

FLIPPABLE TILINGS OF CONSTANT CURVATURE SURFACES

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ABSTRACT. We call “flippable tilings” of a constant curvature surface a tiling by “black” and “white” faces, so that each edge is adjacent to two black and two white faces (one of each on each side), the black face is forward on the right side and backward on the left side, and it is possible to “flip” the tiling by pushing all black faces forward on the left side and backward on the right side. Among those tilings we distinguish the “symmetric” ones, for which the metric on the surface does not change under the flip. We provide some existence statements, and explain how to parameterize the space of those tilings (with a fixed number of black faces) in different ways. For instance one can glue the white faces only, and obtain a metric with cone singularities which, in the hyperbolic and spherical case, uniquely determines a symmetric tiling. The proofs are based on the geometry of polyhedral surfaces in 3-dimensional spaces modeled either on the sphere or on the anti-de Sitter space.

1. INTRODUCTION

We are interested here in tilings of a surface which have a striking property: there is a simple specified way to re-arrange the tiles so that a new tiling of a non-singular surface appears. So the objects under study are actually pairs of tilings of a surface, where one tiling is obtained from the other by a simple operation (called a “flip” here) and conversely. The definition is given below.

We have several motivations for studying those flippable tilings. One is their intrinsic geometric and dynamic properties. It is not completely clear, at first sight, that those flippable tilings can be “flipped”, or even that there should be many examples, beyond a few simple ones that can be constructed by hand. However the 3-dimensional arguments developed below show that there are indeed many examples and that the space of all those tilings can be quite well understood. Another motivation is that those flippable tilings provide a way to translate 3-dimensional questions (on polyhedral surfaces) into 2-dimensional statements.

One of our motivation when developing flippable tilings was to investigate a polyhedral version of earthquakes. Earthquakes are defined on hyperbolic surfaces endowed with measured laminations. Those earthquakes are then intimately related to pleated surfaces in the AdS space (see [Mes07, ABB⁺07] and [BS09b, BS09a, BKS11] for recent developments).

There are also analogs of this picture for some special smooth surfaces in AdS_3 , in particular maximal surfaces. Earthquakes are then replaced by minimal Lagrangian diffeomorphisms, see [AAW00, KS07, BS10]. Similar arguments can be used — leading to slightly different results — for constant mean curvature or constant Gauss curvature surfaces in AdS_3 or in 3-dimensional AdS manifolds.

It is then tempting to investigate what happens for polyhedral surfaces, rather than pleated or smooth surfaces. This leads naturally to the notion of flippable tilings as developed here, although the analogy with earthquakes is only limited, as can be seen in Theorem 1.5.

1.1. Definitions, basic property.

1.1.1. Definitions and first examples.

Definition 1.1. Let (S, g) be a closed (oriented) surface with a constant curvature metric. A *right flippable tiling* of (S, g) is a triple $T = (F_b, F_w, E)$, where

- (1) F_b and F_w are two sets of closed convex polygons homeomorphic to disks in S , called the “black” and “white” faces of T , with disjoint interiors, which cover S .
- (2) E is a set of segments in (S, g) , called the *edges* of the tiling, such that each edge of any black or white polygon is contained in some $e \in E$.

We demand that for each oriented edge $e \in E$:

- e is the union of the edges (in the usual sense) of a black and a white face on its right side, and similarly on its left side,
- the black polygon is forward on the right-hand side of e , and backward on the left-hand side,

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- the lengths of the intersections with e of the two black faces (resp. the two white faces) are equal.

The vertices of the black and white faces are the *vertices* of the tiling, while the black (resp. white) polygons are the *black* (resp. *white*) *faces* of the tiling.

A *left* flippable tiling is defined in the same way, but, for each edge e , the black polygon is forward on the left-hand side of e and backward on the right-hand side. See Figure 1.

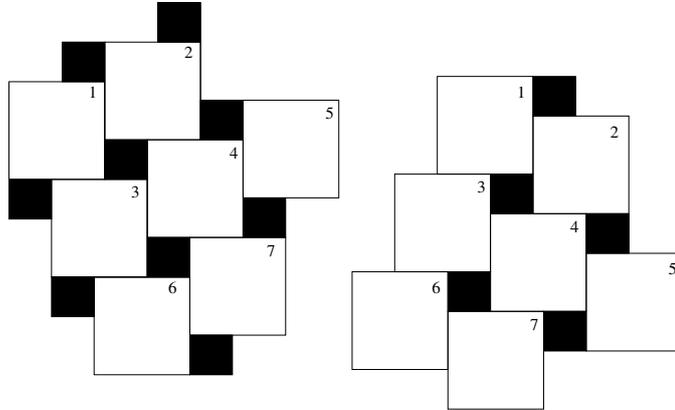


FIGURE 1. A flippable tiling, sometimes found in bathrooms — left and right versions.

Note that the definition does not depend on the orientation of the edges: changing the orientation exchanges both the left and the right side, and the forward and the backward direction.

We assume that some faces of a flippable tiling on the sphere can be digons (see the definitions in Subsection 2.1.1).

Example 1.2. Let us consider two distinct geodesics on the round sphere S^2 . They divide the sphere into 4 faces. Paint in black two non-adjacent faces, and paint in white the other two. We get a flippable tiling with two edges, and there are two choices for the edges: one leading to a left flippable tiling and the other to a right flippable tiling.

More generally, consider on S^2 a convex polygon P with n edges as well as its antipodal $-P$. Let v_1, \dots, v_n be the vertices of P , so that $-v_1, \dots, -v_n$ are the vertices of $-P$. For each $i \in \{1, \dots, n\}$, let e_i be the edge of P joining v_i to v_{i+1} , and let $-e_i$ be the edge of $-P$ joining $-v_i$ to $-v_{i+1}$. For each i , draw the half-geodesic starting from v_i along e_i , and ending at $-v_i$. Then it is not hard to see that we get a decomposition of the sphere into P , $-P$ and digons. The vertices of each digon are a vertex v_i of P and its antipodal $-v_i$. One edge of the digon contains an edge of P and the other contains an edge of $-P$. If we paint in black P and P' , and in white all the digons, then we get a flippable tiling with n edges. There is another possibility: drawing for each i the half-geodesic starting from v_i along e_{i-1} . One possibility leads to a left flippable tiling and the other to a right flippable tiling (depending on the orientation of P).

Corollary 2.10 will state that the examples above are the “simplest” on the sphere. But there exist flippable tilings with only one black and/or white face on higher genus surfaces, as the following example shows.

Example 1.3. Consider a square. Its diagonals divide it in 4 triangles. Paint in black two non-adjacent triangles. For a suitable choice of the edges, we obtain a flippable tiling of the torus with two faces, two edges (which are made of parts of the diagonals) and two vertices. This construction also provides examples of flippable tiling for surfaces of higher genus.

1.1.2. *Flipping a flippable tiling.* The terminology used for those tilings is explained by the following basic definition and statements.

Definition 1.4. Let (S, g_r) be a constant curvature closed surface with a right flippable tiling T_r . If there is a metric g_l on S of the same curvature as g_r with a left flippable tiling T_l , such that:

- there is a one-to-one orientation-preserving map ϕ between the faces of T_r and the faces of T_l , sending each black (resp. white) face f of T_r to a black (resp. white) face of T_l isometric to f ,
- there is a one-to-one map ψ between the edges of T_r and the edges of T_l ,
- if an edge e of a face f of T_r is adjacent to an edge E of T_r , then the corresponding edge in $\phi(f)$ is adjacent to $\psi(E)$, and conversely,

we say that T_l is obtained by *flipping* T_r . Similarly we define the flipping of a left flippable tiling.

In other terms, the left flippable tiling T_l on (S, g_l) has the same combinatorics as the right flippable tiling T_r and its faces are isometric to the corresponding (black or white) faces of T_l , but each edge has been “flipped” in the sense that on the right side of each edge, the black face, which was forward on T_r , is backward on T_l , see Figure 2. Note that the hypothesis that ϕ is orientation-preserving is relevant since otherwise simply changing the orientation on S would yield a left flippable tiling with the desired properties.

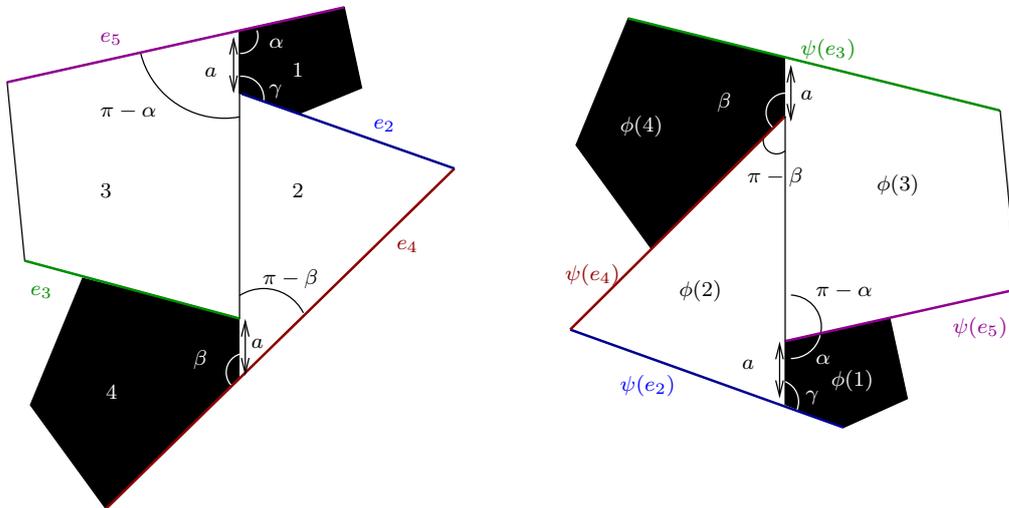


FIGURE 2. The “flip” of an edge.

Theorem S1. *Let S^2 be the round sphere with a right flippable tiling T_r . There is a unique left flippable tiling T_l on S^2 , such that T_l is obtained by flipping T_r .*

Similarly, there is a right flippable tiling obtained by flipping any left flippable tiling, and flipping a tiling T twice yields T .

Theorem H1. *Let (S, g_r) be a hyperbolic surface with a right flippable tiling T_r . There is a unique hyperbolic metric g_l on S with a left flippable tiling T_l , such that T_l is obtained by flipping T_r .*

Similarly, there is a right flippable tiling obtained by flipping any left flippable tiling, and flipping a tiling T twice yields T .

Those statements are not completely obvious since one has to check that, after flipping, the faces “match” at the vertices. The proof is in Section 2 for the spherical case, in Section 3 for the hyperbolic case. In both cases it is based on

- a notion of duality between equivariant polyhedral surfaces in a 3-dimensional space, either the 3-dimensional sphere or the 3-dimensional anti-de Sitter space,
- some key geometric properties shared by S^3 and AdS_3 , basically coming from the fact that they are isometric to Lie groups with a bi-invariant metric.

There is an easy way to get a right (resp. left) flippable tiling from a left (resp. right) flippable tiling T . It suffices to color in white all black faces of T and to color in black all white faces of T . The flipping obtained in this way is denoted by T^* . Obviously $(T^*)^* = T$.

1.1.3. *Flipping from a hyperbolic metric to another.* Flippable tilings can be considered as ways to deform from one hyperbolic metric — the metric underlying the left tiling — to another one — the metric underlying the right tiling obtained after tiling. Recall that \mathcal{T} is the Teichmüller space of S .

Theorem 1.5. *Let $h_l, h_r \in \mathcal{T}$ be two hyperbolic metrics on S . There is a left flippable tiling T_l on h_l such that flipping T_l yields a right flippable tiling T_r on h_r .*

The proof can be found in Section 3.5.5. It should be clear from the proof that T_l is not unique, on the opposite there are many possible choices, and finding a “simplest” such tiling remains an open question.

1.2. Moduli spaces of flippable tilings. We now consider more globally the space of flippable tilings, first on the sphere and then on closed hyperbolic surfaces of a fixed genus $g \geq 2$.

Let $n \geq 3$, consider a right flippable tiling T with n black faces on S^2 . We can associate to T a graph Γ embedded in S^2 (defined up to isotopy) as follows:

- the vertices of Γ correspond to the black faces of T ,
- the faces of Γ (connected components of the complement of Γ in S^2) correspond to the white faces of T ,
- the edges of Γ correspond to the edges of T (which bound a black and a white face of T on each side), so that a vertex of Γ is adjacent to a face of Γ if and only if the corresponding black and white faces of T are adjacent to the same edge of T .

We will call Γ the *incidence graph* of the flippable tiling T .

Definition 1.6. Let $n \geq 3$. We call $\mathcal{T}_n^{r,1}$ the set of right flippable tilings of S^2 with n black faces, considered up to isotopy. If Γ is an embedded graph in S^2 (considered up to isotopy) with n vertices, we call $\mathcal{T}_n^{r,1}(\Gamma)$ the set of right flippable tilings of S^2 with incidence graph Γ .

For each graph Γ there is a natural topology on $\mathcal{T}_n^{r,1}(\Gamma)$, where two tilings in $\mathcal{T}_n^{r,1}(\Gamma)$ are close if, after applying an isometry of S^2 , their corresponding vertices are close. However there is also a topology on $\mathcal{T}_n^{r,1}$. We consider that given a flippable tiling $t \in \mathcal{T}_n^{r,1}(\Gamma)$ and a sequence of tilings $t_n \in \mathcal{T}_n^{r,1}(\Gamma')$, then (t_n) converges to t if, after applying a sequence of isometries of S^2 , the union of white (resp. black) faces of t_n converges to the union of white (resp. black) faces of t in the Hausdorff topology.

For instance Γ could be obtained from Γ' by erasing an edge e between two faces f, f' of Γ' . Then f and f' correspond to white faces of t_n , for all $n \in \mathbb{N}$. In the limit as $n \rightarrow 0$, those faces merge to one white face of t , while the two black faces of the t_n adjacent to the same edge of the flippable tiling “lose” an edge. In a dual way, Γ could be obtained from Γ' by collapsing an edge e , so that its two vertices v, v' become one. In this case the two black faces corresponding to v and v' merge in the limit flippable tiling.

Theorem S2. *With the notations above:*

- (1) *When Γ is 3-connected, $\mathcal{T}_n^{r,1}(\Gamma)$ is non-empty and homeomorphic to an open ball of dimension equal to the number of edges of Γ .*
- (2) *$\mathcal{T}_n^{r,1}$, with the topology described above, is homeomorphic to a manifold of dimension $3n - 6$.*

Recall that a graph is 3-connected if it cannot be disconnected by removing at most 2 vertices — for instance, the 1-skeleton of a triangulation of a simply connected surface is 3-connected.

We now turn to hyperbolic surfaces. We consider a surface S of genus $g \geq 2$, and use the notion of incidence graph introduced above for a flippable tiling.

