

On the infinity category of homotopy Leibniz algebras

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Abstract

We discuss various concepts of ∞ -homotopies, as well as the relations between them (focussing on the Leibniz type). In particular ∞ - n -homotopies appear as the n -simplices of the nerve of a complete Lie ∞ -algebra. In the nilpotent case, this nerve is known to be a Kan complex [Get09]. We argue that there is a quasi-category of ∞ -algebras and show that for truncated ∞ -algebras, i.e. categorified algebras, this ∞ -categorical structure projects to a strict 2-categorical one. The paper contains a shortcut to $(\infty, 1)$ -categories, as well as a review of Getzler's proof of the Kan property. We make the latter concrete by applying it to the 2-term ∞ -algebra case, thus recovering the concept of homotopy of [BC04], as well as the corresponding composition rule [SS07]. We also answer a question of [Sho08] about composition of ∞ -homotopies of ∞ -algebras.

Keywords : Infinity algebra, categorified algebra, higher category, quasi-category, Kan complex, Maurer-Cartan equation, composition of homotopies, Leibniz algebra

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1 Introduction

1.1 General background

Homotopy, sh, or infinity algebras [Sta63] are homotopy invariant extensions of differential graded algebras. They are of importance, e.g. in BRST of closed string field theory, in Deformation Quantization of Poisson manifolds ... Another technique to increase the flexibility of algebraic structures is categorification [CF94], [Cra95] – a sharpened viewpoint that leads to astonishing results in TFT, bosonic string theory ... Both methods, homotopification and categorification are tightly related: the 2-categories of 2-term Lie (resp., Leibniz) infinity algebras and of Lie (resp., Leibniz) 2-algebras turned out to be equivalent [BC04], [SL10] (for a comparison of 3-term Lie infinity algebras and Lie 3-algebras, as well as for the categorical definition of the latter, see [KMP11]). However, homotopies of ∞ -morphisms and their compositions are far from being fully understood. In [BC04], ∞ -homotopies are obtained from categorical homotopies, which are God-given. In [SS07], (higher) ∞ -homotopies are (higher) derivation homotopies, a variant of infinitesimal concordances, which seems to be the wrong concept [DP12]. In [Sho08], the author states that ∞ -homotopies of sh Lie algebra morphisms can be composed, but no proof is given and the result is actually not true in whole generality. The

objective of this work is to clarify the concept of (higher) ∞ -homotopies, as well as the problem of their compositions.

1.2 Structure and main results

In Section 2, we provide explicit formulae for Leibniz infinity algebras and their morphisms. Indeed, although a category of homotopy algebras is simplest described as a category of quasi-free DG coalgebras, its original nature is its manifestation in terms of brackets and component maps.

We report, in Section 3, on the notions of homotopy that are relevant for our purposes: concordances, i.e. homotopies for morphisms between quasi-free DG (co)algebras, gauge and Quillen homotopies for Maurer-Cartan (MC for short) elements of pronilpotent Lie infinity algebras, and ∞ -homotopies, i.e. gauge or Quillen homotopies for ∞ -morphisms viewed as MC elements of a complete convolution Lie infinity algebra.

Section 4 starts with the observation that vertical composition of ∞ -homotopies of DG algebras is well-defined. However, this composition is not associative and cannot be extended to the ∞ -algebra case – which suggests that ∞ -algebras actually form an ∞ -category. To allow independent reading of the present paper, we provide a short introduction to ∞ -categories, see Subsection 4.2. In Subsection 4.3.1, the concept of ∞ - n -homotopy is made precise and the class of ∞ -algebras is viewed as an ∞ -category. Since we apply the proof of the Kan property of the nerve of a nilpotent Lie infinity algebra to the 2-term Leibniz infinity case, a good understanding of this proof is indispensable: we detail the latter in Subsection 4.3.2.

To be complete, we give an explicit description of the category of 2-term Leibniz infinity algebras at the beginning of Section 5. We show that composition of ∞ -homotopies in the nerve- ∞ -groupoid, which is defined and associative only up to higher ∞ -homotopy, projects to a well-defined and associative vertical composition in the 2-term case – thus obtaining the Leibniz counterpart of the strict 2-category of 2-term Lie infinity algebras [BC04], see Subsection 5.2, Theorem 3 and Theorem 4.

Eventually, we provide, in Section 6, the definitions of the strict 2-category of Leibniz 2-algebras, which is 2-equivalent to the preceding 2-category.

An ∞ -category structure on the class of ∞ -algebras over a quadratic Koszul operad is being investigated independently of [Get09] in a separate paper.

