REVISITING BEURLING'S THEOREM FOR DUNKL TRANSFORM

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Abstract. We prove an analogue of Beurling's theorem in the setting of Dunkl transform, which improves the theorem of Kawazoe-Mejjaoli ([5]).

1. Introduction

Uncertainty principles in Euclidean spaces says that a nonzero function f and its Euclidean Fourier transform $\widehat{f}$ can not have arbitrary decay. There are many theorems depending on how the decay of a function is measured. However the remarkable result in recent time is due to Hörmander ([4]) where decay has been measured in terms of integrability of f and its Fourier transform $\widehat{f}$.

Theorem 1.1. (*Hörmander 1991*) *Let* $f \in L^2(\mathbb{R})$ *such that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)||\widehat{f}(y)|e^{|xy|}\, dx\, dy < \infty. \tag{1.1}$$

Then $f = 0$ *a.e.*

Hörmander attributes this theorem to A. Beurling. The beauty of this theorem is that many other theorems like Hardy's theorem, Cowling-Price theorem (inequality case) follows from this theorem. The above theorem was further generalized by Bonami et al ([1]) which can characterize the Gaussian function.

Theorem 1.2. (*Bonami, Demange and Jaming*) *A function* $f \in L^2(\mathbb{R}^d)$ *satisfies the condition*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)|\,|\widetilde{f}(\xi)|}{(1+|x|+|\xi|)^N} e^{|\langle x,\xi\rangle|}\, dx\, d\xi < \infty$$

for some $N \geq 0$ *if and only if* $f(x) = p(x)e^{-a|x|^2)}$ *for some polynomial* p *of degree* $< \frac{N-d}{2}$ *and* $a > 0$.

We may call this as a master theorem as the (equality case of) theorem of Hardy, theorem of Cowling-Price and the theorem of Gelfand-Shilov can also be obtained from this generalized version of Beurling's theorem. For the statement of these theorems and further results in this direction we refer the excellent book of Thangavelu ([10]). There are many attempts has been made to find the suitable version of the theorem above (with or without denominator), in different context such as on $\mathrm{SL}(2,\mathbb{R})$ ([8]), on symmetric spaces of noncompact type ([9]), on Heisenberg groups and two step nilpotent lie groups ([6]) and also on theory of Heckman and Opdam ([2]). Recently these theorems are also considered in the context of Dunkl transform by Gallardo and Trieméche ([3]), Kawazoe and Mejjaoli ([5]). The Dunkl Kernel $E_k(x,y)$ is a generalization of $e^{\langle x,y\rangle}$. In the statement of Beurling's theorem Kawazoe and Mejjaoli measure the decay of f and its Dunkl transform in terms of integral against $e^{|x||y|}$. Our point of departure is to find the exact analogue of the theorem of Bonami, Demange and Jaming in the context of Dunkl transform where $e^{|x||y|}$ is replaced by $E_k(x,\pm y)$ and therefore our theorem improves the theorem of Kawazoe and Mejjaoli. To prove the

2000 *Mathematics Subject Classification.* Primary 33C80; Secondary 51F15, 43A32.
Key words and phrases. Beurling's theorem, Dunkl kernel, Dunkl transform.

theorem we adapt the proof of Bonami et al. The crux of the proof is the existance of convolution of a function with a radial function (thanks to [11]) and the explicit expression of the translation of the heat kernel (see equation (3.1)). These machinaries are not available in the theory of Heckman and Opdam. Even the explicit expression of the translation of the heat kernel is not available in semisimple Lie groups. As in the Euclidean case we obtain several corollaries to prove an analogue of Hardy's theorem, Gelfand-Shilov's theorem and Cowling-Price's theorem.

2. Preliminaries

Let Σ be a root system in $\mathbb{R}^d$ and let G be the finite reflection group associated to the root system Σ. Let $k : \Sigma \to \mathbb{C}$ be a multiplicity function i.e., $k : \Sigma \to \mathbb{C}$ is invariant under the action of the group G. For each $\xi \in \mathbb{R}^d$ and a multiplicity function k, the Dunkl operator (which is a differential reflection operator) is given by

$$T_\xi^k f(x) = \partial_\xi f(x) + \sum_{\alpha \in \Sigma^+} k(\alpha)(\alpha, \xi) \frac{f(x) - f(\sigma_\alpha x)}{(\alpha, x)}.$$

We assume that our multiplicity functions are nonnegative, i.e. $k(\alpha) \geq 0$ for all $\alpha \in \Sigma$. Then it is known that $\xi \mapsto T_\xi^k$ is a commutative algebra of differential reflection operators.

Definition 2.1. The Dunkl kernel $E_k(\cdot, y)$ for a fixed spectral parameter $y \in \mathbb{C}^d$ is defined as the unique real analytic function such that

$$T_\xi^k E_k(\cdot, y) = (\xi, y) E_k(\cdot, y) \text{ for all } \xi \in \mathbb{C}^d$$

and $E_k(0, y) = 1$.