Definition 1.7. Let $n \geq 1$. We call $\mathcal{T}_{S,n}^{r,-1}$ the set of right flippable tilings of a hyperbolic metric on S with n black faces, considered up to isotopy. If Γ is an embedded graph in S (considered up to isotopy) with n vertices, we call $\mathcal{T}_{S,n}^{r,-1}(\Gamma)$ the set of right flippable tilings of a hyperbolic metric on S with incidence graph Γ .

As in the spherical case, given a graph Γ embedded in S , there is a natural topology on $\mathcal{T}_{S,n}^{r,-1}(\Gamma)$, for which two hyperbolic h, h' on S and two flippable tilings T and T' respectively in (S, h) and in (S, h') , T and T' are close if there is a diffeomorphism $u : (S, h) \rightarrow (S, h')$ with bilipschitz constant close to 1 and which sends each face of T close to the corresponding face of T' (in the Hausdorff topology).

However, still as in the spherical case, there is also a natural topology on $\mathcal{T}_{S,n}^{r,-1}$, where two flippable tilings T and T' respectively on (S, h) and on (S, h') are close if there is a diffeomorphism $u : (S, h) \rightarrow (S, h')$ with bilipschitz constant close to 1 sending the union of the white (resp. black) faces of T close to the union of the white (resp. black) faces of T' in the Hausdorff topology. The same phenomena of merging of two white (resp. black) faces can happen as in the spherical case, as described above.

Theorem H2. *With the notations above:*

- (1) *$\mathcal{T}_{S,n}^{r,-1}(\Gamma)$ is non-empty as soon as the universal cover of Γ is 3-connected.*
- (2) *Under this condition, $\mathcal{T}_{S,n}^{r,-1}(\Gamma)$ is homeomorphic to the interior of a manifold with boundary of dimension $6g - 6 + e$, where g is the genus of S and e the number of edges of Γ .*
- (3) *With the topology described above, $\mathcal{T}_{S,n}^{r,-1}$ is homeomorphic to a manifold of dimension $12g + 3n - 12$.*
- (4) *The mapping-class group $MCG(S)$ of S has a canonical properly discontinuous action on $\mathcal{T}_{S,n}^{r,-1}$ compatible with the decomposition as the union of the $\mathcal{T}_{S,n}^{r,-1}(\Gamma)$.*

Note that it appears likely that $\mathcal{T}_{S,n}^{r,-1}(\Gamma)$ is always homeomorphic to a ball. This would follow from an extension to surfaces of higher genus and to equivariant polyhedra of a theorem of Steinitz on the topology of the space of convex polyhedra with given combinatorics in Euclidean 3-space.

1.3. Parameterization with fixed areas of the faces. We now describe more specific results, on the parameterizations of the space of flippable tilings for which the area of the black faces are fixed.

1.3.1. *On the sphere.*

Definition 1.8. For $n > 1$, let $k_1, \dots, k_n \in (0, \pi)$ have sum less than 4π , and be such that for all $j \in \{1, \dots, n\}$, $\sum_{i \neq j} k_i > k_j$. We denote by $\mathcal{T}_n^{r,1}(K)$, $K = (k_1, \dots, k_n)$, the space of right flippable tilings on S^2 with n black faces of area k_1, \dots, k_n .

Theorem S3. *Under the conditions of the previous definition, if $n \neq 2$, there is a natural parameterization of $\mathcal{T}_n^{r,1}(K)$ by the space of configurations of n distinct points on S^2 .*

1.3.2. *Hyperbolic surfaces.*

Definition 1.9. Let $n \geq 1$, let $k_1, \dots, k_n \in \mathbb{R}_{>0}$ of sum less than $2\pi|\chi(S)|$, let (S, h) be a hyperbolic surface, and let $K = (k_1, \dots, k_n)$. We denote by $\mathcal{T}_{S,h,n}^{r,-1}(K)$ the space of right flippable tilings on (S, h) with n black faces of area k_1, \dots, k_n .

To give a satisfactory hyperbolic analog of Theorem S3 we need to consider a special class of flippable tilings. This notion is significant only in the hyperbolic or the Euclidean settings.

Definition 1.10. A right flippable tiling T_r of a surface (S, g_r) is *symmetric* if the metric g_l underlying the left tiling obtained by flipping T_r is isotopic to g_r . We call $\overline{\mathcal{T}}_{S,h,n}^{r,-1}$ the space of symmetric right flippable tilings of a hyperbolic surface (S, h) with n black faces, and its intersection with $\mathcal{T}_{S,h,n}^{r,-1}(K)$ is denoted by $\overline{\mathcal{T}}_{S,h,n}^{r,-1}(K)$.

Clearly a spherical flippable tiling is always symmetric, since there is only one spherical metric on S^2 , up to isometry.

Theorem H3. *In the setting of the previous definition, there is a natural parameterization of $\overline{\mathcal{T}}_{S,h,n}^{r,-1}(K)$ by the space of configurations of n distinct points on S .*

The proof is based on an analog in the AdS space of a classical Alexandrov Theorem on polyhedra with given curvature at its vertices, see Section 4.

1.4. The black and white metrics of a flippable tiling.

1.4.1. *Definitions.* Consider a flippable tiling of a constant curvature surface (S, g) . Let F_b, F'_b be two adjacent black faces. Then F_b and F'_b are adjacent to an edge e which is also adjacent to two white faces F_w and F'_w . The definition of a flippable tiling shows that $F_w \cap e$ and $F'_w \cap e$ have the same length. So there is a well-defined way to glue F_w to F'_w along their intersections with e , isometrically, preserving the orientation of e . Idem for F_b and F'_b .

Definition 1.11. Let (S, g_l) be a closed surface with a constant curvature metric, and let T_l be a left flippable tiling on (S, g_l) . The *black metric* h_b of T_l is a constant curvature metric with cone singularities obtained by gluing isometrically the black faces of T_l along their common edges. The *white metric* h_w of T_l is the metric with cone singularities obtained by similarly gluing the white faces of T_l .

Remark 1.12. If (S, g) is a hyperbolic surface, then h_b is a hyperbolic metric with cone singularities of negative singular curvature (angle larger than 2π). Each cone singularity v of h_b is associated to a white face f of T_l , and the angle excess at v is equal to the area of f .

If (S, g) is a flat surface, then h_b is a flat surface (with no cone singularity).

If (S, g) is a spherical surface, then h_b is a spherical surface with cone singularities of positive singular curvature (angle less than 2π). The singular curvature at a cone singularity is equal to the area of the corresponding white face of T_l .

1.4.2. *Spherical surfaces.* In the sphere, flippable tilings are in one-to-one correspondence with spherical metrics with cone singularities of positive singular curvature.

Theorem S4. *Let g be a spherical metric on S^2 with $n > 2$ cone singularities of angle less than 2π . There exists a unique left flippable tiling of S^2 which has g as its black metric.*

1.4.3. *Hyperbolic surfaces.* We now consider a closed surface S of negative Euler characteristic. For those hyperbolic surfaces the analog of Theorem S4 involves symmetric tilings.

Theorem H4. *Let h_+ be a hyperbolic metric with cone singularities of negative singular curvature on a closed surface S . There is a unique symmetric flippable tiling T_+ on a hyperbolic surface such that h_+ is the black metric of T_+ .*

The proof of Theorem H4 can be found in Section 3.7.

1.5. **Flippable tilings and polyhedral surfaces.** The results on flippable tilings presented here are consequences, sometimes very direct, of a correspondence between flippable tilings on the sphere (resp. a hyperbolic surface) and convex, polyhedral surfaces in the 3-dimensional sphere (resp. convex, space-like polyhedral surfaces in the 3-dimensional anti-de Sitter space).

To each convex polyhedron P in S^3 , we can associate a left flippable tiling T on S^2 . The black faces of T correspond to the faces of P , while the white faces of T correspond to the vertices of P , so that the incidence graph of T is the 1-skeleton of P . More geometrically, the black faces of T are isometric to the faces of P , while the white faces of T are isometric to the spherical duals of the links of the vertices of P .

The same relation holds between flippable tilings on a hyperbolic surface on one hand, and equivariant polyhedral surfaces in AdS_3 on the other. Symmetric flippable tilings then correspond to Fuchsian polyhedral surfaces. The results on the black metrics on tilings correspond to statements on the induced metrics on polyhedra, going back to Alexandrov [Ale05] in the spherical case and proved more recently by the first author [Fil07, Fil11] in the hyperbolic/AdS case. The results on tilings with black faces of fixed area correspond to statements on polyhedra with vertices having fixed curvature, which are proved in Section 4.

1.6. **Remarks about Euclidean surfaces.** The reader has probably noticed that the results described above concern hyperbolic surfaces and the sphere, but not flat metrics on the torus. After a first version of the present paper circulated, it was noticed by François Guéritaud that the analog of theorems S1 and H1 is true for these metrics. The proof can be done in an intrinsic way, and moreover the flip can be described as a continuous path in the Teichmüller space of the torus [Gué].

Maybe the 3-dimensional spherical or AdS arguments used here are still valid in this case, considering the space of isometries of the Euclidean plane. We hope that this question will be the subject of further investigations.

Acknowledgements. The authors would like to thank François Guéritaud for useful conversations related to the results presented here.

2. SPHERICAL POLYHEDRA

2.1. Basics about the sphere S^3 .

2.1.1. *Spherical polyhedra.* Let S^n be the unit sphere in \mathbb{R}^{n+1} , endowed with the usual scalar product $\langle \cdot, \cdot \rangle$.

- (1) A (spherical) *convex polygon* is a compact subset of S^2 obtained by a finite intersection of closed hemispheres, and contained in an open hemisphere of S^2 . We suppose moreover that a convex polygon has non-empty interior.
- (2) A *digon* is a spherical polygon is the intersection of two closed hemispheres, having two vertices and two edges of lengths π , making an angle $< \pi$.
- (3) A (spherical) *convex polyhedron* is a compact subset of S^3 obtained as a finite intersection of half-spaces of the sphere, and contained in an open hemisphere of S^3 . We suppose moreover that a convex polyhedron has a non-empty interior. Faces of a convex polyhedron are convex polygons.
- (4) A *dihedron* satisfies the definition of convex polyhedron except that it has empty interior. It is a convex polygon p in a totally geodesic plane in S^3 . We consider its boundary to be made of two copies of p glued isometrically along their boundary, so that this boundary is homeomorphic to a sphere.
- (5) A *hosohedron* is the non-empty intersection of a finite number of half-spaces with only two vertices. The vertices have to be antipodal, and faces of a hosohedron are digons. A hosohedron is contained in the closure of an open hemisphere, it is not a convex polyhedron in regard of the definition above. We also call digon the degenerated hosohedron made of two copies of the same digon.

Up to a global isometry, we can consider that all the objects defined above are contained in the (closure of) the same open hemisphere S_+^3 .

The *polar link* $Lk(x)$ of a vertex x of a convex polyhedron P is the convex spherical polygon in $T_x S^3$ given by the outward unit normals of the faces of P meeting at x . The lengths of the edges of $Lk(x)$ are the exterior

dihedral angles of P at the edges having x as endpoint, and the interior angles of $\text{Lk}(x)$ are π minus the interior angles on the faces of P at x . The definition of polar link extends to dihedra and hosohedra.

2.1.2. Polar duality. Let $x \in S^3_+$. We denote by x^* the hyperplane orthogonal to the vector $x \in \mathbb{R}^4$ for $\langle \cdot, \cdot \rangle$. We also denote by x^* the totally geodesic surface intersection between this hyperplane and S^3 . Conversely, if H is a totally geodesic surface of S^3 , we denote by H^* the unique point of S^3_+ orthogonal as a vector to the hyperplane H (up to an isometry, we can avoid considering H such that its orthogonal unit vectors belong to the boundary of S^3_+).

The *polar dual* P^* of a convex polyhedron P is the convex polyhedron defined as the intersection of the half-spaces $\{y | \langle x, y \rangle > 0\}$ for each vertex x of P . Each face of P is contained in the dual x^* of a vertex x of P . Equivalently, P^* is a convex polyhedron whose vertices are the H^* for H a totally geodesic surface containing a face of P . It follows that $(P^*)^* = P$ and that the polar link $\text{Lk}(x)$ of a vertex x of P is isometric to the face of P^* dual to x and vice-versa.

The definitions of the polar link and polar dual applied to dihedra and hosohedra indicate that they are in duality with each other (excepted for digons which are dual of digons).

2.1.3. A projective model. Up to a global isometry, we can consider S^3_+ as the open subset of S^3 made of vectors of \mathbb{R}^4 with positive first coordinate. By the central projection which sends $e := (1, 0, 0, 0)$ to the origin, S^3_+ is identified with an affine model of the projective space $P^3(\mathbb{R})$ (the plane at infinity is e^*). In this affine model, a spherical convex polyhedron is drawn as an Euclidean convex polytope, a dihedron is drawn as a convex Euclidean polygon in a plane, a hosohedron is a convex polyhedral infinite cylinder, and a digon is the strip between two parallel lines.

2.1.4. Multiplicative structure on the sphere. The following isometry gives a multiplicative structure on \mathbb{R}^4 :

$$(\mathbb{R}^4, \langle \cdot, \cdot \rangle) \rightarrow (\mathcal{M}, \det), \quad (x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$

where \mathcal{M} is the space of matrix $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$. Note that the neutral element is e . It follows that S^3 is isometric to $SU(2)$, and that multiplication on the right or on the left by an element of S^3 is an isometry. As the kernel of the classical homomorphism $SU(2) \rightarrow SO(3)$ is $\{-\text{Id}, +\text{Id}\}$, $P^3(\mathbb{R})$ is isometric to $SO(3)$, and then $SO(3)$ acts by isometries on $P^3(\mathbb{R})$ by left and right multiplication.

Lemma 2.1. (1) *The left multiplication by x sends y^* to $(xy)^*$.*

(2) *If $x \in e^*$, then $\langle x, y \rangle = -\langle x, y^{-1} \rangle$. In particular $\langle x, y \rangle = 0$ if and only if $\langle x, y^{-1} \rangle = 0$.*

Proof. (1) If $z \in y^*$ then $\langle z, y \rangle = 0$, and the left multiplication is an isometry, so $\langle xz, xy \rangle = 0$, i.e. the image of z by the left multiplication by x belongs to $(xy)^*$.

(2) It is easy to check that $x = (x_1, x_2, x_3, x_4) \in e^*$ is equivalent to $x_1 = 0$ and that if $y = (y_1, y_2, y_3, y_4)$ then $y^{-1} = (y_1, -y_2, -y_3, -y_4)$. □

2.2. Flippable tilings on the sphere and convex polyhedra.

2.2.1. The left and right projections of an angle. Let a, b be in S^3_+ , with a, b, e and their antipodals mutually distinct. Let A be the angle formed by a^* and b^* , with edge E :

$$\begin{aligned} A &= a^*_c \cup b^*_c \cup E \\ &= \{x \in S^3 | \langle x, a \rangle = 0, \langle x, b \rangle > 0\} \cup \{x \in S^3 | \langle x, b \rangle = 0, \langle x, a \rangle > 0\} \cup \{x \in S^3 | \langle x, a \rangle = \langle x, b \rangle = 0\}. \end{aligned}$$

The *left projection* of the angle A is the following map:

$$L_A : A \setminus E \longrightarrow e^*, x \longmapsto \begin{cases} a^{-1}x & \text{if } x \in a^*_c \\ b^{-1}x & \text{if } x \in b^*_c \end{cases}.$$

The *right projection* of the angle A is defined similarly:

$$R_A : A \setminus E \longrightarrow e^*, x \longmapsto \begin{cases} xa^{-1} & \text{if } x \in a^*_c \\ xb^{-1} & \text{if } x \in b^*_c \end{cases}.$$

In the following we will identify e^* with S^2 : L_A and R_A have now values in the sphere S^2 .

