^{(1)}, x_n^{(1)}))ST_n(x_2^{(1)}, x_n^{(1)})) = \Gamma_n(z_1, \ldots, z_n), \)
- \( \phi_t^{-1}(z_1 \Gamma_n^{-1}(z_2, \ldots, z_n))ST_n(x_2^{(1)}, x_n^{(1)}))ST_n(x_2^{(1)}, x_n^{(1)})) = \Gamma_n(z_1, \ldots, z_n), \)
this leads to

\[ h_t \hat{G}'_{n-1}d^B(x_1|\cdots|x_n) = \hat{G}_n(x_1|\cdots|x_n) \]

Thus, by an obvious change of variables, we get that

\[
\int_{[0,1]^n} dt_1 \cdots dt_n \hat{G}_n(x_1|\cdots|x_n) = \int_{[0,1]^n} dt_1 \cdots dt_n h_t \hat{G}'_{n-1}d^B(x_1|\cdots|x_n) \\
= h \int_{[0,1]^{n-1}} dt_1 \cdots dt_{n-1} \hat{G}_{n-1}d^B(x_1|\cdots|x_n)
\]

which proves that the right-hand side of \( 3 \) satisfies the induction relation (2) and concludes the proof of theorem 2.2.3.

As a consequence of theorem 2.2.3 and proposition 2.2.2, we have the following

**Corollary 2.2.5.** For any \( U_{\mathfrak{g}} \)-bimodule \( M \), the pair of maps

\[
\array{rcl}
M \otimes_{U_{\mathfrak{g}} \otimes U_{\mathfrak{g}}^{op}} B_+(U_{\mathfrak{g}}) & \xrightarrow{\text{Id}_M \otimes F_*} & M \otimes_{U_{\mathfrak{g}} \otimes U_{\mathfrak{g}}^{op}} \text{CK}_+(\mathfrak{g}) \\
& \xleftarrow{\text{Id}_M \otimes G_*} & \\
\array{c}
\end{array}
\]

is a deformation retract of chain complexes.

Note that the preceding corollary means that \( G_* \circ F_* = \text{Id}_{\text{CK}_+(\mathfrak{g})} \), which follows easily from the properties of the eulerian idempotent and the \( B_n \)'s, and that there exists a graded map of \( U_{\mathfrak{g}} \)-bimodules \( H_* : B_*(U_{\mathfrak{g}}) \to B_{*+1}(U_{\mathfrak{g}}) \) of degree +1 such that

\[ H_* \circ d^B + d^B \circ H_* = F_* \circ G_* - \text{Id}_{B_*(U_{\mathfrak{g}})} \]

with the convention \( B_{-1}(U_{\mathfrak{g}}) := \{0\} \).

If the existence of \( H_* \) is a consequence of the fundamental lemma of calculus of derived functors, one may ask for an explicit formula for it, in view of further applications. It turns out that once again, the answer relies on the knowledge of some explicit contracting homotopy. Let us first recall the following standard result:

**Definition-Proposition 2.2.6.**

1. The degree +1 graded \( \mathbb{K} \)-linear map \( h^B : B_*(U_{\mathfrak{g}}) \to B_{*+1}(U_{\mathfrak{g}}) \) defined in degree \( n \) by

\[ h^B(a < x_1|\cdots|x_n >) := 1 < a|x_1|\cdots|x_n > b \]

for all \( a, b, x_1, \ldots, x_n \) in \( U_{\mathfrak{g}} \), is a contracting homotopy of the bar resolution \( B_*(U_{\mathfrak{g}}) \).

2. Moreover, the graded map \( \hat{h} : B_*(U_{\mathfrak{g}}) \to B_{*+1}(U_{\mathfrak{g}}) \) defined from \( h^B \) by

\[ \hat{h} := h^B \circ d^B \circ h^B, \]

is still a contracting homotopy of \( B_*(U_{\mathfrak{g}}) \), which satisfies in addition the gauge condition

\[ \hat{h}^3 = 0 \]

**Definition-Proposition 2.2.7.** Let \( H_* : B_*(U_{\mathfrak{g}}) \to B_{*+1}(U_{\mathfrak{g}}) \) be the unique graded endomorphism of \( U_{\mathfrak{g}} \)-bimodule of degree +1 defined by induction on the degree \( n \) via

\[ H_0 := 0 : B_0(U_{\mathfrak{g}}) \to B_1(U_{\mathfrak{g}}) \]

and

\[ H_{n+1}(x_1|\cdots|x_{n+1}) := \hat{h} \circ (F_{n+1}G_{n+1} - \text{Id}_{B_{n+1}(U_{\mathfrak{g}})}) - H_nd^B(x_1|\cdots|x_{n+1}) , \quad n \geq 0 \] (4)

for all \( x_1, \ldots, x_n \) in \( U_{\mathfrak{g}} \). Then \( H_* \) is a homotopy between \( F_*G_* \) and \( \text{Id}_{B_*(U_{\mathfrak{g}})} \).