If $k = 0$, then $E_0(x, y) = e^{(x,y)}$ for all $x, y \in \mathbb{C}^d$. It is known that there exists a probability measure μ_x^k such that

$$E_k(x, y) = \int_{\mathbb{R}^d} e^{(\xi, y)} \, d\mu_x^k(\xi)$$

where μ_x^k is supported in the closed ball $B(0, \|x\|)$.

We have the following well known properties of the Dunkl kernel:

(1) $E_k(gz_1, gz_2) = E_k(z_1, z_2)$ for all $g \in G$ and $z_1, z_2 \in \mathbb{C}^d$.
(2) $E_k(z_1, z_2) = E_k(z_2, z_1)$ for all $z_1, z_2 \in \mathbb{C}^d$.
(3) $|E_k(x, u + iv)| \leq E_k(x, u)$ for all $x, u, v \in \mathbb{R}^d$.
(4) $E_k(x, y) \leq \max_{g \in G} e^{\langle x, gy \rangle}$, for $x, y \in \mathbb{R}^d$.
(5) $|D_z^\nu E_k(x, z)| \leq |x|^\nu \exp(|x||\Re z|)$ where $D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \partial z_2^{\nu_2} \cdots \partial z_n^{\nu_n}}$.

For $z \in \mathbb{C}^d$, let $l(z) = z_1^2 + z_2^2 + \cdots + z_d^2$. Then for $z, w \in \mathbb{C}^d$ we have

$$\int_{\mathbb{R}^d} E_k(z, x) E_k(w, x) e^{-\frac{|x|^2}{2}} w_k(x) \, dx = c_k e^{\frac{l(z)+l(w)}{2}} E_k(z, w) \tag{2.1}$$

where $c_k = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2}} w_k(x) \, dx$.

Definition 2.2. For a function $f \in L^1(\mathbb{R}^d, w_k)$ the Dunkl transform is defined by

$$\mathcal{D}_k f(\xi) = c_k^{-1} \int_{\mathbb{R}^d} f(x) E_k(-i\xi, x) w_k(x) \, dx$$

where $w_k(x) = \prod_{\alpha \in \Sigma^+} |(\alpha, x)|^{2k(\alpha)}$.

It follows from equation (2.1) that $\mathcal{D}_k(e^{-|x|^2/2})(\xi) = e^{-|\xi|^2/2}$. Also by the change of variable it follows that, $\mathcal{D}_k(e^{-\delta|x|^2})(\xi) = \frac{1}{(2\delta)^{\nu+\frac{d}{2}}} e^{-\frac{|\xi|^2}{4\delta}}$ for $\delta > 0$.

We have the following well known properties of the Dunkl transform:

(1) If $f \in L^1(\mathbb{R}^d, w_k)$, then $\mathcal{D}_k f \in C_0(\mathbb{R}^d)$.

(2) If $f \in L^1(\mathbb{R}^d, w_k)$ and $\mathcal{D}_k f \in L^1(\mathbb{R}^d, w_k)$, then the following inversion formula holds:

$$f(x) = c_k^{-2} \int_{\mathbb{R}^d} \mathcal{D}_k f(\xi) E_k(i\xi, x) w_k(\xi)\, d\xi.$$

(3) The Dunkl transform on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ extends uniquely to an isometric isomorphism from $L^2(\mathbb{R}^d, w_k)$ onto $L^2(\mathbb{R}^d, w_k)$.

(4) For $j = 1, 2, \cdots, d$ and $f \in \mathcal{S}(\mathbb{R}^d)$, $\mathcal{D}_k(ix_j f) = -T_{e_j}(\mathcal{D}_k f)$ where e_j is the j-th standard basis. Therefore for any polynomial p,

$$\mathcal{D}_k(p(x)e^{-\delta|x|^2})(\xi) = r(\xi)e^{-\frac{|\xi|^2}{4\delta}}$$

where r is a polynomial of degree equal to $\deg p$.

For $f \in L^1(\mathbb{R}^d, w_k)$ we have

$$\int_{\mathbb{R}^d} f(x) w_k(x)\, dx = \int_0^\infty \left(\int_{S^{d-1}} f(r\beta) w_k(\beta)\, d\sigma_d(\beta) \right) r^{2\nu+d-1}\, dr,$$

where $d\sigma_d$ is the normalized surface measure on the unit sphere S^{d-1} and $\nu = \sum_{\alpha\in\Sigma^+} k(\alpha)$. Therefore, if f is a radial integrable function then there exists a function F on $\mathbb{R}$ such that $f(x) = F(\|x\|) = F(r)$ for $\|x\| = r$ and

$$\int_{\mathbb{R}^d} f(x) w_k(x)\, dx = d_k \int_0^\infty F(r) r^{2\nu+d-1}\, dr \tag{2.2}$$

where $d_k = \int_{S^{d-1}} w_k(\beta)\, d\sigma_d(\beta)$.