- Lemma 2.2.** (1) *The half-planes a_c^* and b_c^* are isometrically mapped by L_A to two non-intersecting open hemispheres of S^2 delimited by the great circle $E^l := a^{-1}E = b^{-1}E$. If $x \in E$ then the distance between $a^{-1}x$ and $b^{-1}x$ is the exterior dihedral angle of A . Moreover, if $L_A(a_c^*)$ is on the left of E^l and $L_A(b_c^*)$ is on the right of E^l , then $a^{-1}x$ is forward and $b^{-1}x$ is backward.*
- (2) *The half-planes a_c^* and b_c^* are isometrically mapped by R_A to two non-intersecting open hemispheres of S^2 delimited by the great circle $E^r := Ea^{-1} = Eb^{-1}$. If $x \in E$ then the distance between xa^{-1} and xb^{-1} is the exterior dihedral angle of A . Moreover, if $R_A(a_c^*)$ is on the left of E^r and $R_A(b_c^*)$ is on the right of E^r , then $b^{-1}x$ is forward and $a^{-1}x$ is backward.*

Proof. We prove the first part of the lemma, the second is proved using similar arguments. We first check that L_A is an injective map. It suffices to show that $L_A(a_c^*) \cap L_A(b_c^*) = \emptyset$. Suppose that the intersection is non-empty. Hence there exists $x_1 \in a_c^*$ and $x_2 \in b_c^*$ such that $a^{-1}x_1 = b^{-1}x_2$. From $\langle x_1, b \rangle > 0$ it comes $\langle b^{-1}x_2, a^{-1}b \rangle > 0$, but $b^{-1}x_2 \in e^*$, so by Lemma 2.1 we obtain $\langle x_2, a \rangle < 0$, that is a contradiction.

Let $x \in E$. Lemma 2.1 implies that $a^{-1}x \in (a^{-1}b)^* \cap e^*$, $b^{-1}x \in (b^{-1}a)^* \cap e^*$, and that these two spaces are equal. By definition, the cosine of the exterior dihedral angle $\alpha < \pi$ of A is $\cos \alpha = \langle a, b \rangle = \langle a^{-1}, b^{-1} \rangle = \langle a^{-1}x, b^{-1}x \rangle = \cos d_{S^2}(a^{-1}x, b^{-1}x)$.

It is not hard to see that $a^{-1}b$ is a vector orthogonal to E^l on the same side as $L_A(a_c^*)$. It remains to prove that, for $x \in E$, $\det(b^{-1}x, a^{-1}x, a^{-1}b, e) > 0$. This determinant never vanishes. This comes from the facts that $a^{-1}b$ and e are orthogonal to $E^l \subset e^*$, $a^{-1}x$ and $b^{-1}x$ span E^l , and $a^{-1}b$ is not collinear to e . Hence the sign of the determinant is constant, and it can be checked on examples. \square

2.2.2. Convex polyhedra as flippable tilings. Let P be a convex polyhedron and E an edge of P . The planes containing the faces of P adjacent to E make an angle $A(E)$. The *left projection* of P is, for each edge E of P , the left projection of the angle $A(E)$ restricted to the faces of P adjacent to E . The right projection of P is defined in a similar way.

The following two propositions are the keystone of your study of flippable tilings in the spherical case, since they relate flippable tilings to convex polyhedra.

Proposition 2.3. *The left (resp. right) projection of a convex polyhedron P is the set of the white faces of a right (resp. left) flippable tiling of the sphere. The black faces are isometric to the polar links of the vertices of P . The 1-skeleton of P is isotopic to the incident graph of the tiling.*

Proof. Let us consider the case of the left projection. The case of the right projection is proved in a similar way. The left projection of the faces of P adjacent to an edge E gives two convex polygons w_1 and w_2 of S^2 , that we paint in white. From Lemma 2.2, w_1 and w_2 have an edge on a common geodesic. Let us consider the part e of this geodesic which contains the edge of each polygon and which is such that one of them is forward on the left.

Let s be a vertex of P and E_1, \dots, E_n be (oriented) edges of P having s as an endpoint. As an element of the edge E_i , the left projection sends s to two points ext_i and int_i of the segment e_i of S^2 (e_i is defined as above). ext_i is an endpoint of e_i and int_i belongs to the interior of e_i . Similarly, if E_j is another edge from s adjacent to a face having E_i has an edge, s is sent to two points ext_j and int_j of e_j , and $\text{ext}_j = \text{int}_i$ or $\text{int}_j = \text{ext}_i$. Finally s is sent by the left projection to n distinct points of S^2 . The segments between ext_i and int_i for any i define a polygon p . Lemma 2.2 says that the lengths of the edges of p are equal to the dihedral angles of P . Moreover, by construction, any interior angle of p is supplementary to the interior angle of a white face, which is the angle of a face of P at s . By definition of the polar link, p is isometric to the polar link of P at s . In particular it is a convex polygon. We paint it in black.

If we do the same for all the vertices of P , we get a set of white faces and a set of black faces on S^2 . By construction all faces have disjoint interior. The Gauss–Bonnet formula says exactly that the area of P plus the area of the polar duals of the vertices of P is 4π , the area of S^2 . Hence our white and black faces cover the whole sphere. The definition of right flippable tiling is satisfied, and it clearly satisfies the properties stated in the proposition. \square

We denote by $T_l(P)$ (resp. $T_r(P)$) the left (resp. right) flippable tiling obtained from P . Notice that if Q is a convex polyhedron isometric to P , then $T_l(Q)$ is isometric to $T_l(P)$. In the following, we may write such equivalence as: if $P = Q$ then $T_l(P) = T_l(Q)$. It comes from the definitions that

$$T_l(P^*) = T_r(P)^*; \quad T_r(P^*) = T_l(P)^*.$$

Remark 2.4. Proposition 2.3 remains true for hosohedra which are not digons. It is actually also true for dihedra and digons — up to adapting Lemma 2.2 in the case where $a = -b$. The flippable tilings obtained are the ones described in Example 1.2.

2.2.3. *Flippable tilings as convex polyhedra.* Let T_l be a left flippable tiling of S^2 . It is always possible to isometrically embed in S^3 the white faces of T_l adjacent to a black face b , such that the resulting gluing is a convex cone with boundary, with polar link at the apex isometric to b , see Figure 3. Moreover, up to congruence, the resulting cone is unique.

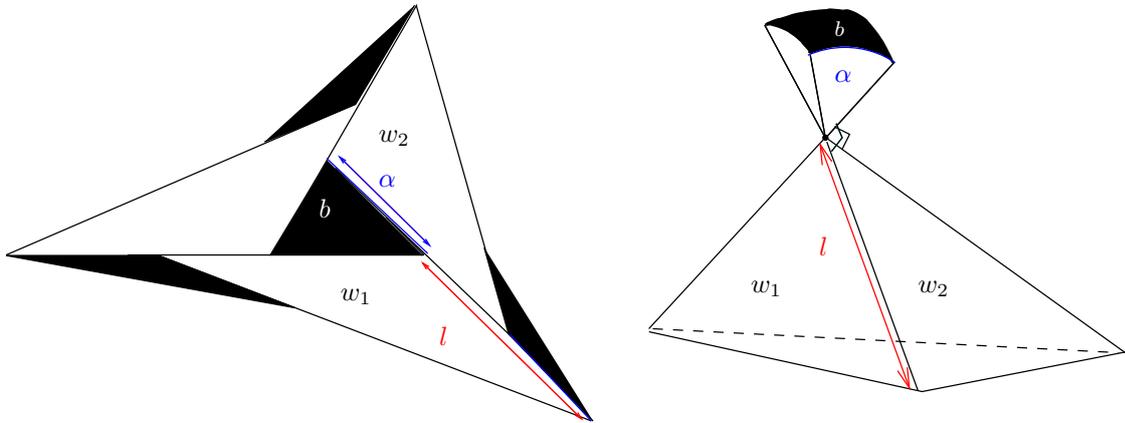


FIGURE 3. White faces around a black face define a convex cone.

We repeat this procedure for the black faces connected to b , and so on. By the principle of global continuation of functions on simply connected manifolds (see for example [Jos06, Lemma 1.4.1]), this provides a well defined map from T_l to S^3 . This map is clearly continuous, and, as it is clear from the definition that the incidence graph of a flippable tiling is connected, we obtain a closed (i.e. compact without boundary) polyhedral surface of S^3 , called the *white polyhedron* of T_l and denoted by $P_w(T_l)$.

Proposition 2.5. *The white polyhedron $P_w(T_l)$ of a left flippable tiling T_l of the sphere S^2 is a convex polyhedron (or a dihedron or a hosohedron) of S^3 , unique up to a global isometry. The 1-skeleton of $P_w(T_l)$ is isotopic to the incident graph of T_l . Moreover a white face of T_l is isometric to its corresponding face of $P_w(T_l)$ and a black face of T_l is isometric to the polar link of the corresponding vertex of $P_w(T_l)$.*

It will be clear from the proof that if T_l has exactly two black (resp. white) faces then $P_w(T_l)$ is a hosohedron (resp. a dihedron).

Proof. By construction, $P := P_w(T_l)$ satisfies the properties stated in the proposition. It remains to check that this polyhedral surface is convex.

Let us denote by M the white metric of T_l (see 1.4.1). Then P is an isometric polyhedral immersion of M into S^3 . Let us denote by i this immersion. Let p be a cone singularity of M and let V be the set of white faces on M having p as vertex. By definition $i(V)$ is a convex cone in S^3 . Note that as the metric on T_l is the round metric on S^2 , the distance between two conical singularities of M is $\leq \pi$.

Let H be an open hemisphere which contains $i(V)$ and such that $i(p)$ belongs to the boundary ∂H (i.e. ∂H is a support plane of $i(V)$). Suppose that the distance between p and another conical singularity p' belonging to the closure \bar{V} of V is π . As the faces of the tiling are supposed to be convex spherical polygons or digons, this means that p and p' are two vertices of a digon. Each edge of the digon is the edge of a white face, and by definition of the tiling the white faces next to this one have to be digons. Finally the cone with apex $i(p)$ is a hosohedron and this is actually the whole P .

Now we suppose that $i(\bar{V})$ is contained in H . We will prove that in this case $P \setminus \{i(p)\}$ is contained in H . Suppose it is true. Then there exists an affine chart of S^3 in which P is a locally convex compact polyhedral surface, and it follows that P is convex [VH52]. Hence P is a dihedron or a convex polyhedron, depending if it has empty interior or not. Let us prove that $P \setminus \{i(p)\}$ is contained in H . This is a straightforward adaptation to the polyhedral case of the argument of [dCW70], which concerns differentiable surfaces. For convenience we repeat it.

We denote by π the map from H to the affine chart of S^3 defined by H and by N the connected component of $i^{-1}(H)$ containing $V \setminus \{p\}$. $\pi(i(N))$ is a locally convex subset of the Euclidean space, and as M is compact, $\pi(i(N))$ is complete for the metric induced by the Euclidean metric [dCW70, 2.5]. Moreover $\pi(i(N))$ contains the image of a neighborhood of a conical singularity q of M different from p (take a conical singularity contained in the boundary of V). In the Euclidean space, the image of q is the apex of a truncated convex cone contained

in the image of N . Finally $\pi(i(N))$ is unbounded (π sends $i(p)$ to infinity). By [VH52], $\pi(i(N))$ is homeomorphic to a plane, then N is homeomorphic to a plane. It follows that, as ∂V is homeomorphic to a circle, its image for $\pi \circ i$ separates $\pi(i(N))$ into two connected components. Let us denote by W the bounded one. It is easy to see that if $m \in N \setminus V$, then $\pi(i(m)) \in W$ (suppose the converse and look at the image of a segment between m and p). This implies that no point of ∂H , except $i(p)$, is a limit point of $i(N)$. As M is connected it follows that $P \setminus \{i(p)\}$ is contained in H . \square

The two constructions above (from a polyhedron to a flippable tiling and from a flippable tiling to a polyhedron) are of course inverse one from the other, for example for left flippable tilings:

$$(1) \quad T_l(P_w(T_l)) = T_l, \quad P_w(T_l(P)) = P.$$

The *black polyhedron* $P_b(T_l)$ of a left flippable tiling T_l is constructed similarly, but the faces of $P_b(T_l)$ are isometric to the black faces of T_l , and the polar links at the vertices of $P_b(T_l)$ are isometric to the white faces of T_l . The definitions of white and black polyhedron for a right flippable tiling are straightforward. We have for example:

$$P_b(T_l) = P_w(T_l)^* = P_w(T_l^*).$$

2.3. Proof of Theorem S1. Let T_r be a right flippable tiling on the sphere. The left flippable tiling

$$T'_r := T_l(P_w(T_r))$$

is obtained by flipping T_r . Similarly, if T_l is a left flippable tiling on the sphere, the right flippable tiling

$$T'_l := T_r(P_w(T_l))$$

is obtained by flipping T_l . Moreover $(T'_r)' = T_r$ due to (1).

2.4. Proof of Theorem S2.

2.4.1. Flippable tilings vs convex polyhedra. Proposition 2.3 and Proposition 2.5 determine a one-to-one map between flippable tilings on the sphere S^2 and convex polyhedra in S^3 . Moreover the incidence graph of a flippable tiling is equal to the 1-skeleton of the corresponding convex polyhedron.

There is a strong relation between on the one hand the topology on the space of flippable tilings on the sphere (as defined right before Theorem S2) and on the other hand the natural topology on the space of convex polyhedra in S^3 (the Hausdorff topology on the interior of the polyhedra, considered up to global isometry).

Lemma 2.6. *Let $(T_{l,n})_{n \in \mathbb{N}}$ be a sequence of left flippable tilings on S^2 , with the same number of black faces. This sequence converges to a left flippable tiling T_l if and only if the sequence of the white polyhedra $P_w(T_{l,n})$ converges to a convex polyhedron P . P is then the white polyhedron of the limit T_l .*

In the statement above and in the proof below, P is a spherical convex polyhedron in a wide sense: it can also be a dihedron or a hosohedron.

Proof. Clearly we can suppose that all the polyhedra $P_w(T_{l,n})$ have the same combinatorics. However the combinatorics of the limit P might be different.