2 Category of Leibniz infinity algebras

Let P be a quadratic Koszul operad. Surprisingly enough, P_∞ -structures on a graded vector space V (over a field \mathbb{K} of characteristic zero), which are essentially sequences ℓ_n of n -ary brackets on V that satisfy a sequence R_n of defining relations, $n \in \{1, 2, \dots\}$, are 1:1 [GK94] with codifferentials

$$D \in \text{CoDer}^1(\mathcal{F}_{P_i}^{\text{gr},c}(s^{-1}V)) \quad (|\ell_n| = 2 - n) \quad \text{or} \quad D \in \text{CoDer}^{-1}(\mathcal{F}_{P_i}^{\text{gr},c}(sV)) \quad (|\ell_n| = n - 2), \quad (1)$$

or, also, (if V is finite-dimensional) 1:1 with differentials

$$d \in \text{Der}^1(\mathcal{F}_{P_i}^{\text{gr}}(sV^*)) \quad (|\ell_n| = 2 - n) \quad \text{or} \quad d \in \text{Der}^{-1}(\mathcal{F}_{P_i}^{\text{gr}}(s^{-1}V^*)) \quad (|\ell_n| = n - 2). \quad (2)$$

Here $\text{Der}^1(\mathcal{F}_{P^!}^{\text{gr}}(sV^*))$ (resp., $\text{CoDer}^1(\mathcal{F}_{P^!}^{\text{gr},c}(s^{-1}V))$), for instance, denotes the space of endomorphisms of the free graded algebra over the Koszul dual operad $P^!$ of P on the suspended linear dual sV^* of V , which have degree 1 (with respect to the grading of the free algebra that is induced by the grading of V) and are derivations for each binary operation in $P^!$ (resp., the space of endomorphisms of the free graded coalgebra over the Koszul dual cooperad $P^!$ on the desuspended space $s^{-1}V$ that are coderivations) (by differential and codifferential we mean of course a derivation or coderivation that squares to 0).

Although the original nature of homotopified or oidified algebraic objects is their manifestation in terms of brackets [BP12], the preceding coalgebraic and algebraic settings are the most convenient contexts to think about such higher structures.

2.1 Zinbiel (co)algebras

Since we take an interest mainly in the case where P is the operad Lei (resp., the operad Lie) of Leibniz (resp., Lie) algebras, the Koszul dual $P^!$ to consider is the operad Zin (resp., Com) of Zinbiel (resp., commutative) algebras. We now recall the relevant definitions and results.

Definition 1. A graded Zinbiel algebra (GZA) (resp., graded Zinbiel coalgebra (GZC)) is a \mathbb{Z} -graded vector space V endowed with a multiplication, i.e. a degree 0 linear map $m : V \otimes V \rightarrow V$ (resp., a comultiplication, i.e. a degree 0 linear map $\Delta : V \rightarrow V \otimes V$) that verifies the relation

$$m(\text{id} \otimes m) = m(m \otimes \text{id}) + m(m \otimes \text{id})(\tau \otimes \text{id}) \quad (\text{resp.}, (\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta + (\tau \otimes \text{id})(\Delta \otimes \text{id})\Delta), \quad (3)$$

where $\tau : V \otimes V \ni u \otimes v \mapsto (-1)^{|u||v|}v \otimes u \in V \otimes V$.

When evaluated on homogeneous vectors $u, v, w \in V$, the Zinbiel relation for the multiplication $m(u, v) =: u \cdot v$ reads,

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w + (-1)^{|u||v|}(v \cdot u) \cdot w.$$

Example 1. The multiplication \cdot on the reduced tensor module $\bar{T}(V) := \bigoplus_{n \geq 1} V^{\otimes n}$ over a \mathbb{Z} -graded vector space V , defined, for homogeneous $v_i \in V$, by

$$\begin{aligned} (v_1 \dots v_p) \cdot (v_{p+1} \dots v_{p+q}) &= \sum_{\sigma \in \text{Sh}(p, q-1)} (\sigma^{-1} \otimes \text{id})(v_1 \dots v_{p+q}) = \\ &= \sum_{\sigma \in \text{Sh}(p, q-1)} \varepsilon(\sigma^{-1}) v_{\sigma^{-1}(1)} v_{\sigma^{-1}(2)} \dots v_{\sigma^{-1}(p+q-1)} v_{p+q}, \end{aligned} \quad (4)$$

where we wrote tensor products of vectors by simple juxtaposition, where $\text{Sh}(p, q-1)$ is the set of $(p, q-1)$ -shuffles, and where $\varepsilon(\sigma^{-1})$ is the Koszul sign, endows $\bar{T}(V)$ with a GZA structure.