3. Beurling's theorem

Kawazoe and Mejjaoli (in [5]) proved the following analogue of the Beurling's theorem in the setting of Dunkl transform:

Theorem 3.1. (*Kawazoe and Mejjaoli*) *Let $N \in \mathbb{N}, \delta > 0$ and $f \in L^2(\mathbb{R}^d, w_k)$ satisfies*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)|\, |\mathcal{D}_k f(\xi)|\, |P(\xi)|^\delta}{(1 + |x| + |\xi|)^N} e^{|x|\,|\xi|} w_k(x) dx\, d\xi < \infty$$

where P is a polynomial of degree m. If $N \geq d + m\delta + 2$, then

$$f(x) = \sum_{|s| < \frac{N-d-m\delta}{2}} a_s^k W_s^k(x, r) \ \text{a.e.},$$

where $r > 0, a_s^k \in \mathbb{C}$. Otherwise $f(x) = 0$ a.e..

Here $W_s^k(x, r)$ is defined as

$$W_l^k(x, r) = \frac{i^{|l|}}{c_k^2} \int_{\mathbb{R}^d} \xi_1^{l_1} \cdots \xi_d^{l_d} e^{-r|\xi|^2} E_k(i\xi, x) w_k(\xi)\, d\xi.$$

Then $\mathcal{D}_k(W_l^k(\cdot, r))(\xi) = \frac{i^{|l|}}{c_k} \xi_1^{l_1} \cdots \xi_d^{l_d} e^{-r|\xi|^2}$.

We prove the following analogue of Beurling's theorem for Dunkl transform, which improves the theorem above. To prove the theorem we adapt the proof of Bonami, Demange and Jaming ([1]).

Theorem 3.2. *Let $f \in L^2(\mathbb{R}^d, w_k)$, be such that,*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)|\,|\mathcal{D}_k f(\xi)|}{(1+|x|+|\xi|)^N} E_k(x, \pm\xi) w_k(x) w_k(\xi)\, dx\, d\xi < \infty.$$

Then

$$f(x) = p(x)e^{-\delta|x|^2}$$

for some polynomial p of $\deg p < \frac{N-d}{2} - \nu$ and for some $\delta > 0$.

Proof. **Step 1:** Since f satisfies the condition above, for almost every $\xi \in \mathbb{R}^d$ we have

$$|\mathcal{D}_k f(\xi)| \int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^N} E_k(x, \pm\xi) w_k(x)\, dx < \infty.$$

Let $\{\xi_1, \xi_2, \cdots, \xi_d\}$ be a basis of $\mathbb{R}^d$ such that $\mathcal{D}_k f(\xi_i) \neq 0$ for all $i = 1, \cdots, d$. Now

$$\begin{aligned}
\sum_{i=1}^d E_k(x, \xi_i) + \sum_{i=1}^d E_k(x, -\xi_i) &= \sum_{i=1}^d \int_{\mathbb{R}^d} e^{\langle y, x\rangle}\, d\mu_{\xi_i}(y) + \sum_{i=1}^d \int_{\mathbb{R}^d} e^{\langle y, -x\rangle}\, d\mu_{\xi_i}(y) \\
&= \sum_{i=1}^d \int_{\mathbb{R}^d} \left(e^{\langle y, x\rangle} + e^{\langle y, -x\rangle}\right) d\mu_{\xi_i}(y) \\
&\geq C \sum_{i=1}^d \int_{\mathbb{R}^d} e^{|\langle y, x\rangle|}\, d\mu_{\xi_i}(y) \\
&\geq C \sum_{i=1}^d \int_{\mathbb{R}^d} (1+|x|)^N\, d\mu_{\xi_i}(y) \\
&= C(1+|x|)^N.
\end{aligned}$$

Therefore we have $f \in L^1(\mathbb{R}^d, w_k)$. Similarly we can prove that $\mathcal{D}_k f \in L^1(\mathbb{R}^d, w_k)$

Step 2: We consider the heat kernel $q_t(x) = \frac{1}{(2t)^{\nu+\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$ for $t > 0$. Then we have $\mathcal{D}_k q_t(\xi) = e^{-t|\xi|^2}$. The translation of a suitable function f is given by

$$\tau_y^k f(x) = \int_{\mathbb{R}^d} \mathcal{D}_k f(\xi) E_k(ix, \xi) E_k(-iy, \xi) w_k(\xi)\, d\xi.$$