Suppose that $(P_w(T_{l,n}))$ converges to a limit P . For all n , the white faces of $T_{l,n}$ correspond to the faces of $P_w(T_{l,n})$, and the set of faces of $P_w(T_{l,n})$ converges to the set of faces of P . The black faces of $T_{l,n}$ correspond to the polar links of the vertices of $P_w(T_{l,n})$, which are the faces of the dual polyhedron $P_w(T_{l,n})^*$. However the polar map is continuous [RH93, 2.8] and thus $P_w(T_{l,n})^* \rightarrow P^*$, so that the faces of the polyhedra $P_w(T_{l,n})^*$ converge to the faces of P^* . It follows that the flippable tilings $T_{l,n}$ converge, in the topology defined before Theorem S2, to a left flippable tiling T_l , with $P = P_w(T_l)$.

The same argument can be used conversely, for a sequence of flippable tilings $(T_{l,n})$ converging to a limit T . Then both the faces and the dual faces of the corresponding polyhedra converge, so that the sequence of those polyhedra has to converge. \square

The same lemma, along with its proof, holds also for flippable tilings on hyperbolic surfaces, in relation to equivariant polyhedral surfaces in AdS_3 .

2.4.2. *The proof.* To prove that $\mathcal{T}_n^{r,1}(\Gamma)$ is non-empty it is sufficient to show the existence of a convex polyhedron $P \subset S^3$ with 1-skeleton equal to Γ . However according to a well-known theorem of Steinitz, any 3-connected planar graph is the 1-skeleton of a convex polyhedron $Q \subset \mathbb{R}^3$. One can then consider the image of Q in a projective model of a hemisphere $H \subset S^3$, which is a convex polyhedron in H with the same combinatorics as Q . This proves that $\mathcal{T}_n^{r,1}(\Gamma)$ is non-empty as soon as Γ is 3-connected.

Under this assumption, and according to Lemma 2.6, the topology of $\mathcal{T}_n^{r,1}(\Gamma)$ is the same as the topology of the space of convex polyhedra in S^3 with 1-skeleton equal to Γ , considered up to global isometries. But the same argument based on projective models of the hemisphere shows that it is the same as the topology of the space of convex polyhedra in \mathbb{R}^3 with 1-skeleton equal to Γ , because any convex polyhedron in S^3 is contained in a hemisphere and is therefore the image in the projective model of this hemisphere of a convex Euclidean polyhedron. Finally it is known that the space of convex Euclidean polyhedra of given combinatorics is contractible, this is a refinement of the Steinitz theorem, see [RG96, Section 13.3, p. 144]. So $\mathcal{T}_n^{r,1}(\Gamma)$ is homeomorphic to a ball.

Still by Lemma 2.6, $\mathcal{T}_n^{r,1}$ is homeomorphic to the space of convex polyhedra with n vertices in S^3 , considered up to global isometry. This space is itself homeomorphic to a manifold of dimension $3n - 6$ — the space of n -tuples of points in convex position in the sphere — and point (2) follows.

2.5. Proof of Theorem S4. Theorem S4 is a direct consequence of the following famous theorem.

Theorem 2.7 (Alexandrov Existence and Uniqueness Theorem, [Ale05]). *Let g be a spherical metric with $n > 2$ conical singularities of positive curvature on the sphere S^2 . Then there exists a convex polyhedron P_g such that the induced metric on the boundary of P_g is isometric to g . Moreover P_g is unique up to global isometries.*

Let us be more precise. If P is a convex polyhedron, then it is clear that the black metric of $T_r(P)$ is the induced metric on the boundary of P^* , or equivalently that the black metric of $T_r(P)^*$ is the induced metric on the boundary of P .

Now let g and P_g be as in the statement above. Then $T_r(P_g)^* =: T_l$ is a left flippable tiling of the sphere, and its black metric is the induced metric on P_g , that is g . This proves the existence part of Theorem S4. Moreover the uniqueness part of Theorem 2.7 implies that $P_g = P_b(T_l)$, that implies the uniqueness part of Theorem S4.

Remark 2.8. Theorem 2.7 is false for metrics with two conical singularities. Actually for such metrics the cone points have to be antipodal (see Theorem 2.9 below), and then they are realized in S^3 by hosohedra, but it is easy to construct two non-congruent hosohedra having the same induced metric. It follows that for $n = 2$ the existence part of Theorem S4 is true, but the uniqueness part is false. The case $n = 1$ is meaningless as there does not exist spherical metric on the sphere with only one conical singularity (Theorem 2.9).

Theorem 2.9 ([Tro89, Tro93]). *Let g be a spherical metric on the sphere with two conical singularities of positive curvature. Then the two singularities are antipodal and the curvature of both singularities are equal. Furthermore, any two such surfaces are isometric if and only if they have the same curvature.*

A spherical metric on the sphere cannot have only one conical singularity.

Corollary 2.10. *The only examples of flippable tiling on the sphere with two black faces are the ones described in Example 1.2. Moreover there does not exist any flippable tiling on the sphere with only one black face or only one white face.*

Proof. Consider the white metric of a flippable tiling on the sphere with two black faces. We obtain a spherical metric on the sphere with two conical singularities of positive curvature. Theorem 2.9 says that the singularities have to be antipodal, hence the black faces of the tiling have to be antipodal. \square

2.6. Proof of Theorem S3. Let $\mathcal{M}_n^1(K)$ be the space of spherical metrics on the sphere with n conical singularities of positive curvature k_i (K is as in Definition 1.8).

There is a map from $\mathcal{T}_n^{r,1}(K)$ to $\mathcal{M}_n^1(K)$ which consists of taking the white metric. This map is bijective as its inverse is given by (a suitable version) of Theorem S4. The proof of Theorem S3 follows because by the following theorem $\mathcal{M}_n^1(K)$ is parameterized by the space of configurations of n distinct points on S^2 .

Theorem 2.11 ([LT92]). *Let $k = (k_1, \dots, k_n) \in (\mathbb{R}_+)^n$, $n > 2$, such that $\sum_{i=1}^n k_i < 4\pi$ and $\sum_{i \neq j} k_i > k_j$. For each conformal structure on the n -punctured sphere there exists a unique conformal spherical metric with cone points of curvature k_i .*

(The existence part was obtained in [Tro91].) Note that Theorem S3 is meaningless for $n = 1$ and false for $n = 2$ due to Theorem 2.9.

3. SOME ADS GEOMETRY

In this section we recall the definition and some basic properties of the anti-de Sitter space AdS_n , and in particular of AdS_3 . We point out in particular those properties reminiscent of those of the 3-dimensional sphere, which are most relevant for flippable tilings.

3.1. The n -dimensional anti-de Sitter space and its projective model. The anti-de Sitter (AdS) space can be considered as the Lorentzian analog of the hyperbolic space. It can be defined as a quadric in $\mathbb{R}^{n-1,2}$, which is simply \mathbb{R}^{n+1} with a symmetric bilinear form of signature $(n-1, 2)$:

$$\langle x, y \rangle_2 = x_1 y_1 + \cdots + x_{n-1} y_{n-1} - x_n y_n - x_{n+1} y_{n+1} ,$$

then

$$AdS_n := \{x \in \mathbb{R}^{n-1,2} \mid \langle x, x \rangle_2 = -1\} ,$$

with the induced metric. Here are some of its key properties.

- AdS_n is a Lorentzian space of constant curvature -1 ,
- it is geodesically complete,
- it is *not* simply connected, its fundamental group is \mathbb{Z} and its universal cover is denoted by $Ad\tilde{S}_n$,
- its totally geodesic space-like hyperplanes are all isometric to the hyperbolic space H^{n-1} ,
- its time-like geodesics are closed curves of length 2π ,
- there is a well-defined notion of “distance” between two points, defined implicitly by the relation:

$$\cosh d(x, y) = -\langle x, y \rangle_2 .$$

This distance is real and positive when x and y are connected by a space-like geodesic, imaginary when x and y are connected by a time-like geodesic. It is in both cases equal to the length of the geodesic segment connecting them.

It is sometimes also convenient to consider the quotient of AdS_n by the antipodal map (sending x to $-x$ in the quadric model above), we will simply denote it by AdS_n/\mathbb{Z}_2 here. In this space, time-like geodesics are closed curves of length π .

As for H^n , there is a useful projective model of AdS_n , obtained by projecting in $\mathbb{R}^{n-1,2}$ from the quadric to any tangent plane in the direction of the origin. However this yields a projective model of only “half” of AdS_n (as for the n -dimensional sphere). For instance, the projection on the hyperplane H of equation $x_{n+1} = 1$ is a homeomorphism from the subset of AdS_n of points where $x_{n+1} > 0$ to its image, which is the interior of the intersection with H of the cone of equation $\langle x, x \rangle_2 = 0$ in $\mathbb{R}^{n-1,2}$. This intersection is again a quadric, but now in \mathbb{R}^n .

This model can be considered in the projective n -space $\mathbb{R}P^n$, it provides a projective model of the whole quotient AdS_n/\mathbb{Z}_2 as the interior of a quadric of signature $(n-1, 1)$. Lifting to the double cover of $\mathbb{R}P^n$ we find a projective model of the whole space AdS_n as the interior of a quadric in S^n .

This projective model naturally endows AdS_n with a “boundary at infinity”, topologically $S^{n-2} \times S^1$. This boundary at infinity comes with a conformal Lorentz structure, namely the conformal class of the second fundamental form of the quadric in \mathbb{R}^n (or equivalently $\mathbb{R}P^n$, resp. S^n).

3.2. The 3-dimensional space AdS_3 . As a special case of the n -dimensional case we find the 3-dimensional AdS space which will be of interest to us in the sequel:

$$AdS_3 := \{x \in \mathbb{R}^{2,2} \mid \langle x, x \rangle_2 = -1\} .$$

It has some specific features which are not present in the higher-dimensional case. In some respects AdS_3 can be considered as a Lorentz analog of the 3-dimensional sphere.

- AdS_3 is isometric to $SL(2, \mathbb{R})$ with its bi-invariant Killing metric. (Similarly, S^3 is identified with $SU(2)$).
- Therefore, it has an isometric action of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, where the left factor acts by left multiplication and the right factor acts by right multiplication. The kernel of this action is reduced to $\{(I, I), (-I, -I)\}$ and this identifies the identity component of the isometry group of AdS_3 with $SL(2, \mathbb{R}) \times SL(2\mathbb{R})/\mathbb{Z}_2$. (There is a similar isometric action of $O(3) \times O(3)$ on S^3 .)
- There is a notion of duality in AdS_3 reminiscent of the polar duality in S^3 . It associates to each oriented totally geodesic plane in AdS_3 a point, and conversely. The dual of a convex polyhedron is a convex polyhedron, with the edge lengths of one equal to the exterior dihedral angles of the other.

The decomposition of the isometry group of AdS_3 as $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})/\mathbb{Z}_2$ can also be understood as follows. In the projective model above, AdS_3/\mathbb{Z}_2 is identified with the interior of a quadric $Q \subset \mathbb{R}P^3$. Q is foliated by two families of projective lines, which we call \mathcal{L}_l and \mathcal{L}_r and call the “left” and “right” families of lines. Each line of each family intersects each line of the other family at exactly one point, and this provides both \mathcal{L}_l and \mathcal{L}_r with a projective structure. The isometries in the identity component of $\text{Isom}(AdS_3/\mathbb{Z}_2)$ permute the lines of \mathcal{L}_l and of \mathcal{L}_r , and the corresponding action on \mathcal{L}_l and \mathcal{L}_r are projective, so they define two elements of $PSL(2, \mathbb{R})$.

3.3. Globally hyperbolic AdS manifolds. We recall in this section some known facts on globally hyperbolic AdS 3-manifolds, mostly from [Mes07], which are useful below.

Let S be a closed surface of genus at least 2. We denote by $\text{Isom}_0(AdS_3)$ the identity component of the isometry group of AdS_3 .

An AdS 3-manifold is a Lorentz manifold with a metric locally modelled on the 3-dimensional AdS space, AdS_3 . It is *globally hyperbolic* (GH) if it contains a closed space-like surface S , and any inextensible time-like curve intersects S at exactly one point. It is *globally hyperbolic maximal* (GHM) if it is maximal — for the inclusion — under those conditions.

Globally hyperbolic AdS manifolds are never geodesically complete. However, a GHM AdS manifold is always the quotient by $\pi_1(S)$ of a convex subset Ω of AdS_3 . This defines a representation ρ of $\pi_1(S)$ in $\text{Isom}_0(AdS_3)$, the identity component of the isometry group of AdS_3 . This identity component decomposes as $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$, so ρ decomposes as a pair of representations (ρ_l, ρ_r) of $\pi_1(S)$ in $PSL(2, \mathbb{R})$. Mess [Mes07, ABB⁺07] proved that those two representations have maximal Euler number, so that they are holonomy representations of hyperbolic metrics on S .

He also proved the following theorem, which can be considered as an AdS analog of the Bers Double Uniformization Theorem for quasifuchsian hyperbolic manifolds.

Theorem 3.1 (Mess [Mes07]). *Given two hyperbolic metrics $h_l, h_r \in \mathcal{T}$ on S , there is a unique globally hyperbolic AdS structure on $S \times (0, 1)$ such that ρ_l (resp. ρ_r) is the holonomy representation of h_l (resp. h_r).*

We will say that M is *Fuchsian* if it contains a totally geodesic, closed, space-like, embedded surface. This happens if and only if the representations ρ_l and ρ_r are conjugate.

Let M be a GHM AdS 3-manifold. Then M contains a smallest non-empty convex subset $C(M)$, called its *convex core*. If M is Fuchsian, then $C(M)$ is the (unique) totally geodesic space-like surface in M . Otherwise, $C(M)$ is topologically the product of S by an interval. Its boundary is the disjoint union of two space-like surfaces each homeomorphic to S . Those two surfaces are pleated surfaces, their induced metric is hyperbolic and they are bent along a measured lamination.

There are many future convex surfaces (surfaces for which the future is convex) in the past of $C(M)$, and many past convex surfaces in the future of $C(M)$. Given a strictly future convex surface $S_c \subset M$, we can define the dual surface (using the duality in AdS_3 , see 3.5.1) as the set of points dual to the tangent planes of S_c , it is a past convex surface, and conversely.

3.4. Equivariant polyhedral surfaces. Consider a globally hyperbolic AdS 3-manifold M , and a closed, space-like surface Σ in M . Then Σ lifts to an *equivariant surface* in the universal cover of M , considered as a subset of AdS_3 . We recall here the basic definitions and key facts on those equivariant surfaces, focussing on the polyhedral surfaces which are relevant below.

3.4.1. Equivariant embeddings. Given a closed polyhedral surface S in an AdS 3-manifold M , one can consider a connected component of the lift of S to the universal cover of M . It is a polyhedral surface invariant under the action of a surface group. This leads to the definition of an equivariant embedding of a surface in AdS_3 , as follows.