Similarly, the comultiplication Δ on $\bar{T}(V)$, defined, for homogeneous $v_i \in V$, by

$$\Delta(v_1 \dots v_p) = \sum_{k=1}^{p-1} \sum_{\sigma \in \text{Sh}(k, p-k-1)} \varepsilon(\sigma) (v_{\sigma(1)} \dots v_{\sigma(k)}) \otimes (v_{\sigma(k+1)} \dots v_{\sigma(p-k-1)} v_p), \quad (5)$$

is a GZC structure on $\bar{T}(V)$.

As for the GZA multiplication on $\overline{T}(V)$, we have in particular

$$\begin{aligned} v_1 \cdot v_2 &= v_1 v_2 ; & (v_1 v_2) \cdot v_3 &= v_1 v_2 v_3 ; \\ v_1 \cdot (v_2 v_3) &= v_1 v_2 v_3 + (-1)^{|v_1||v_2|} v_2 v_1 v_3 ; & (((v_1 \cdot v_2) \cdot v_3) \dots) \cdot v_k &= v_1 v_2 \dots v_k . \end{aligned}$$

Proposition 1. *The GZA $(\overline{T}(V), \cdot)$ (resp., the GZC $(\overline{T}(V), \Delta)$) defined in Example 1 is the free GZA (resp., free GZC) over V . We will denote it by $\text{Zin}(V)$ (resp., $\text{Zin}^c(V)$).*

Definition 2. *A differential graded Zinbiel algebra (DGZA) (resp., a differential graded Zinbiel coalgebra) (DGZC) is a GZA (V, m) (resp., GZC (V, Δ)) together with a degree 1 (-1 in the homological setting) derivation d (resp., coderivation D) that squares to 0. More precisely, d (resp., D) is a degree 1 (-1 in the homological setting) linear map $d : V \rightarrow V$ (resp., $D : V \rightarrow V$), such that*

$$dm = m(d \otimes \text{id} + \text{id} \otimes d) \quad (\text{resp.,} \quad \Delta D = (D \otimes \text{id} + \text{id} \otimes D) \Delta)$$

and $d^2 = 0$ (resp., $D^2 = 0$).

Since the GZA $\text{Zin}(V)$ (resp., GZC $\text{Zin}^c(V)$) is free, any degree 1 linear map $d : V \rightarrow \text{Zin}(V)$ (resp., $D : \text{Zin}^c(V) \rightarrow V$) uniquely extends to a derivation $d : \text{Zin}(V) \rightarrow \text{Zin}(V)$ (resp., coderivation $D : \text{Zin}^c(V) \rightarrow \text{Zin}^c(V)$).

Definition 3. *A quasi-free DGZA (resp., a quasi-free DGZC) over V is a DGZA (resp., DGZC) of the type $(\text{Zin}(V), d)$ (resp., $(\text{Zin}^c(V), D)$).*

2.2 Leibniz infinity algebras

In the present text we use homological (i -ary map of degree $i - 2$) and cohomological (i -ary map of degree $2 - i$) infinity algebras. Let us recall the definition of homological Leibniz infinity algebras.

Definition 4. *A (homological) Leibniz infinity algebra is a graded vector space V together with a sequence of linear maps $l_i : V^{\otimes i} \rightarrow V$ of degree $i - 2$, $i \geq 1$, such that for any $n \geq 1$, the following higher Jacobi identity holds:*

$$\begin{aligned} \sum_{i+j=n+1} \sum_{j \leq k \leq n} \sum_{\sigma \in \text{Sh}(k-j, j-1)} (-1)^{(n-k+1)(j-1)} (-1)^{j(v_{\sigma(1)} + \dots + v_{\sigma(k-j)})} \varepsilon(\sigma) \text{sign}(\sigma) \\ l_i(v_{\sigma(1)}, \dots, v_{\sigma(k-j)}, l_j(v_{\sigma(k-j+1)}, \dots, v_{\sigma(k-1)}, v_k), v_{k+1}, \dots, v_n) = 0 , \end{aligned} \tag{6}$$

where $\text{sign} \sigma$ is the signature of σ and where we denoted the degree of the homogeneous $v_i \in V$ by v_i instead of $|v_i|$.

Theorem 1. *There is a 1:1 correspondence between Leibniz infinity algebras, in the sense of Definition 4, over a graded vector space V and quasi-free DGZC-s $(\text{Zin}^c(sV), D)$ (resp., in the case of a finite-dimensional graded vector space V , quasi-free DGZA-s $(\text{Zin}(s^{-1}V^*), d)$).*

In the abovementioned 1:1 correspondence between infinity algebras over a quadratic Koszul operad P and quasi-free DGP^iC (resp., quasi-free DGP^iA) (self-explaining notation), a P_∞ -algebra structure on a graded vector space V is viewed as a representation on V of the DG operad P_∞ – which is defined as the cobar construction ΩP^i of the Koszul dual cooperad P^i . Theorem 1 makes this correspondence concrete in the case $P = \text{Lei}$; a proof can be found in [AP10].