Then $\mathcal{D}_k \tau_y^k f(\xi) = \mathcal{D}_k f(\xi) E_k(-iy, \xi)$. For $k = 0$, the translation reduces to $\tau_y^0 f(x) = f(x-y)$. It follows from equation (2.1) that the translation of the heat kernel is given by

$$\tau_y^k q_t(x) = \frac{c_k}{(2t)^{\nu+\frac{d}{2}}} e^{-(\frac{|x|^2+|y|^2}{4t})} E_k(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}) \tag{3.1}$$

(see [7]). Let $F(x) = (f * q_{1/2})(x)$. Such convolution exists and $F \in L^1 \cap L^2(\mathbb{R}^d, w_k)$ (thanks to [11, Theorem 4.1]). Also, $\mathcal{D}_k F(\xi) = \mathcal{D}_k f(\xi) e^{-\frac{|\xi|^2}{2}}$. Then,

$$F(x) = \int_{\mathbb{R}^d} f(y) \tau_y^k q_{1/2}(x) w_k(y) dy = C \int_{\mathbb{R}^d} f(y) e^{-(\frac{|x|^2+|y|^2}{2})} E_k(x, y) w_k(y) dy.$$

Now

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|F(x)||\mathcal{D}_kF(\xi)|}{(1+|x|+|\xi|)^N} E_k(x,\pm\xi)w_k(x)w_k(\xi)\,dx\,d\xi$$

$$= C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{(1+|x|+|\xi|)^N}\left|\int_{\mathbb{R}^d} f(y)e^{-(\frac{|x|^2+|y|^2}{2})}E_k(x,y)w_k(y)dy\right| |\mathcal{D}_kf(\xi)|e^{-\frac{1}{2}|\xi|^2}$$

$$E_k(x,\pm\xi)w_k(x)w_k(\xi)\,dx\,d\xi$$

$$\leq C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|f(y)|\,|\mathcal{D}_kf(\xi)|\int_{\mathbb{R}^d}\frac{e^{-\left(\frac{|x|^2+|y|^2+|\xi|^2}{2}\right)}}{(1+|x|+|\xi|)^N}E_k(x,y)E_k(x,\pm\xi)w_k(x)\,dx\,w_k(y)w_k(\xi)\,dy\,d\xi.$$

We let $\alpha = 1+|y|+|\xi|$ and $\mathcal{M} = \{x\in\mathbb{R}^d \mid \min_{\pm,g,g'\in G}|gy-x\pm g'x| \geq b\alpha\}$ for some fixed $\frac{1}{\sqrt{2}} < b < 1$.

Then

$$\int_{\mathcal{M}} e^{-\left(\frac{|x|^2+|y|^2}{2}\right)}\frac{1}{(1+|x|+|\xi|)^N}E_k(x,y)E_k(x,\pm\xi)e^{-\frac{|\xi|^2}{2}}w_k(x)\,dx$$

$$\leq \max_{\widetilde{g}\in G}\int_{\mathcal{M}} e^{-\frac{|\xi|^2}{2}}e^{-\frac{|\widetilde{g}y|^2}{2}}e^{\langle x,\widetilde{g}y\rangle}e^{-\frac{|x|^2}{2}}E_k(x,\pm\xi)w_k(x)\,dx$$

$$= \max_{\widetilde{g}\in G}\int_{\mathcal{M}} e^{-\left(\frac{|\widetilde{g}y-x|^2}{2}\right)}e^{-\frac{|\xi|^2}{2}}E_k(x,\pm\xi)w_k(x)\,dx$$

$$\leq \max_{\widetilde{g},g'\in G}\int_{\mathcal{M}} e^{-\left(\frac{|\widetilde{g}y-x|^2}{2}\right)}e^{-\frac{|g'\xi|^2}{2}}e^{\pm\langle x,g'\xi\rangle}w_k(x)\,dx$$

$$= \max_{\widetilde{g},g'\in G} e^{\pm\langle\widetilde{g}y,g'\xi\rangle}\int_{\mathcal{M}} e^{-\left(\frac{|\widetilde{g}y-x\pm g'\xi|^2}{2}\right)}w_k(x)\,dx$$

Then the expression is less than equals to

$$\max_{\widetilde{g},g'\in G} e^{\pm\langle\widetilde{g}y,g'\xi\rangle}p_1(|y|)p_2(|\xi|)e^{-\frac{b^2}{2}(1+|\xi|+|y|)^2}$$

for some polynomials p_1,p_2. This is bounded by a constant C. Also, from step 1, we have

$$E_k(y,\xi)+E_k(y,-\xi) \geq C(1+|y|+|\xi|)^N.$$

This implies that,

$$\int_{\mathcal{M}} e^{-\left(\frac{|x|^2+|y|^2}{2}\right)}\frac{1}{(1+|x|+|\xi|)^N}E_k(x,y)E_k(x,\pm\xi)e^{-\frac{|\xi|^2}{2}}w_k(x)\,dx \leq C\frac{E_k(y,\xi)+E_k(y,-\xi)}{(1+|y|+|\xi|)^N}.$$