Definition 3.2. An *equivariant polyhedral embedding* of S in AdS_3 is a couple (ϕ, ρ) , where:

- ϕ is a locally convex, space-like polyhedral immersion of \tilde{S} in AdS_3 ,
- $\rho : \pi_1(S) \rightarrow \text{Isom}_0(AdS_3)$ is a homomorphism,

satisfying the following natural compatibility condition:

$$\forall x \in \tilde{S}, \forall \gamma \in \pi_1(S), \phi(\gamma.x) = \rho(\gamma)\phi(x) .$$

An equivariant polyhedral surface is a surface of the form $\phi(\tilde{S})$, where (ϕ, ρ) is an equivariant polyhedral embedding of S in AdS_3 .

The assumption that $\phi(\tilde{S})$ is space-like means that it is made of pieces of space-like planes, but also that each support plane is space-like. With this assumption we get the following classical results, see [Mes07, ABB⁺07, BBZ07] and references therein. The *boundary at infinity* ∂F of a surface F of AdS_3 is the intersection, in an affine chart, of the closure of the surface with the boundary at infinity of AdS_3 .

Lemma 3.3. *Let (ϕ, ρ) be an equivariant polyhedral embedding of S in AdS_3 . Then*

- (1) ϕ is an embedding,
- (2) each time-like geodesic of AdS_3 meets $\phi(\tilde{S})$ exactly once,
- (3) each light-like geodesic of AdS_3 meets $\phi(\tilde{S})$ at most once,
- (4) there exists an affine chart which contains $\phi(\tilde{S})$,
- (5) and if a light-like geodesic has one of its endpoints on $\partial\phi(\tilde{S})$ then the geodesic does not meet $\phi(\tilde{S})$.

A key remark, basically obtained by G. Mess [Mes07], is that the holonomy representation is the “product” of the holonomy representations of two hyperbolic surfaces.

Proposition 3.4. *Let (ϕ, ρ) be an equivariant polyhedral embedding of S in AdS_3 , let $\rho = (\rho_l, \rho_r) \in PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$. Then ρ_l and ρ_r have maximal Euler number, so that they are the holonomy representations of hyperbolic metrics on S .*

Proof. Consider the polyhedral surface $\phi(\tilde{S})$. It has a free and properly discontinuous cocompact action of $\rho(\pi_1(S))$. Therefore it follows from Lemma 3.3 that there is a neighborhood U of $\phi(\tilde{S})$ in AdS_3 on which $\rho(\pi_1(S))$ also acts freely and properly discontinuously [BBZ07, 5.22]. The quotient $M = U/\rho(\pi_1(S))$ is an AdS 3-manifold which contains a closed space-like surface (namely, $\phi(\tilde{S})/\rho(\pi_1(S))$).

It follows (see [Mes07, ABB⁺07]) that M has an isometric embedding in a (unique) globally hyperbolic AdS 3-manifold N . Moreover $N = \Omega/\rho(\pi_1(S))$, where Ω is a maximal convex subset of AdS_3 on which $\rho(\pi_1(S))$ acts properly discontinuously. Finally, it is shown in [Mes07] that $\rho = (\rho_l, \rho_r)$, where ρ_l and ρ_r have maximal Euler number. \square

3.4.2. *Fuchsian equivariant surfaces.* Among the equivariant surfaces, some are simpler to analyze.

Definition 3.5. An equivariant polyhedral embedding (ϕ, ρ) of S in AdS_3 is *Fuchsian* if its representation ρ globally fixes a totally geodesic space-like plane in AdS_3 . A *Fuchsian polyhedral surface* is a polyhedral surface of the form $\phi(\tilde{S})$, where (ϕ, ρ) is a Fuchsian equivariant polyhedral embedding.

Equivalently, (ϕ, ρ) is Fuchsian if, in the identification of $\text{Isom}_0(AdS_3/\mathbb{Z}_2)$ with $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$, ρ corresponds to a couple $(\rho_l, \rho_r) \in PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ such that ρ_l is conjugate to ρ_r .

3.4.3. *Equivariant polyhedral surfaces as convex hulls.* Let (ϕ, ρ) be a convex polyhedral embedding of S . If V is the set of the vertices of $\phi(\tilde{S})$, we denote by $\partial CH(V)$ the boundary of the convex hull of V (there exists an affine chart which contains $\phi(\tilde{S})$, see Lemma 3.3, hence it is meaningful to speak about convex hull). Let v_1, \dots, v_n be a set of representatives of the equivalence classes for the action of $\rho(\pi_1(S))$ on V . The embedding is said to have n fundamental vertices.

Lemma 3.6. *An equivariant polyhedral embedding (ϕ, ρ) of S is determined by ρ and v_1, \dots, v_n . Moreover any sufficiently small perturbation of the v_i gives another equivariant polyhedral embedding of S with n fundamental vertices.*

Proof. It is clear that the data of ρ and v_1, \dots, v_n gives V .

Recall that the *convex core* $C(\rho)$ for ρ is the minimal (for the inclusion) non-empty convex set in AdS_3 globally invariant under the action of ρ . If (ϕ, ρ) is Fuchsian, the convex core is reduced to a totally geodesic surface, otherwise it has non-empty interior and two boundary components, which are pleated space-like surfaces, isometric to the hyperbolic plane for the induced metric. The convex core is the convex hull of its boundary at infinity, which is a (topological) circle $c(\rho)$ drawn on the boundary at infinity of AdS_3 . Actually this boundary at infinity is exactly $\partial\phi(\tilde{S})$. Note that $\phi(\tilde{S})$ does not meet the interior of $C(\rho)$, otherwise the intersection between $C(\rho)$ and the half-spaces delimited by the planes containing the faces of $\phi(\tilde{S})$ would define a convex set invariant under ρ and contained in $C(\rho)$, this is impossible. It follows that $\phi(\tilde{S})$ is contained in one side of $C(\rho)$. Let us denote by $\partial C(\rho)_-$ the component of the boundary of $C(\rho)$ opposite to $\phi(\tilde{S})$.

In an affine chart, $\phi(\tilde{S})$ and $\partial C(\rho)_-$ are both locally convex surfaces. The intersection of the closure of both with $\partial_\infty AdS_3$ is the curve $c(\rho)$. Since $c(\rho)$ is the boundary at infinity of a space-like surface in AdS_3 , it is a weakly space-like curve in $\partial_\infty AdS_3$, for its conformal Lorentz structure. (Weakly space-like means that no point of $c(\rho)$ is in the future of another.) Given a point $x \in c(\rho)$, there is a unique light-like plane p_x in

AdS_3 through x . The intersection of p_x with $\partial_\infty AdS_3$ is the union of the two light-like geodesics of $\partial_\infty AdS_3$ through x (again for the conformal Lorentz structure on $\partial_\infty AdS_3$). Therefore $c(\rho)$, since it is weakly space-like, is contained in the half-space bounded by p_x , and so are $\phi(\tilde{S})$ and $\partial C(\rho)_-$. So p_x is a support plane of the closure of $\phi(\tilde{S}) \cup \partial C(\rho)_-$. So this closure is locally convex at each point of $c(\rho)$, and it is therefore locally convex, and it is therefore a convex surface.

It follows that the elements of V are extreme points for the convex hull of V . As the elements of V accumulates on $c(\rho)$, the boundary of the convex hull of V is made of the union of $\phi(\tilde{S})$ and $\partial C(\rho)_-$ plus a circle at infinity.

Now perturb slightly the v_1, \dots, v_n , this gives new points v'_1, \dots, v'_n . Denote by V' the orbits of the v'_i under ρ and by $\phi'(\tilde{S})$ the boundary of the convex hull of V' minus $\partial C(\rho)_-$. If the v'_i are sufficiently near the v_i , all the v'_i are extreme points of $\phi'(\tilde{S})$. Moreover $\phi'(\tilde{S})$ is also disjoint from $C(\rho)$. We saw that $\phi(\tilde{S})$ is disjoint from the interior of $C(\rho)$, but it is also disjoint from its boundary. Otherwise, if it touched it at a point x , by convexity $\phi(\tilde{S})$ should contain the geodesic line of the boundary of $C(\rho)$ containing x , this is impossible as $\phi(\tilde{S})$ is a polyhedral surface (in particular with compact faces). As the action of $\rho(\pi_1(S))$ is cocompact on $\phi(\tilde{S})$ and $C(\rho)$, the distance between both is bounded from below. Hence if v'_1, \dots, v'_n are sufficiently close to v_1, \dots, v_n , $\phi'(\tilde{S})$ is made of totally geodesic faces. It is also space-like (with space-like support planes) as the fact that two points are on a same space-like geodesic is also an open condition.

Note finally that if F is a face of the convex hull of V , it has a finite number of vertices in V by definition of a polyhedral embedding. Moreover, in a projective model, the boundary at infinity of the space-like plane P containing F is disjoint from the intersection with $\partial_\infty AdS_3$ of the closure of $\phi(\tilde{S})$. It follows that all points in V which are not a vertex of F are outside a small neighborhood of P in the projective model of AdS_3 .

Now consider a deformation with the v'_i close enough to the v_i . Given three vertices x, y, z of F , let x', y', z' be the corresponding points of V' , and let p' be the totally geodesic plane containing x', y', z' . For a small enough deformation, any point of V which is not a vertex of F remains in the past of p' , and so does $\partial_\infty \phi(\tilde{S})$. It follows that the face F corresponds, in the deformed polyhedral surface, to a union of faces having as vertices the vertices of F . In other terms all faces of the deformed polyhedral surface remain compact, with a finite number of vertices. \square

3.5. The left and right multiplications. This part is very similar to subsections 2.1 and 2.2, so we skip some details.

3.5.1. Polar duality and multiplication. The *polar link* $Lk(x)$ of a vertex x of a convex space-like polyhedral surface P in AdS_3 is the convex hyperbolic polygon in $T_x AdS_3$ given by the outward unit time-like normals of the faces of P meeting at x . The *exterior dihedral angle* of an edge of P with endpoint x is the length of the corresponding edge of $Lk(x)$. The interior angles of $Lk(x)$ are π minus the interior angles on the faces of P at x .

Let $x \in AdS_3$. We denote by x^* the hyperplane orthogonal to the vector $x \in \mathbb{R}^{2,2}$ for $\langle \cdot, \cdot \rangle_2$. The intersection between x^* and AdS_3 is space-like and has two connected components, which are antipodal. We also denote by x^* the intersection between the hyperplane and AdS_3 . Conversely, if H is a totally geodesic plane in AdS_3 , we denote by H^* the unique point of AdS_3/\mathbb{Z}_2 orthogonal as a vector to the hyperplane containing H .

The *polar dual* P^* of a convex space-like polyhedral surface P is a convex polyhedral surface defined as the intersection of the half-spaces $\{y \in AdS_3 | \langle x, y \rangle_2 < 0\}$ for each vertex x of P . Faces of P are contained in x^* . Actually this defines two convex polyhedral surfaces, which are identified in AdS_3/\mathbb{Z}_2 . Equivalently P^* can be defined as the convex hull (in an affine chart) of the H^* for H a plane containing a face of P . It follows that $(P^*)^* = P$ and that the polar link $Lk(x)$ of a vertex x of P is isometric to the face of P^* dual to x , and vice-versa. It is easy to see that the dual of a space-like polyhedral surface is space-like. Moreover if P is a convex polyhedral surface equivariant under the action of ρ , as ρ acts by isometries, P^* is also equivariant under the action of ρ .

We will identify the two models of AdS_3 through the following isometry:

$$AdS_3 \subset (\mathbb{R}^4, \langle \cdot, \cdot \rangle_2) \rightarrow (SL(2, \mathbb{R}), -\det), \quad (x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_2 + x_4 & x_1 + x_3 \\ x_1 - x_3 & x_4 - x_2 \end{pmatrix}.$$

The neutral element as a vector of \mathbb{R}^4 is then $e = (0, 0, 0, 1)$, and if $y = (y_1, y_2, y_3, y_4)$ then $y^{-1} = (-y_1, -y_2, -y_3, y_4)$.

Let a, b be in AdS_3/\mathbb{Z}_2 , with a, b, e mutually different, and such that a^* and b^* meet along a 2-plane E in $\mathbb{R}^{2,2}$. Let A be the following angle in AdS_3 :

$$\begin{aligned} A &= a_c^* \cup b_c^* \cup E \\ &= \{x \in AdS_3/\mathbb{Z}_2 \mid \langle x, a \rangle_2 = 0, \langle x, b \rangle_2 < 0\} \cup \\ &\quad \{x \in AdS_3/\mathbb{Z}_2 \mid \langle x, b \rangle_2 = 0, \langle x, a \rangle_2 < 0\} \cup \{x \in AdS_3/\mathbb{Z}_2 \mid \langle x, a \rangle_2 = \langle x, b \rangle_2 = 0\}. \end{aligned}$$

The *left projection* of the angle A is the following map:

$$L_A : A \setminus E \longrightarrow e^*, x \longmapsto \begin{cases} a^{-1}x & \text{if } x \in a_c^* \\ b^{-1}x & \text{if } x \in b_c^* \end{cases}.$$

The *right projection* of the angle A is defined similarly:

$$R_A : A \setminus E \longrightarrow e^*, x \longmapsto \begin{cases} xa^{-1} & \text{if } x \in a_c^* \\ xb^{-1} & \text{if } x \in b_c^* \end{cases}.$$

We assume that R_A and L_A have values in AdS_3/\mathbb{Z}_2 , so that (the quotient of) e^* is isometric to the hyperbolic plane H^2 : we consider that L_A and R_A have values in H^2 . By arguments very similar to the proof of Lemma 2.2 for the spherical case, it is easy to see that the left (resp. right) projection sends isometrically the faces of the angle to two half-parts of the hyperbolic plane, in such a way that the part on the left (resp. right) is forward and the part on the right (resp. left) is backward.

3.5.2. Equivariant polyhedral surfaces as flippable tilings. The left and right projections of a space-like convex polyhedral surface P of AdS_3/\mathbb{Z}_2 are defined similarly to the spherical case. Let (ϕ, ρ) be an equivariant polyhedral embedding of S with $\rho = (\rho_r, \rho_l)$ and $\phi(\tilde{S}) = P$. The action of $\rho(\pi_1(S))$ sends a point $x \in P$ to a point $\rho(\gamma)(x) = \rho_l(\gamma)x\rho_r(\gamma)$ of P . If x belongs to a face of P contained in a plane a^* , the left projection of P sends x to $a^{-1}x$ and sends $\rho(\gamma)(x)$ to $\rho_r(\gamma)^{-1}a^{-1}x\rho_r(\gamma)$. If we identify e^* with H^2 , we obtain an action of $\rho_r(\pi_1(S))$ on H^2 . By Proposition 3.4, $H^2/\rho_r(\pi_1(S))$ endows S with a hyperbolic metric g_l . We denote by $T_l(P)$ this hyperbolic surface together with the tiling given by the images of the faces of P . We paint these images in white and the remainder of the surface in black, and by arguments similar to the proof of Proposition 2.3, we get that $T_l(P)$ is a left flippable tiling of the surface.

Similarly, the right projection of P endows S with a hyperbolic metric g_r together with a right flippable tiling denoted by $T_r(P)$. Note that P is Fuchsian if and only if there is an isometry isotopic to the identity between g_r and g_l .