**Proof.** Let us prove that \( H_* \) satisfies

\[ H_{n-1}d^B + d^B H_n = F_nG_n - \text{Id}_{B_n(U_{\mathfrak{g}})} , \quad n \geq 0 \] (5)

by induction on \( n \). For \( n = 0 \) we have

\[ F_0G_0 - \text{Id}_{B_0(U_{\mathfrak{g}})} = 0 = d^B H_0. \]
Assuming that \( [5] \) is true for all \( 0 \leq n \leq k \), using that \( d^B\tilde{h} + \tilde{h}d^B = \text{Id} \) in strictly positive degrees, we get
\[
d^B H_{k+1}(<x_1|\cdots|x_{k+1}>) = d^B \tilde{h}(F_{k+1}G_{k+1} - \text{Id}_{B_{k+1}(Ug)} - H_k d^B)(<x_1|\cdots|x_{k+1}>)
\]
\[
= (F_{k+1}G_{k+1} - \text{Id}_{B_{k+1}(Ug)} - H_k d^B)(<x_1|\cdots|x_{k+1}>)
\]
\[
- \tilde{h}d^B(F_{k+1}G_{k+1} - \text{Id}_{B_{k+1}(Ug)} - H_k d^B)(<x_1|\cdots|x_{k+1}>)
\]
But, because \( F_*G_* \) is an endomorphism of chain complex and thanks to the induction hypothesis:
\[
d^B (F_{k+1}G_{k+1} - \text{Id}_{B_{k+1}(Ug)} - H_k d^B) = (F_k G_k - \text{Id}_{B_k(Ug)} - d^B H_k) d^B = H_{k-1}(d^B)^2 = 0
\]
Thus
\[
d^B H_{k+1}(<x_1|\cdots|x_{k+1}>) = (F_{k+1}G_{k+1} - \text{Id}_{B_{k+1}(Ug)} - H_k d^B)(<x_1|\cdots|x_{k+1}>)
\]
which proves that \( [5] \) is true for \( n = k + 1 \), when applied to tensors of the form \( <x_1|\cdots|x_{k+1}> \). As both sides of \( [5] \) are morphisms of bimodules, this implies that they have to coincide on the whole \( B_{k+1}(Ug) \).

One could ask why, in the preceding proposition, we have used \( h^B \) instead of \( \tilde{h} \) to define the homotopy \( H_* \), since the proof doesn’t involve the gauge condition \( \tilde{h}^2 = 0 \). This choice of particular contraction is in fact motivated by the following result:

**Proposition 2.2.8.** Denote by \( C_* : B_*(Ug) \to B_*(Ug) \) the endomorphism of graded bimodule \( F_*G_* - \text{Id}_{B_*(Ug)} \). The homotopy \( H_* \) defined in \( 2.2 \) satisfies
\[
H_n(<x_1|\cdots|x_n>) = \sum_{i=1}^{n-1} (-1)^{i+1} \tilde{h}(x_1(x_2|\cdots|x_{n-i})(x_{n-i+1}<x_i|\cdots|x_n>)) \cdots)
\]
for all \( x_1, \ldots, x_n \) in \( Ug \).

**Proof.** Let \( \tilde{H}_* : B_*(Ug) \to B_{n+1}(Ug) \) be the degree +1 endomorphism of bimodule defined by the right hand side of \( [6] \) on tensors of the form \( <x_1|\cdots|x_n> \).

As \( d^B(<x_1|\cdots|x_n>) = x_1 <x_2|\cdots|x_n> + R \), where \( R \) is a sum of tensors of the form \( <y_2|\cdots|y_n> \) on which \( \tilde{h} \circ H_{n-1} \) vanishes because \( \tilde{h}^2 = 0 \), we see that
\[
\tilde{h}(C_n - \tilde{H}_{n-1} d^B)(<x_1|\cdots|x_n>) = \tilde{h}C_n(<x_1|\cdots|x_n>) - \tilde{h}(x_1 \tilde{H}_{n-1}(<x_2|\cdots|x_n>)) = \tilde{H}_n(<x_1|\cdots|x_n>)
\]
which proves that, as \( H_* \), \( \tilde{H}_* \) satisfies the induction relation \( [4] \). Since \( \tilde{H}_0 = H_0 = 0 \), they have to coincide on the whole \( B_*(Ug) \).

**Corollary 2.2.9.** For all \( x \) in \( Ug \),
\[
H_1(<x>) = \int_0^1 dt <\phi_t(x^{(1)})|pr(x^{(2)}) > \phi_{1-t}(x^{(3)}) - <1|x>
\]

**Proof.** Let \( x \) be an element of \( Ug \). Then
\[
H_1(<x>) = \tilde{h} C_1(<x>)
\]
\[
= h^B d^B h^B \left( \int_0^1 dt \phi_t(x^{(1)}) <pr(x^{(2)}) > \phi_{1-t}(x^{(3)}) - <x> \right)
\]
\[
= h^B \left( \int_0^1 dt \phi_t(x^{(1)}) <pr(x^{(2)}) > \phi_{1-t}(x^{(3)}) - <\phi_t(x^{(1)})|pr(x^{(2)}) > \phi_{1-t}(x^{(3)}) + <\phi_t(x^{(1)}) > pr(x^{(2)}) \phi_{1-t}(x^{(3)}) \right.
\]
\[
\left. - <1|x> \right)
\]
\[
= h^B \left( \int_0^1 dt \phi_t(x^{(1)}) <pr(x^{(2)}) > \phi_{1-t}(x^{(3)}) - \int_0^1 dt \frac{d}{dt} \left( <\phi_t(x^{(1)}) > \phi_{1-t}(x^{(2)}) \right) - <1|x> \right)
\]
\[
= \int_0^1 dt <\phi_t(x^{(1)})|pr(x^{(2)}) > \phi_{1-t}(x^{(3)}) - <1|x>
\]

**Remark 2.2.10.** In fact, \( h^B C_1 = \tilde{h} C_1 \) so choosing \( h^B \) instead of \( \tilde{h} \) in the definition of \( H_* \) would have led to the same result in degree 1. This doesn’t seem to be true any longer in higher degrees, and it is not clear whether a compact formula like \( [6] \) could be obtained without the gauge condition.
Appendix A  The Poincaré-Birkhoff-Witt theorem

References