Therefore we have

$$\int_{\mathcal{M}} e^{-\left(\frac{|x|^2+|y|^2}{2}\right)}\frac{1}{(1+|x|+|\xi|)^N}E_k(x,y)E_k(x,\pm\xi)e^{-\frac{|\xi|^2}{2}}w_k(x)\,dx \leq C\max_{\pm}\frac{E_k(y,\pm\xi)}{(1+|y|+|\xi|)^N}.$$

Let $g_0, g_0' \in G$ be such that $\min_{\pm, g, g' \in G} |gy - x \pm g'\xi| = |g_0 y - x - g_0'\xi|$. Then on $\mathbb{R}^d \setminus \mathcal{M}$, we have

$$\begin{aligned} 1 + |x| + |\xi| &= 1 + |g_0 y - (g_0 y - x)| + |g_0'\xi| \\ &\geq 1 + \tfrac{1}{2}|g_0 y - (g_0 y - x)| + |g_0'\xi| \\ &\geq \tfrac{1}{2}(1 + |g_0 y| + |g_0'\xi|) - 1/2|g_0 y - x - g_0'\xi| \\ &\geq \tfrac{1-b}{2}\alpha. \end{aligned}$$

If $g_0, g_0' \in G$ be such that $\min_{\pm, g, g' \in G} |gy - x \pm g'\xi| = |g_0 y - x + g_0'\xi|$, then similarly on $\mathbb{R}^d \setminus \mathcal{M}$, we have

$$1 + |x| + |\xi| \geq \frac{1}{2}(1 + |g_0 y| + |g_0'\xi|) - 1/2|g_0 y - x + g_0'\xi| \geq \frac{1-b}{2}\alpha.$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^d \setminus \mathcal{M}} e^{-\left(\frac{|\xi|^2 + |y|^2}{2}\right)} \frac{1}{(1+|x|+|\xi|)^N} E_k(x,y) E_k(x, \pm\xi) e^{-\frac{|x|^2}{2}} w_k(x)\, dx \\ &\leq C \frac{1}{(1+|y|+|\xi|)^N} e^{-\left(\frac{|\xi|^2 + |y|^2}{2}\right)} \int_{\mathbb{R}^d} E_k(x,y) E_k(x, \pm\xi) e^{-\frac{|x|^2}{2}} w_k(x)\, dx \\ &= C \frac{1}{(1+|y|+|\xi|)^N} e^{-\left(\frac{|\xi|^2 + |y|^2}{2}\right)} E_k(y, \pm\xi) e^{\left(\frac{|\xi|^2 + |y|^2}{2}\right)} \\ &= C \frac{E_k(y, \pm\xi)}{(1+|y|+|\xi|)^N} \end{aligned}$$

The last but one step follows from equation (2.1). Hence

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|F(x)|\,|\mathcal{D}_k F(\xi)|}{(1+|x|+|\xi|)^N} E_k(x, \pm\xi) w_k(x) w_k(\xi)\, dx\, d\xi \\ &\leq C \max_{\pm} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(y)||\mathcal{D}_k f(\xi)|}{(1+|y|+|\xi|)^N} E_k(y, \pm\xi) w_k(y) w_k(\xi)\, dy\, d\xi \\ &< \infty. \end{aligned}$$

Step 3: We have $F \in L^1(\mathbb{R}^d, w_k)$ and $\mathcal{D}_k F \in L^1(\mathbb{R}^d, w_k)$. Therefore,

$$F(x) = c\mathcal{D}_k(\mathcal{D}_k F)(-x) \in \mathcal{C}_0(\mathbb{R}^d).$$

Also, since $w_k(x) \to \infty$ as $|x| \to \infty$, there exists $M > 0$ such that $w_k(x) > 1$ for all $|x| > M$. Then from Step 2, we get that

$$\begin{aligned} &\int_{|x|>M} \int_{\mathbb{R}^d} \frac{|F(x)|\,|\mathcal{D}_k F(\xi)|}{(1+|x|+|\xi|)^N} E_k(x, \pm\xi) w_k(\xi)\, dx\, d\xi \\ &\leq \int_{|x|>M} \int_{\mathbb{R}^d} \frac{|F(x)|\,|\mathcal{D}_k F(\xi)|}{(1+|x|+|\xi|)^N} E_k(x, \pm\xi) w_k(x) w_k(\xi)\, dx\, d\xi \\ &< \infty. \end{aligned}$$