3.5.3. Flippable tilings as equivariant polyhedral surfaces. Let T_l be a left flippable tiling of a compact hyperbolic surface S . We denote by \tilde{T}_l the universal cover of T_l . As in the spherical case, there exists a locally convex polyhedral immersion of the white faces of \tilde{T}_l in AdS_3 , such that the polar links of the vertices are isometric to the black faces. This polyhedral immersion is defined only up to global isometries. We denote by ϕ the choice of one immersion. Let $\phi(x)$ belong to a face of $\phi(\tilde{T}_l)$. There is a unique isometry in the connected component of the identity of $\text{Isom}(AdS_3)$ sending the plane containing this face to the plane containing the face containing $\phi(\gamma x)$, and sending the vertices of the face containing $\phi(x)$ to the corresponding vertices of the face containing $\phi(\gamma x)$. This provides a representation ρ of $\pi_1(S)$ into $\text{Isom}_0(AdS_3)$. Because the polar link at each vertex is isometric to a compact hyperbolic polygon, all the support planes at each vertex are space-like. It follows that (ϕ, ρ) is an equivariant polyhedral embedding of S in AdS_3 in the sense of Definition 3.2. We call it the *white polyhedron* of T_l and denote it by $P_w(T_l)$.

3.5.4. Proof of Theorem H1. By construction, the proof is word by word the same as the one of Theorem S1, considering a closed hyperbolic surface instead of the sphere.

3.5.5. Proof of Theorem 1.5. Given $h_l, h_r \in \mathcal{T}$, there is by Theorem 3.1 a unique globally hyperbolic AdS manifold M with left and right representations ρ_l and ρ_r equal to the holonomy representations of h_l and of h_r , respectively.

A convex polyhedral surface Σ in M determines an equivariant polyhedral surface $\tilde{\Sigma}$ with associated representation $\rho = (\rho_l, \rho_r)$. According to the content of Section 3.5.2, $\tilde{\Sigma}$ determines in turn a left flippable tiling T_l on (S, h_l) such that flipping T_l yields a right flippable tiling T_r on (S, h_r) . So the proof of Theorem 1.5 reduces to showing that M contains a convex polyhedral surface.

However the existence of such a polyhedral surface is quite easy to check. Given a globally hyperbolic AdS manifold M , consider a point x_1 in the future of the convex core $C(M)$. Since $C(M)$ is convex, there is a totally geodesic plane P_1 passing through x_1 and in the future of $C(M)$, and let H_1 be the future of P_1 . Then repeat this procedure with a point x_2 which is not in H_1 , then with a point x_2 not in $H_1 \cup H_2$, etc. After a finite

number of steps, the boundary of the intersection of the complements of the H_i is a closed, convex polyhedral surface in M .

3.5.6. Fuchsian polyhedral surfaces and symmetric tilings. We have already noticed at the end of Subsection 3.5.2 that P is Fuchsian if and only if there is an isometry isotopic to the identity sending $T_r(P)$ to $T_l(P)$. As $T_l(P)$ is obtained by flipping $T_r(P)$, it follows that if P is Fuchsian then $T_r(P)$ is symmetric in the sense of Definition 1.10. Conversely, let T_r be a symmetric right flippable tiling of a surface (S, g_r) . This means that $T'_r = T_l(P_w(T_r))$ is isometric to $T_r = T_r(P_w(T_r))$, i.e. that $P_w(T_r)$ is Fuchsian.

3.6. Proof of Theorem H2. Let Γ be a graph embedded in a closed surface S of hyperbolic type, such that the universal cover of Γ is 3-connected. We will prove that $\mathcal{T}_{S,n}^{r,-1}(\Gamma)$ is non-empty. Let us denote by Γ^* the dual graph of Γ . Recall Thurston's extension to hyperbolic surfaces of Koebe's circle packing theorem (see e.g. [Sch, Sch03]): there exists a unique hyperbolic metric h on S and a unique circle packing in (S, h) such that:

- there is a circle for each vertex of Γ^* , and the circles bound non-intersecting open discs,
- the circles are tangent if the corresponding vertices are joined by an edge of Γ^* ,
- faces of Γ^* are associated to interstices (connected components of S minus the open discs),
- and to each interstice is associated a circle orthogonal to all the circles corresponding to the vertices of this face.

Let H be a totally geodesic space-like plane in AdS_3 . It is isometric to the hyperbolic plane H^2 , and can be identified isometrically with the universal cover of (S, h) . This yields a circle packing \mathcal{C} with incidence graph Γ^* on the quotient of H by an isometric action ρ of $\pi_1(S)$ in the isometry group of H . We extend ρ to an isometric action on AdS_3 .

Let $r \in (0, \pi/2)$, and let H_r be the set of points at distance r in the future of H . For each circle C of \mathcal{C} corresponding to a vertex of Γ^* , consider the set C' of points of H_r which project orthogonally on H to a point of C . C' is the intersection with H_r of a totally geodesic space-like plane H_c , let P_C be the past of H_c . The intersection of the half-spaces P_C for all $C \in \mathcal{C}$ is a convex polyhedron in AdS_3 , invariant under $\rho(\pi_1(S))$, and its boundary S_C is a space-like polyhedral surface in AdS_3 , invariant under ρ .

This equivariant polyhedral surface S_C has the same combinatorics as Γ . To see this, it suffices to note that for all circles of H_r corresponding to faces around a same vertex of Γ , the corresponding planes meet at a same point, namely the apex of the cone tangent to H_r along the image of the circle orthogonal to the previous ones. (This is a projective property. It is easy to see if one considers H in the hyperbolic space, where H_r is a part of the boundary at infinity. Then the projection from H to H_r preserves orthogonality, and the result follows using the polar duality.)

Through the content of Section 3.5.2, we obtain a flippable tiling on (S, h) with incidence graph Γ , this proves that $\mathcal{T}_{S,n}^{r,-1}(\Gamma)$ is non-empty. This proves point (1).

For point (3) note that the space of flippable tilings with n black faces is homeomorphic, by Lemma 2.6, to the space of equivariant polyhedral surfaces with n vertices. By Lemma 3.6, this space is parameterized by the holonomy representation of the fundamental group of the surface in the isometry group of AdS_3 and by the positions of the vertices, so that it is homeomorphic to a manifold of dimension $12g - 12 + 3v$.

We now turn to the proof of part (2), and consider a fixed graph Γ . By Lemma 2.6, it is sufficient to prove that the space of equivariant polyhedral surfaces in AdS_3 , with combinatorics given by Γ , is homeomorphic to a manifold of dimension $6g - 6 + e$.

Denote by n_f the sum, over all faces of Γ , of the number of edges minus 3. We get

$$n_f = 2e - 3f .$$

As the combinatorics is given, the vertices belonging to a same face have to stay in the same totally geodesic plane during the deformation. For each face, given three vertices of the face, the other vertices of the face have to belong to the affine plane (in the projective model of AdS_3) spanned by the three vertices. This gives n_f equations, so $\mathcal{T}_{S,n}^{r,-1}(\Gamma)$ is homeomorphic to a manifold of dimension $12g - 12 + 3n - n_f$. Using the Euler relation $n - e + f = 2 - 2g$ we obtain that the dimension is also equal to $6g - 6 + e$.

Finally point (4) is a direct consequence of the constructions, since the mapping-class group of S clearly acts on the space of flippable tilings.

3.7. Proof of Theorem H4. The proof is an immediate adaptation of the proof of Theorem S4 in Subsection 2.5, using the following result instead of Alexandrov Theorem (Theorem 2.7).

Theorem 3.7 ([Fil11]). *Let (S, g) be a hyperbolic metric with conical singularities of negative curvature on a closed surface g of genus > 1 . Then there exists a Fuchsian equivariant polyhedral embedding (ϕ, ρ) of S such*

that the induced metric on $\partial\phi(\tilde{S})/\pi_1(S)$ is isometric to (S, g) . Moreover, up to global isometries, (ϕ, ρ) is unique among Fuchsian equivariant polyhedral embedding.

4. ALEXANDROV AND MINKOWSKI TYPE RESULTS IN ADS

In this section, we will prove a AdS analogue of a standard result of Euclidean geometry, known as Alexandrov curvature theorem, see for example [Ale05, 9.1] or [Pak]. Roughly speaking, the problem is to prescribe geodesics containing vertices of a convex polyhedron together with the singular curvature at the vertices. It will appear that it is equivalent to an analogue of the Minkowski theorem for Euclidean convex polytopes.

4.1. Two equivalent statements. Let (ϕ, ρ) be a Fuchsian equivariant polyhedral embedding of a surface S in AdS_3 , and let $G = \rho(\pi_1(S))$. By definition G fixes a totally geodesic space-like plane H in AdS_3 , and it follows from the proof of Lemma 3.6 that $\phi(\tilde{S})$ does not meet H . Without loss of generality, suppose that $\phi(\tilde{S})$ is in the future of H . G also fixes the dual H^* of H (H^* is chosen to be in the future of H). The boundary at infinity ∂H of H is also the boundary at infinity of $\phi(\tilde{S})$, and all light-like geodesics passing through a point of ∂H also pass through H^* . By Lemma 3.3 it follows that $\phi(\tilde{S})$ is contained in the past cone of H^* , and that each geodesic orthogonal to H (i.e. with endpoint H^*) meets $\phi(\tilde{S})$ exactly once.

Let GR be a set of future-directed times-like segments of length $\pi/2$ orthogonal to H , invariant under the action of G . The elements of GR are called *half-rays*. We denote by $R = (r_1, \dots, r_n), n \geq 1$ a subset of GR in a fundamental domain for the action of G . Let $\mathcal{P}(G, R)$ be the set, up to global isometries, of Fuchsian polyhedral surfaces invariant under the action of G and such that the vertices are on GR . Recall that the (singular) curvature of a vertex is 2π minus the sum of the face angles around this vertex.

Theorem 4.1. *If k_1, \dots, k_n are negative numbers such that*

$$(2) \quad \sum_{i=1}^n k_i > 2\pi\chi(g)$$

then there exists a unique Fuchsian polyhedral surface in $\mathcal{P}(G, R)$ such that the vertex on r_i has curvature k_i .

Theorem 4.2. *Under the assumptions of Theorem 4.1, there exists a unique Fuchsian polyhedral surface with faces contained in planes orthogonal to GR , such that the face in the plane orthogonal to r_i has area $-k_i$.*

By duality the two statements are equivalent:

Lemma 4.3. *If P is a Fuchsian polyhedral surface given by one of the two theorems above, then the dual P^* of P satisfies the other statement.*

Proof. Let P be the unique Fuchsian polyhedral surface satisfying Theorem 4.1 for the data (r_1, \dots, r_n) and (k_1, \dots, k_n) . Let x be a vertex of P on the half-ray r_i . By definition of the polar dual, the plane x^* is orthogonal to each time-like geodesic passing through x . In particular, the geodesic containing r_i is orthogonal to x^* . More precisely, x^* is orthogonal to a half-ray \tilde{r}_i obtained from r_i by a symmetry with respect to H . By definition, P^* is the intersection of all the x_i^* for each vertex x_i of P , and by the Gauss–Bonnet Formula and by definition of the polar link, the area of the corresponding face of P^* is $-k_i$. Hence P^* satisfies the existence part of Theorem 4.2 for the data $(\tilde{r}_1, \dots, \tilde{r}_n)$ and (k_1, \dots, k_n) .

The reversed direction for the existence part is done in the same way, and this implies the equivalence of the uniqueness parts. □

Note also that condition (2) is necessary in Theorem 4.1 because of Gauss–Bonnet formula, and by the preceding lemma it is also necessary in Theorem 4.2. Conversely to the Euclidean Alexandrov curvature theorem, there is no other condition on the curvatures. Actually Theorem 4.1 can be thought as the AdS analog of the Alexandrov curvature theorem for convex caps, in which Gauss–Bonnet is the only condition on the curvatures [Pak].

The analog of Theorem 4.1 for Fuchsian polyhedral surfaces in Minkowski space was proved in [Isk00]. It was extended to general Fuchsian surfaces in this space, and to higher dimensions, in [Ber10].

4.2. Proof of Theorem H3 from Theorem 4.1. Let T_r be a symmetric flippable tiling of $\overline{\mathcal{T}}_{S,h,n}^{r,-1}(K)$. From Subsubsection 3.5.6 we can choose a Fuchsian polyhedral surface P constructed from T_r , such that the singular curvatures of P (i.e. minus the area of the polar links of its vertices) are equal to the area of the black faces of T_r . This gives us a group $G \subset \text{Isom}(AdS_3)$ fixing a totally geodesic space-like plane H . Hence $\overline{\mathcal{T}}_{S,h,n}^{r,-1}(K)$ is in bijection with $\mathcal{P}(G, R)$. Theorem 4.1 says that $\mathcal{P}(G, R)$ is parameterized by the orthogonal projection of the

vertices onto H , which is the same as the configuration of n points on $H/G \simeq S$. This is exactly the content of Theorem 4.1.

4.3. Proof of Theorem 4.1. The proof uses a classical continuity method. We recall here a basic topology theorem at the heart of this continuity method.

Lemma 4.4. *Let A, B be two manifolds of the same finite dimension with a continuous map $I : A \rightarrow B$. If*

- (1) A is connected,
- (2) B is connected and simply connected,
- (3) I is locally injective,
- (4) I is proper,

then I is a homeomorphism.

Proof. As I is continuous and locally injective, it is a local homeomorphism by the invariance of domain theorem. The conclusion follows because a local homeomorphism between a pathwise connected Hausdorff space and a simply connected Hausdorff space is a global homeomorphism if and only if the map is proper (see for example [Ho75]). \square

Let $P \in \mathcal{P}(G, R)$. The *height* h_i of P is the distance from its vertex on r_i to the hyperbolic plane H . We know from Lemma 3.6 that an element of $\mathcal{P}(G, R)$ is determined by its heights (h_1, \dots, h_n) , and that $\mathcal{P}(G, R)$ is a manifold of dimension n . We need a slightly more here. The matter is simplified because in the Fuchsian case the convex core has empty interior.

Lemma 4.5. *Let (h_1, \dots, h_n) , $0 < h_1 < \pi/2$. If the orbits for G of the points on the segments r_i at distance h_i from H are in convex position, then the convex hull of these orbits has two connected components, one is H and the other belongs to $\mathcal{P}(G, R)$.*

Moreover $\mathcal{P}(G, R)$ is contractible.

Proof. Let P be the convex hull of the orbits less H . By construction P is invariant under the action of G and has its vertices on GR . To prove that it belongs to $\mathcal{P}(G, R)$, it remains to check that it is a space-like polyhedral surface.

Let us consider the case when P is made only of the orbit of one vertex. All the vertices of P belong to a space-like surface at constant distance from H . By Lemma 3.3, this surface meets at most once every time-like and light-like geodesic hence P is space-like. The fact that the support planes are space-like comes from the cocompactness of the action of G on H , the argument is the same as for Fuchsian surfaces in the Minkowski space [Fil]. Now consider the surface Q which is the boundary of the intersection of half-spaces bounded by planes orthogonal to the half-rays at the vertices of P . It is not hard to see that the orthogonal projection of the 1-skeleton of Q onto H is a Dirichlet tessellation of H for the group G , hence Q is a polyhedral surface as G acts cocompactly on H . The dual of Q is a polyhedral surface, equal to P , up to translate the vertices along the r_i .

The case with more than one orbit follows as in this case the surface can be seen as the boundary of the intersection of the convex hull of each orbit.

Hence $\mathcal{P}(G, R)$ can be identified with the set of heights such that the vertices are in convex position, and it is not hard to see that it is homeomorphic to a convex subset of \mathbb{R}^n (see [Fil11] where a very similar argument is used). \square

Let us define $\mathcal{K}(n) = \{(k_1, \dots, k_n) \in (\mathbb{R}_-)^n \mid \sum k_i > 2\pi\chi(S)\}$. It is obviously a non-empty contractible (convex) open subset of \mathbb{R}^n .

We have a natural map which to each Fuchsian polyhedral surface associates its singular curvatures:

$$C : \begin{array}{ccc} \mathcal{P}(G, R) & \longrightarrow & \mathcal{K}(n) \\ (h_1, \dots, h_n) & \longmapsto & (k_1, \dots, k_n) \end{array} .$$

Theorem 4.1 says exactly that C is a bijective map. We will prove that C is proper (Lemma 4.6) and locally injective (Lemma 4.7). Then by Lemma 4.4, C is a homeomorphism, in particular a bijective map.

Lemma 4.6. *The map C is proper: Let $(K_i)_i$ be a converging sequence of $\mathcal{K}(n)$ such that for all i , there exists $P_i \in \mathcal{P}(G, n)$ with $K_i = C(P_i)$. Then a subsequence of $(P_i)_i$ converges in $\mathcal{P}(G, n)$.*

Proof. We will first prove that the sequence of heights defining the polyhedra P_i has a converging subsequence in $]0, \pi/2[^n$. Actually, the sequence of heights belongs to $[0, \pi/2]^n$ which is compact, so $(P_i)_i$ has a converging subsequence in this set (that we again denote by $(P_i)_i$), hence it suffices to check that no height goes to 0 or to $\pi/2$.

First suppose that all the heights converge to 0. In this case $(P_i)_i$ converges to a degenerated polyhedron (the hyperbolic plane H). This means that all the curvatures go to zero, that is impossible, as the sequence of curvatures is supposed to converge in $\mathcal{K}(n)$. So there exists at least one half-ray x such that the sequence of heights on this half-ray doesn't go to 0. If the heights on another ray y ($y \notin Gx$) go to zero, then, if i is sufficiently large, the vertex of P_i on y has to be inside the convex hull of the points Gx_i , where x_i is the vertex of P_i lying on x , this contradicts the fact that P_i is convex.

Now suppose that all the heights converge to $\pi/2$. This means that all the vertices converge to the point H^* dual of the hyperbolic plane H . In particular, the face areas in a fundamental domain for G go to zero, and by the Gauss–Bonnet formula, the sum of the curvatures go to $2\pi\chi(g)$. But this is impossible as by assumption the sequence of curvatures converges inside $\mathcal{K}(n)$.

Therefore there exists at least one half-ray x such that the sequence of heights on this ray doesn't go to $\pi/2$. For each i , let x_i be the vertex of P_i on x . As P_i is convex, it is contained between the plane H and the cone of vertex x_i formed by the planes containing the faces of P_i meeting at x . Each half-ray containing a vertex meets those planes, and this prevents the other heights from going to $\pi/2$. Indeed, if L is a plane containing a face of P_i , as it is space-like, then it meets every time-like geodesic. Moreover the orbit of x accumulates on the boundary at infinity of H , so L does not meet H , as P_i is on one side of L . It is then clear that every time-like geodesic from H to H^* has to meet L .

We proved that the heights of $(P_i)_i$ converge in $]0, \pi/2[$. Now as each P_i is a convex polyhedron and as the sequence of induced curvatures converges, it is clear that $(P_i)_i$ converges inside $\mathcal{P}(G, n)$. \square

Lemma 4.7. *The map C is locally injective.*

Proof. The idea is to study the Jacobian of C and to invoke the local inverse theorem. Unfortunately, it is not clear whether C is C^1 , as the combinatorics of the polyhedra may change under deformation. Let $P \in \mathcal{P}(G, n)$. Triangulate the faces of P such that no new vertex arises and such that the triangulation is invariant under the action of G . The new edges added are called *false edges* and the older ones *true edges*. We denote by Γ the combinatorics of the resulting triangulation and by $\mathcal{P}^\Gamma(G, R)$ the set of polyhedral surfaces invariant under the action of G with vertices on GR with a triangulation of combinatorics Γ (the elements of $\mathcal{P}^\Gamma(G, R)$ are not asked to be convex). We denote by C_Γ the map from $\mathcal{P}^\Gamma(G, n)$ to \mathbb{R}^n which associates to each polyhedral surface the curvature at the vertices. In Subsection 4.5 we will prove that, at a point $P \in \mathcal{P}(G, R)$, C_Γ is C^1 and its Jacobian is non-degenerate. (Actually we will study the map which associates to the heights the cone-angles, instead of curvatures, this does not change the result.) Hence C_Γ is locally injective around P .

Now suppose that C is not locally injective. This means that there exists $P' \in \mathcal{P}(G, R)$, arbitrarily close to P , with the same curvature at the corresponding vertices. As P' is sufficiently close to P , they can be endowed with a triangulation of same combinatorics Γ , such that, if new vertices appear, they appear on a true edge. We don't consider the heights of the false new vertices as variables, and we don't compute the possible variation of the cone-angles (which are equal to 2π) at them. Only the angles on the faces around the false vertices enter the computations. So the corresponding map C_Γ is not locally injective, that is a contradiction. \square

4.4. Interlude: The case of the sphere. In this section we switch from the geometry of AdS to that of the sphere. Although the results presented here are of independent interest, our main motivation is to present the (rather technical) arguments of the next section in a context in which most readers are more comfortable. Following the same arguments in the AdS setting should then be relatively easy.

We wonder whether there exists an analogue of Theorem 4.1 for the sphere. As in the AdS case, the question can be reduced to proving that the map sending the heights h_x determining a convex polyhedron to the curvatures k_x (actually the cone-angle ω_x) at its vertices is a bijection (between suitable spaces)— here the half-rays should be from a point o contained in the interior of the polyhedron. We will compute the Jacobian of this map — in a simplified case where we assume that the polyhedron has only triangular faces.

We need the following lemma of spherical trigonometry.

Lemma 4.8. *Let a, b, c be the side lengths of a spherical triangle with angles α, β, γ . Considering b as a function of a, c, β , we get*

$$(3) \quad \frac{\partial b}{\partial a} = \cos \gamma.$$

Considering α as a function of a, c, β , we get

$$(4) \quad \frac{\partial \alpha}{\partial a} = \frac{\sin \gamma}{\sin b},$$

$$(5) \quad \frac{\partial \alpha}{\partial c} = -\frac{\sin \alpha \cos b}{\sin b}.$$

Proof. The derivative of the cosine law

$$(6) \quad \cos b = \cos c \cos a + \sin c \sin a \cos \beta$$

gives

$$\begin{aligned} \frac{\partial b}{\partial a} &= \frac{\sin a \cos c - \cos a \sin c \cos \beta}{\sin b} \\ &\stackrel{(6)}{=} \frac{\cos c - \cos b \cos a}{\sin a \sin b} \stackrel{*}{=} \frac{\sin a \sin b \cos \gamma}{\sin a \sin b} = \cos \gamma, \end{aligned}$$

and (3) is proved (we used another cosine law at *).

Now we consider α as a function of the side-lengths a, b, c , and we denote it by $\tilde{\alpha}$ to avoid confusion with α appearing in (4) and (5). The derivative of the suitable cosine law gives that

$$(7) \quad \frac{\partial \tilde{\alpha}}{\partial a} = \frac{1}{\sin b \sin \gamma}.$$

and that

$$(8) \quad \frac{\partial \tilde{\alpha}}{\partial b} = -\frac{\cos c - \cos a \cos b}{\sin^2 b \sin c \sin \alpha} \stackrel{**}{=} -\frac{\sin c \cos \gamma}{\sin b \sin c \sin \gamma} = -\frac{\cotan \gamma}{\sin b}$$

(we used another cosine law and a sine law at (**)). Now if α is a function of a, c, β we have

$$(9) \quad \frac{\partial \alpha}{\partial a} = \frac{\partial \tilde{\alpha}}{\partial a} + \frac{\partial \tilde{\alpha}}{\partial b} \frac{\partial b}{\partial a}$$

and (4) follows by using (7), (8) and (3) in the equation above. Equation (5) is proved similarly. \square

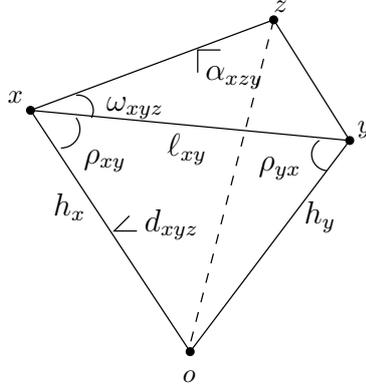


FIGURE 4. Notations in a pyramid.

Let ω_x be the cone-angle at the vertex on the ray r_x . For each face triangle xyz of P we get a pyramid $oxyz$. The cone-angle ω_x is decomposed as a sum of angles of the form ω_{xyz} (see notations on Figure 4). The link of the pyramid at x is a triangle in which ω_{xyz} is a function of ρ_{xy}, ρ_{xz} and d_{xyz} when a height varies. Note that d_{xyz} remains fixed by hypothesis.

We denote $a_{xy} := \frac{\partial \omega_x}{\partial h_y}$. There is at most one edge between two different vertices. If there is no edge then $a_{xy} = 0$ ($y \neq x$). If there is one edge xy , shared by the triangles xyz and xyz' , then

$$a_{xy} = \frac{\partial \omega_x}{\partial h_y} = \frac{\partial \omega_x}{\partial \rho_{xy}} \frac{\partial \rho_{xy}}{\partial h_y} = \left(\frac{\partial \omega_{xyz}}{\partial \rho_{xy}} + \frac{\partial \omega_{xyz'}}{\partial \rho_{xy}} \right) \frac{\partial \rho_{xy}}{\partial h_y} \stackrel{(3),(4)}{=} (\cos \alpha_{xyz} + \cos \alpha_{xyz'}) \frac{\sin \rho_{yx}}{\sin \ell_{xy}}$$

By convexity α_{xyz} and $\alpha_{xyz'}$ are both positive and $\alpha_{xyz} + \alpha_{xyz'} < \pi$, so $a_{xy} > 0$.

Remark 4.9. The fact that $a_{xy} > 0$ has a clear geometric meaning: consider the convex spherical polygon spanned by the edges at x . If the height of y increase, we get another convex spherical polygon which is strictly greater than the former one. As they are convex, the perimeter of the new polygon is greater than the former one, and the perimeter is exactly the cone-angle at x .

This also means that if the support number h_y^* of the dual of P increases, then the area of the face dual to x increases (recall that $h_y^* = \pi/2 - h_y$).

We can also compute the diagonal terms. E is the set of edges of the triangulation, and we imply that $\ell_{xy} = 0$ if there is no edge between x and y .

$$\begin{aligned} a_{xx} &:= \frac{\partial \omega_x}{\partial h_x} = \sum_{xy \in E} \frac{\partial \omega_x}{\partial \rho_{xy}} \frac{\partial \rho_{xy}}{\partial h_x} \stackrel{(4),(5)}{=} - \sum_{xy \in E} \cos \ell_{xy} \frac{\sin \rho_{xy}}{\sin \rho_{yx}} \frac{\partial \omega_x}{\partial \rho_{xy}} \frac{\partial \rho_{xy}}{\partial h_y} \\ &= - \sum_{xy \in E} \cos \ell_{xy} \frac{\sin \rho_{xy}}{\sin \rho_{yx}} \frac{\partial \omega_x}{\partial h_y} = - \sum_{x \neq y} \cos \ell_{xy} \frac{\sin \rho_{xy}}{\sin \rho_{yx}} a_{xy} = - \sum_{x \neq y} \cos \ell_{xy} a_{yx}. \end{aligned}$$

Remark 4.10. This formula has also a clear geometric meaning. If a face x^* of the dual of P has all its dihedral angles $< \pi/2$ (resp. $>$), then if h_x^* increases, the area of x^* decreases (resp. increases). The case when the dihedral angles are $\pi/2$ is a kind of critical point.

It is not clear if the matrix $(a_{xy})_{xy}$ is non-degenerate. Further study would lead us too far from the scope of this paper. So we address the following question:

Question 1. *Is there an analog of the Alexandrov prescribed curvature theorem in the sphere? More precisely, let r_1, \dots, r_n be half-rays in the sphere S^3 starting from a same point and (k_1, \dots, k_n) real positive numbers. Does there exist conditions on the r_i and k_i such that the following holds: there exists a unique convex polyhedron having its vertices on the rays r_1, \dots, r_n , with curvature k_i at the vertex on r_i ?*

An obvious condition on the k_i is that they have to satisfy the Gauss–Bonnet formula. Note that an answer to the question would give information about the existence of an analog of the Minkowski theorem in the sphere (from an argument very similar to the proof of Lemma 4.3). The corresponding statement in the hyperbolic case is stated without proof in [Ale05, 9.3.1].

4.5. End of the proof of Lemma 4.7.

4.5.1. *Hyperbolic-de Sitter trigonometry.* We denote by $\mathbb{R}^{n,1}$ the Minkowski space of dimension n , that is, the space \mathbb{R}^{n+1} endowed with the bilinear form

$$\langle x, y \rangle_1 = x_1 y_1 + \dots + x_{n-1} y_{n-1} - x_{n+1} y_{n+1}.$$

We denote by $\|\cdot\|_1$ the associated pseudo-norm: $\|x\|_1 := \sqrt{\langle x, x \rangle_1}$. The pseudo-norm of time-like vectors is chosen as a positive multiple of i . A model of the hyperbolic space is the upper-sheet of the two-sheeted hyperboloid:

$$H^n = \{x \in \mathbb{R}^{n,1} \mid \|x\|_1^2 = -1, x_{n+1} > 0\};$$

and a model of the de Sitter space is the one-sheeted hyperboloid:

$$dS^n = \{x \in \mathbb{R}^{n,1} \mid \|x\|_1^2 = 1\}.$$

We define the “de Sitter-Hyperbolic space” HS^n as the union of the two spaces, and we denote by d its pseudo-distance. It is well-known that

$$\begin{aligned} \langle x, y \rangle_1 &= -\cosh(d(x, y)), (x, y) \in H^n \times H^n, \\ \langle x, y \rangle_1 &= \cos(d(x, y)), (x, y) \in dS^n \times dS^n, \\ \langle x, y \rangle_1 &= \sinh(d(x, y)), (x, y) \in H^n \times dS^n, \end{aligned}$$

where in the last equation the distance between $x \in H^n$ and $y \in dS^n$ is defined as follows. Let y^* be the hyperplane orthogonal to y for $\langle \cdot, \cdot \rangle_1$. We also denote by y^* its intersection with the hyperbolic space, that is a totally geodesic hypersurface. The distance $d(x, y)$ is the oriented hyperbolic distance between x and y^* : it is positive if x is lying in the same half-space delimited by y^* than y , and negative otherwise. See [Thu97, Sch98] for more details.