Also

$$\begin{aligned}
&\int_{|x|\le M}\int_{\mathbb{R}^d}\tfrac{|F(x)|\,|\mathcal{D}_kF(\xi)|}{(1+|x|)^N(1+|\xi|)^N}E_k(x,\pm\xi)w_k(\xi)\,dx\,d\xi\\
&\le \int_{|x|\le M}\tfrac{|F(x)|}{(1+|x|)^N}\int_{\mathbb{R}^d}\tfrac{|\mathcal{D}_kf(\xi)|}{(1+|\xi|)^N}e^{-\frac{1}{2}|\xi|^2}e^{|x|\,|\xi|}w_k(\xi)\,d\xi\,dx\\
&\le \int_{|x|\le M}\tfrac{|F(x)|}{(1+|x|)^N}\int_{\mathbb{R}^d}\tfrac{|\mathcal{D}_kf(\xi)|}{(1+|\xi|)^N}e^{-\frac{1}{2}|\xi|^2}e^{M|\xi|}w_k(\xi)\,d\xi\,dx\\
&\le C\int_{|x|\le M}\tfrac{|F(x)|}{(1+|x|)^N}\int_{\mathbb{R}^d}\tfrac{|\mathcal{D}_kf(\xi)|}{(1+|\xi|)^N}w_k(\xi)\,d\xi\,dx\\
&<\infty.
\end{aligned}$$

Therefore we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|F(x)|\,|\mathcal{D}_kF(\xi)|}{(1+|x|)^N(1+|\xi|)^N}E_k(x,\pm\xi)w_k(\xi)\,dx\,d\xi<\infty.$$

Also

$$\int_{|x|>M}|F(x)|\,dx\le\int_{|x|>M}|F(x)|w_k(x)\,dx<\infty$$

and $\int_{|x|\le M}|F(x)|\,dx<\infty$ implies that

$$\int_{\mathbb{R}^d}|F(x)|\,dx<\infty.$$

Step 4: Let $\beta>2$ be a fixed number. Then

$$\begin{aligned}
&\int_{|x|\le R}\int_{\mathbb{R}^d}|F(x)|\,|\mathcal{D}_kF(\xi)|E_k(x,\pm\xi)w_k(\xi)\,dx\,d\xi\\
&\le \int_{|x|\le R}|F(x)|\left[\int_{|\xi|>\beta R}|\mathcal{D}_kf(\xi)|e^{-\frac{1}{2}|\xi|^2}e^{R|\xi|}w_k(\xi)\,d\xi+\int_{|\xi|\le\beta R}|\mathcal{D}_kF(\xi)|E_k(x,\pm\xi)w_k(\xi)\,d\xi\right]dx\\
&\le C\int_{\mathbb{R}^d}|F(x)|\,dx+\int_{|x|\le R}\int_{|\xi|\le\beta R}|F(x)|\,|\mathcal{D}_kF(\xi)|E_k(x,\pm\xi)w_k(\xi)\,dx\,d\xi.
\end{aligned}$$

The first term is finite as $\int_{\mathbb{R}^d}|F(x)|\,dx<\infty$. Also multiplying and dividing the polynomial $(1+|x|)^N(1+|\xi|)^N$ in the second term of the expression above, we get from step 3 that,

$$\int_{|x|\le R}\int_{\mathbb{R}^d}|F(x)|\,|\mathcal{D}_kF(\xi)|E_k(x,\pm\xi)w_k(\xi)\,dx\,d\xi\le C(1+R)^{2N}.$$

Step 5: It is easy to check that

$$|\mathcal{D}_kF(\xi)|\le Ce^{-\frac{1}{2}|\xi|^2}.$$

Using this and inversion formula, it follows that F admits a holomorphic extension to $\mathbb{C}^d$ and also

$$|F(z)|\le Ce^{\frac{1}{2}|z|^2}\text{ for all }z\in\mathbb{C}^d.$$

For $x\in\mathbb{R}^d$ and $\theta\in\mathbb{R}$, we have

$$\begin{aligned}
F(e^{i\theta}x)&=C\int_{\mathbb{R}^d}\mathcal{D}_kF(\xi)E_k(i\xi,e^{i\theta}x)w_k(\xi)\,d\xi\\
&=C\int_{\mathbb{R}^d}\mathcal{D}_kF(\xi)E_k(\xi,ie^{i\theta}x)w_k(\xi)\,d\xi
\end{aligned}$$

Hence

$$|F(e^{i\theta}x)|\le C\int_{\mathbb{R}^d}|\mathcal{D}_kF(\xi)|E_k(\xi,-x\sin\theta)w_k(\xi)\,d\xi.$$

But it is easy to check that

$$E_k(\xi, -x\sin\theta) \le E_k(\xi, x) + E_k(\xi, -x).$$

Therefore

$$|F(e^{i\theta}x)| \le C \int_{\mathbb{R}^d} |\mathcal{D}_k F(\xi)| \left(E_k(x,\xi) + E_k(x,-\xi)\right) w_k(\xi)\, d\xi. \tag{3.2}$$