Similarly to the Euclidean case, we define the *angle* θ between two vectors of the Minkowski space as the corresponding distance on the hyperboloids:

$$\begin{aligned}\cosh(\theta) &= \frac{\langle x, y \rangle_1}{\|x\|_1 \|y\|_1}, (\|x\|_1, \|y\|_1) \in i\mathbb{R}_+ \times i\mathbb{R}_+, x_{n+1} > 0, y_{n+1} > 0, \\ \cos(\theta) &= \frac{\langle x, y \rangle_1}{\|x\|_1 \|y\|_1}, (\|x\|_1, \|y\|_1) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ \sinh(\theta) &= i \frac{\langle x, y \rangle_1}{\|x\|_1 \|y\|_1}, (\|x\|_1, \|y\|_1) \in i\mathbb{R}_+ \times \mathbb{R}_+.\end{aligned}$$

In the first and third cases the angle θ is a real number. For the second case we need to describe the different possibilities:

- if x and y span a space-like plane, $\theta \in (0, \pi)$;
- if x and y span a time-like plane and intersect the same connected component of the resulting geodesic on the de Sitter space, then $\theta \in i\mathbb{R}_+$;
- if x and y span a time-like plane and intersect different connected components of the resulting geodesic on the de Sitter space, then $\theta \in \pi - i\mathbb{R}_+$ (this follows from the preceding case as by definition the cosine of the angle between x and y is minus the cosine of the angle between $-x$ and y).

It is also possible to define this angle if x and y belong to different sheets of the hyperboloid with two sheets, but we won't use it, and we won't consider the case where x and y span a light-like plane. There exists other natural ways to define these angles but all the definitions are equivalent, up to an affine transformation (maybe complex), see for example [Sch07, Cho09].

Lemma 4.11. *Let $\langle \cdot, \cdot \rangle'$ be the bilinear form $-\langle \cdot, \cdot \rangle_1$.*

- (1) *If $\|x\|_1 \in \mathbb{R}_+$, then $\|x\|' \in i\mathbb{R}_+$ and $\|x\|_1 = i\|x\|'$.*
- (2) *If $\|x\|_1 \in i\mathbb{R}_+$, then $\|x\|' \in \mathbb{R}_+$ and $\|x\|_1 = -i\|x\|'$.*

Proof. If $\|x\|_1 \in \mathbb{R}_+$ (resp. $i\mathbb{R}_+$), then $\langle x, x \rangle_1 > 0$ (resp. < 0), so $\langle x, x \rangle' < 0$ (resp. > 0) and $\|x\|' \in i\mathbb{R}_+$ (resp. \mathbb{R}_+). It is immediate to check that in both cases $\|x\|_1 = \sqrt{-1}\|x\|'$. The choice of the square root of -1 follows because we know the spaces the terms belong to. \square

Lemma 4.12. *In a Lorentzian space-form M , consider an angle formed by two different space-like half-hyperplanes S_1, S_2 meeting at a codimension 2 plane E , such that there exists a time-like half-hyperplane T containing E . Let α_i be the dihedral angle between S_i and T .*

- (1) *The half-planes S_1 and S_2 belong to the same hyperplane if and only if $\sinh \alpha_1 + \sinh \alpha_2 = 0$.*
- (2) *If S_1 and S_2 are not in the same hyperplane, then T is inside the convex side of the angle if and only if $\sinh \alpha_1 + \sinh \alpha_2 < 0$.*

In the lemma above and in all the remainder of the section, a dihedral angle is an interior dihedral angle: for any $x \in E$, α_i is the angle in $T_x M$ between a vector of $T_x S_i$ orthogonal to $T_x E$ and a vector of $T_x T$ orthogonal to $T_x E$.

Proof. By considering the problem in the Minkowski plane spanned by y_1 and y_2 in $T_x M$, it is easy to see that the half-plane are in the same plane if and only if $y_1 = -y_2$ and that the convexity is equivalent to $\langle x, -y_2 \rangle > \langle x, y_1 \rangle$, that gives the condition of the lemma by definition of $\sinh \alpha_i$. \square

Lemma 4.13. *Consider a contractible triangle in dS^2 , with two space-like edges of length a and c and a time-like edge of length ib and angles $\alpha, i\beta, \gamma$ ($a, b, c, \alpha, \beta, \gamma$ are real numbers). Then*

$$\begin{cases} \cos a = \cosh b \cos c + \sinh b \sin c \sinh \alpha \\ \cosh b = \cos c \cos a + \sin c \sin a \cosh \beta \\ \cos c = \cos a \cosh b + \sin a \sinh b \sinh \gamma \\ \frac{\sin a}{\sinh \alpha} = \frac{\sinh b}{\sinh \beta} = \frac{\sin c}{\cosh \gamma}, \\ \cosh \beta = \sinh \alpha \sinh \gamma + \cosh \alpha \cosh \gamma \cosh b. \end{cases}$$

Proof. The formulas are proved for example in [Cho09]. \square

Lemma 4.14. *Consider a triangle in AdS_3 contained in a time-like totally geodesic hypersurface, with a space-like edge of length b and two time-like edges of length ia, ic . Let α, β, γ be the corresponding angles ($a, b, c, \alpha, \beta, \gamma$ are real numbers). If we consider α as a function of a, c, β , then*

$$(10) \quad \frac{\partial \alpha}{\partial a} = \frac{\cosh \gamma}{\sinh b},$$

$$(11) \quad \frac{\partial \alpha}{\partial c} = -\frac{\cosh b \cosh \alpha}{\sinh b}.$$

If we suppose moreover that the triangle is such that $a = c$, and if we deform it among isosceles triangles we obtain:

$$(12) \quad \frac{\partial \alpha}{\partial a} = \frac{\cosh \alpha (1 - \cosh b)}{\sinh b}.$$

Proof. In the ambient vector space, the triangle is contained in the intersection of the quadric with an hyperplane of signature $(+, -, -)$, that is, orthogonal to a space-like vector. Without loss of generality we consider that this space-like vector is given by the first coordinate in $\mathbb{R}^{2,2}$. So the triangle is contained in

$$\{(x_2, x_3, x_4) \in \mathbb{R}^3 | x_2^2 - x_3^2 - x_4^2 = -1\}$$

which is anti-isometric to dS^2 . It is clear that the corresponding triangle in dS^2 is contractible and from the definitions of the pseudo-distances it has edge lengths a, ib, c . The angle between the two space-like edges of length a and c is the angle between two space-like vectors spanning a time-like plane and meeting the same component of the resulting geodesic on the corresponding de Sitter space (this is because the triangle is contractible and the edge opposite to the angle is time-like). Hence from Lemma 4.11, the angle is $i\beta$, and it also follows that the two other angles are $-\alpha$ and $-\gamma$.

Using formulas of Lemma 4.13, we compute (10) and (11) similarly to the corresponding formulas of Lemma 4.8.

Equation (12) follows by adding (10) and (11) and equalize a and c (that implies $\alpha = \gamma$). \square

Lemma 4.15. *Consider a triangle in HS^2 , with two vertices in dS^2 joined by a space-like edge of length a , and a vertex in H^2 , with edges of length b, c . Denote the corresponding angles by α, β, γ (all the data are real numbers). Then*

$$\begin{cases} \cos a = -\sinh b \sinh c + \cosh b \cosh c \cos \alpha \\ \sinh b = \cos a \sinh c + \sin a \cosh c \sinh \beta \\ \sinh c = \cos a \sinh b + \sin a \cosh b \sinh \gamma \end{cases}.$$

If a is considered as a function of b, c, α , then

$$(13) \quad \frac{\partial a}{\partial b} = \sinh \gamma.$$

Proof. The cosine laws come from [Cho09]. The derivative is computed in the same way as (3). \square

4.5.2. *Jacobian matrix.* The faces of the Fuchsian polyhedral surface P are decomposed in triangles. If we extend each half-ray to the point o , the dual of H antipodal to H^* , each triangle gives a pyramid with forth vertex o . The height h_x from H to the vertex on r_x should become an height $h_x + \pi/2$ from o to this vertex, but this will change nothing in the arguments below. We consider the same notations as in the case of the sphere: these are the one of Figure 4 — up to change h_x and h_y by ih_x and ih_y . The cone-angle ω_x at the vertex x lying on the half-ray r_x is decomposed as a sum of angles of the form ω_{xyz} (note that $\omega_{xyz} = \omega_{xzy}$). The link of the pyramid at x is a triangle of the nature of the one of the statement of Lemma 4.15, and in which ω_{xyz} is a function of ρ_{xy}, ρ_{xz} and d_{xyz} when a height varies. Note that d_{xyz} remains fixed by hypothesis.

We denote $a_{xy} := \frac{\partial \omega_x}{\partial h_y}$.

- (1) If x is not joined to a vertex belonging to Gy , then $a_{xy} = 0$.
- (2) Consider a vertex $y \notin Gx$, such that there is an edge between x and y . We denote by s_1, \dots, s_m the vertices joined by an edge to x , numbered in the direct order around x , and such that $y = s_0 = s_{m+1}$. Then

$$\begin{aligned} a_{xy} &= \frac{\partial \omega_x}{\partial h_y} = \sum_{j=0}^m \frac{\partial \omega_{xs_j s_{j+1}}}{\partial h_y} = \sum_{j=0}^m \frac{\partial \omega_{xs_j s_{j+1}}}{\partial \rho_{xs_j}} \frac{\partial \rho_{xs_j}}{\partial h_y} + \frac{\partial \omega_{xs_j s_{j+1}}}{\partial \rho_{xs_{j+1}}} \frac{\partial \rho_{xs_{j+1}}}{\partial h_y} \\ &= \sum_{j=0}^m \left(\frac{\partial \omega_{xs_{j-1} s_j}}{\partial \rho_{xs_j}} + \frac{\partial \omega_{xs_j s_{j+1}}}{\partial \rho_{xs_j}} \right) \frac{\partial \rho_{xs_j}}{\partial h_y} = \sum_{j=0}^m a_{xy}^j. \end{aligned}$$

- (a) If $s_j \notin Gy$ then $a_{xy}^j = 0$,
 (b) otherwise

$$a_{xy}^j \stackrel{(13),(10)}{=} \left(\sinh \alpha_{xs_{j-1}s_j} + \sinh \alpha_{xs_j s_{j+1}} \right) \frac{\cosh \rho_{s_j x}}{\sinh \ell_{xs_j}}$$

and it follows from Lemma 4.12 that

- (i) if the edge between x and s_j is a false edge then $a_{xy}^j = 0$,
 (ii) otherwise, as P is convex, $a_{xy}^j < 0$.

Remark 4.16. In general, $\rho_{s_j x} \neq \rho_{xs_j}$, then $a_{xy}^j \neq a_{yx}^j$, and so at a first sight $a_{xy} \neq a_{yx}$. This should say that there is no functional f from the set of heights to \mathbb{R} such that the gradient of f is equal to C_Γ . In the Euclidean case there is a famous variational proof of the Minkowski theorem, due to Minkowski himself. It is not clear if a variational proof could be worked out in the AdS case.

Now we compute the diagonal terms a_{xx} .

$$\begin{aligned} a_{xx} &= \frac{\partial \omega_x}{\partial h_x} = \sum_{j=0}^m \frac{\partial \omega_{xs_j s_{j+1}}}{\partial h_x} = \sum_{j=0}^m \frac{\partial \omega_{xs_j s_{j+1}}}{\partial \rho_{xs_j}} \frac{\partial \rho_{xs_j}}{\partial h_x} + \frac{\partial \omega_{xs_j s_{j+1}}}{\partial \rho_{xs_{j+1}}} \frac{\partial \rho_{xs_{j+1}}}{\partial h_x} \\ &= \sum_{j=0}^m \left(\frac{\partial \omega_{xs_{j-1}s_j}}{\partial \rho_{xs_j}} + \frac{\partial \omega_{xs_j s_{j+1}}}{\partial \rho_{xs_j}} \right) \frac{\partial \rho_{xs_j}}{\partial h_x} = \sum_{j=0}^m a_{xx}^j, \end{aligned}$$

and

- (1) if $s_j \in Gy$ with $y \notin Gx$ then

$$\begin{aligned} a_{xx}^j &= \left(\frac{\partial \omega_{xs_{j-1}s_j}}{\partial \rho_{xs_j}} + \frac{\partial \omega_{xs_j s_{j+1}}}{\partial \rho_{xs_j}} \right) \frac{\partial \rho_{xs_j}}{\partial h_x} \\ &\stackrel{(10),(11)}{=} \left(\frac{\partial \omega_{xs_{j-1}s_j}}{\partial \rho_{xs_j}} + \frac{\partial \omega_{xs_j s_{j+1}}}{\partial \rho_{xs_j}} \right) \left(-\cosh \ell_{xy} \frac{\cosh \rho_{xy}}{\cosh \rho_{yx}} \right) \frac{\partial \rho_{xs_j}}{\partial h_y} = -\cosh \ell_{xy} a_{yx}^j, \end{aligned}$$

which is a non-negative number,

- (2) otherwise $s_j \in Gx$ and

$$a_{xx}^j = \left(\frac{\partial \omega_{xs_{j-1}s_j}}{\partial \rho_{xs_j}} + \frac{\partial \omega_{xs_j s_{j+1}}}{\partial \rho_{xs_j}} \right) \frac{\partial \rho_{xs_j}}{\partial h_x} \stackrel{(12),(13)}{=} \left(\sinh \alpha_{xs_{j-1}s_j} + \sinh \alpha_{xs_j s_{j+1}} \right) \frac{\cosh \rho_{xs_j} (1 - \cosh \ell_{xs_j})}{\sinh \ell_{xs_j}}.$$

which is a non-negative number.

Let us note that $a_{xx} > 0$. There are two cases.

- P is made of the orbit of only one vertex. In this case there must exist a true edge between x and another vertex in the orbit of x , and then the formula above says that $a_{xx} > 0$.
- P is made of the orbit of at least two vertices. So there must be a true edge between two vertices x and y lying in different orbits. In this case we have seen that $a_{xy}^j < 0$, hence $a_{xx}^j = -\cosh \ell_{xy} a_{yx}^j > 0$ and the result follows as all the other a_{xy}^j are non-positive.

It follows that

$$|a_{xx}| > \sum_{y \neq x} |a_{yx}|,$$

that means that the matrix $(a_{xy})_{xy}$ is strictly diagonally dominant, and then invertible by an elementary result of linear algebra. Moreover C_Γ is clearly C^1 . This completes the proof of Lemma 4.7.

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