Then as in [1] we will prove that $F(z)F(iz)$ is a polynomial. Let

$$\Gamma(z) = \int_0^{z_1} \cdots \int_0^{z_d} F(u)F(iu)\, du.$$

To prove that $F(z)F(iz)$ is a polynomial we will prove that $\Gamma(z)$ is a polynomial on $\mathbb{C}^d$. For $\eta = (\eta_1, \eta_2, \cdots, \eta_d) \in \mathbb{R}^d$ and $0 < \alpha < \beta_0$, let $\Gamma_\eta^{(\alpha)}$ be defined on $\mathbb{C}$ by

$$\Gamma_\eta^{(\alpha)}(z) = \int_0^{\eta_1 z} \cdots \int_0^{\eta_d z} F(e^{-i\alpha}u)F(iu)\, du.$$

Then using step 4 and equation (3.2), it is easy to check that $\Gamma_\eta^{(\alpha)}$ has polynomial growth on $e^{i\alpha}\mathbb{R}$ and on $i\mathbb{R}$ (which does not depend on α). Therefore the same estimate will valid inside the angular sector by the Phragmén-Lindelhöf theorem and extends to $\Gamma_\eta(z) := \Gamma(z\eta)$. Also the similar estimate will valid on other three quadrants and get that Γ_η is a polynomial. This will imply that Γ is a polynomial on $\mathbb{C}^d$. This proves that $F(z)F(iz)$ is a polynomial.

Step 6: Using [1, Lemma 2.3] we get that,

$$F(x) = \widetilde{p}(x)e^{-a|x|^2}$$

for some polynomial $\widetilde{p}$ and for some $a > 0$. Also it follows from [1, Proposition 2.1] and equation (2.2) that the degree of the polynomial $\widetilde{p}$ satisfies $\deg \widetilde{p} < \frac{N-d}{2} - \nu$. Therefore,

$$f * q_{1/2}(x) = \widetilde{p}(x)e^{-a|x|^2}.$$

Taking Dunkl transform on both side we get

$$\mathcal{D}_k f(\xi) e^{-\frac{1}{2}|\xi|^2} = r(\xi) e^{-\frac{1}{4a}|\xi|^2}$$

where r is a polynomial of degree=$\deg p$. That is

$$\mathcal{D}_k f(\xi) = r(\xi) e^{-(\frac{1}{4a} - \frac{1}{2})|\xi|^2}.$$

Taking again Dunkl transform we get

$$f(x) = p(x) e^{-\frac{a}{1-2a}|x|^2}$$

for some polynomial p of degree=$\deg \widetilde{p} < \frac{N-d}{2} - \nu$ and $0 < a < \frac{1}{2}$.

□

Remark 3.3. This theorem improves the result of Kawazoe and Mejjaoli (Theorem 3.1). In fact if a function f satisfies condition of Theorem 3.1, then it is easy to check that it satisfies the condition of the Theorem 3.2 with N replaced by $N + 2\nu - m\delta$. Therefore by Theorem 3.2 it follows that $f(x) = p(x)e^{-a|x|^2}$ where p is a polynomial of degree $< \frac{N - m\delta - d}{2}$ and $a > 0$.

Corollary 3.4. *Let $f \in L^2(\mathbb{R}^d, w_k)$, be such that,*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)|\, |\mathcal{D}_k f(\xi)|}{(1+|x|)^N (1+|\xi|)^N} e^{|x|\,|\xi|} w_k(x) w_k(\xi)\, dx\, d\xi < \infty.$$

Then

$$f(x) = p(x)e^{-\delta|x|^2}$$

for some polynomial p of $\deg p < N - \nu - \frac{d}{2}$ and for some $\delta > 0$.

Proof. To prove the Corollary we just have to use the facts that $E_k(x, \pm y) \le e^{|x|\,|y|}$ and

$$(1 + |x| + |y|)^{2N} \ge (1 + |x|)^N (1 + |y|)^N.$$

□

Corollary 3.5. (*Hardy's theorem*) *Let $f \in L^2(\mathbb{R}^d, w_k)$ be such that*

$$|f(x)| \le C(1 + |x|)^N e^{-\alpha|x|^2}$$

and

$$|\mathcal{D}_k f(\xi)| \le C(1 + |\xi|)^N e^{-\beta|\xi|^2}.$$

Then

(1) *$f = 0$ if $\alpha\beta > \frac{1}{4}$.*
(2) *$f(x) = p(x)e^{-\delta|x|^2}$ for some $\delta > 0$ and p a polynomial of degree $< N + 1 + \frac{d}{2}$ if $\alpha\beta = \frac{1}{4}$.*

Proof. We have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)|\,|\mathcal{D}_k f(\xi)|}{(1+|x|)^{N_1}(1+|\xi|)^{N_1}} e^{|x|\,|\xi|} w_k(x) w_k(\xi)\, dx\, d\xi$$

$$\le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{e^{-(\sqrt{\alpha}|x| - \sqrt{\beta}|\xi|)^2}}{(1+|x|)^{N_1 - N}(1+|\xi|)^{N_1 - N}} e^{(1 - 2\sqrt{\alpha\beta})|x|\,|\xi|} w_k(x) w_k(\xi)\, dx\, d\xi$$

Then if $\alpha\beta > \frac{1}{4}$, the expression above for $N_1 = 0$ is finite. Hence by Corollary 3.4 $f = 0$.

If $\alpha\beta = \frac{1}{4}$, then the expression above is finite if $N_1 - N - \nu > d$ i.e. if $N_1 > N + \nu + d$ and in that case $f(x) = p(x)e^{-\delta|x|^2}$ for some $\delta > 0$ and for some polynomial p of degree $< N + 1 + \frac{d}{2}$. □

Corollary 3.6. (*Gelfand-Shilov theorem*) *Let $f \in L^2(\mathbb{R}^d, w_k)$ be such that for $1 \le p, q < \infty$,*

$$\int_{\mathbb{R}^d} \frac{|f(x)|}{(1 + |x|)^N} e^{\frac{(\alpha|x|)^p}{p}} w_k(x)\, dx < \infty$$

and

$$\int_{\mathbb{R}^d} \frac{|\mathcal{D}_k f(\xi)|}{(1 + |\xi|)^N} e^{\frac{(\beta|\xi|)^q}{q}} w_k(\xi)\, d\xi < \infty.$$

Then

(1) *$f = 0$ if $\alpha\beta > 1$.*
(2) *$f(x) = p(x)e^{-\delta|x|^2}$ for some $\delta > 0$ and p a polynomial of degree $< N - \nu - \frac{d}{2}$ if $\alpha\beta = 1$.*

To prove the corollary we have to use the inequality $\alpha\beta|x||\xi| \le \frac{\alpha^p}{p}|x|^p + \frac{\beta^q}{q}|\xi|^q$.

Corollary 3.7. (*Cowling-Price theorem*) *Let $f \in L^2(\mathbb{R}^d, w_k)$ be such that for $1 \le p, q < \infty$,*

$$\int_{\mathbb{R}^d} \frac{|f(x)|^p}{(1 + |x|)^N} e^{\alpha p|x|^2} w_k(x)\, dx < \infty$$

and

$$\int_{\mathbb{R}^d} \frac{|\mathcal{D}_k f(\xi)|^q}{(1 + |\xi|)^N} e^{\beta q|\xi|^2} w_k(\xi)\, d\xi < \infty.$$

Then

(1) *$f = 0$ if $\alpha\beta > \frac{1}{4}$.*
(2) *$f(x) = p(x)e^{-\delta|x|^2}$ for some polynomial p and for some $\delta > 0$ if $\alpha\beta = \frac{1}{4}$.*

Proof. Let $M > \max\{2\nu + d + \frac{N-2\nu-d}{p}, 2\nu + d + \frac{N-2\nu-d}{q}\}$. Then using Hölder's inequality we get the following inequalities:

$$\int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^M} e^{\alpha|x|^2} w_k(x)\ dx < \infty$$

and

$$\int_{\mathbb{R}^d} \frac{|\mathcal{D}_k f(\xi)|}{(1+|\xi|)^M} e^{\beta|\xi|^2} w_k(\xi)\ d\xi < \infty.$$

Now

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x)|\,|\mathcal{D}_k f(\xi)| e^{|x|\,|\xi|} w_k(x) w_k(\xi)\, dx\, d\xi$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^M} e^{\alpha|x|^2} \frac{|\mathcal{D}_k f(\xi)|}{(1+|\xi|)^M} e^{\beta q|\xi|^2} e^{-(\sqrt{\alpha}|x|-\sqrt{\beta}|\xi|)^2}$$

$$\times (1+|x|)^M (1+|\xi|)^M e^{(1-2\sqrt{\alpha\beta})|x|\,|\xi|} w_k(x) w_k(\xi)\, dx\, d\xi$$

$$< \infty \text{ if } \alpha\beta > \tfrac{1}{4}.$$

Therefore $f = 0$ if $\alpha\beta > \frac{1}{4}$.

If $\alpha\beta = \frac{1}{4}$ then similarly we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^M} \frac{|\mathcal{D}_k f(\xi)|}{(1+|\xi|)^M} e^{|x|\,|\xi|} w_k(x) w_k(\xi)\, dx\, d\xi < \infty.$$

Therefore we get $f(x) = p(x) e^{-\delta|x|^2}$ for some $\delta > 0$ and for some polynomial of degree $< M - \nu - \frac{d}{2}$. □

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