2.3 Leibniz infinity morphisms

Definition 5. A morphism between Leibniz infinity algebras (V, l_i) and (W, m_i) is a sequence of linear maps $\varphi_i : V^{\otimes i} \rightarrow W$ of degree $i - 1$, $i \geq 1$, which satisfy, for any $n \geq 1$, the condition

$$\begin{aligned} & \sum_{i=1}^n \sum_{k_1+\dots+k_i=n} \sum_{\sigma \in \mathfrak{Sh}(k_1, \dots, k_i)} (-1)^{\sum_{r=1}^{i-1} (i-r)k_r + \frac{i(i-1)}{2}} (-1)^{\sum_{r=2}^i (k_r-1)(v_{\sigma(1)}+\dots+v_{\sigma(k_1+\dots+k_{r-1})})} \varepsilon(\sigma) \text{sign}(\sigma) \\ & m_i \left(\varphi_{k_1}(v_{\sigma(1)}, \dots, v_{\sigma(k_1)}), \varphi_{k_2}(v_{\sigma(k_1+1)}, \dots, v_{\sigma(k_1+k_2)}), \dots, \varphi_{k_i}(v_{\sigma(k_1+\dots+k_{i-1}+1)}, \dots, v_{\sigma(k_1+\dots+k_i)}) \right) \\ & = \\ & \sum_{i+j=n+1} \sum_{j \leq k \leq n} \sum_{\sigma \in \text{Sh}(k-j, j-1)} (-1)^{k+(n-k+1)j} (-1)^{j(v_{\sigma(1)}+\dots+v_{\sigma(k-j)})} \varepsilon(\sigma) \text{sign}(\sigma) \\ & \quad \varphi_i(v_{\sigma(1)}, \dots, v_{\sigma(k-j)}, l_j(v_{\sigma(k-j+1)}, \dots, v_{\sigma(k-1)}, v_k), v_{k+1}, \dots, v_n), \end{aligned} \tag{7}$$

where $\mathfrak{Sh}(k_1, \dots, k_i)$ denotes the set of shuffles $\sigma \in \text{Sh}(k_1, \dots, k_i)$, such that $\sigma(k_1) < \sigma(k_1 + k_2) < \dots < \sigma(k_1 + k_2 + \dots + k_i)$.

Theorem 2. There is a 1:1 correspondence between Leibniz infinity algebra morphisms from (V, l_i) to (W, m_i) and DGC morphisms $\text{Zin}^c(sV) \rightarrow \text{Zin}^c(sW)$ (resp., in the finite-dimensional case, DGA morphisms $\text{Zin}(s^{-1}W^*) \rightarrow \text{Zin}(s^{-1}V^*)$), where the quasi-free DGZC-s (resp., the quasi-free DGZA-s) are endowed with the codifferentials (resp., differentials) that encode the structure maps l_i and m_i .

In literature, infinity morphisms of P_∞ -algebras are usually defined as morphisms of quasi-free DGP^iC -s. However, no explicit formulae seem to exist for the Leibniz case. A proof of Theorem 2 can be found in the first author's thesis. Let us also stress that the concept of infinity morphism of P_∞ -algebras does not coincide with the notion of morphism of algebras over the operad P_∞ .

2.4 Composition of Leibniz infinity morphisms

Composition of infinity morphisms between P_∞ -algebras corresponds to composition of the corresponding morphisms between quasi-free DGP^iC -s: the categories $P_\infty\text{-Alg}$ and $\text{qfDGP}^i\text{CoAlg}$ (self-explaining notation) are isomorphic. Explicit formulae can easily be computed.

3 Leibniz infinity homotopies

3.1 Concordances and their compositions

Let us first look for a proper concept of homotopy in the category $\text{qfDGP}^i\text{CoAlg}$, or, dually, in qfDGP^1Alg .

3.1.1 Definition and characterization

The following concept of homotopy – referred to as concordance – first appeared in an unpublished work by Stasheff and Schlessinger, which was based on ideas of Bousfield and Gugenheim. It can

also be found in [SSS07], for homotopy algebras over the operad Lie (algebraic version), as well as in [DP12], for homotopy algebras over an arbitrary operad P (coalgebraic version).

It is well-known that a C^∞ -homotopy $\eta : I \times X \rightarrow Y$, $I = [0, 1]$, connecting two smooth maps p, q between two smooth manifolds X, Y , induces a cochain homotopy between the pullbacks p^*, q^* . Indeed, in the algebraic category,

$$\eta^* : \Omega(Y) \rightarrow \Omega(I) \otimes \Omega(X) ,$$

and $\eta^*(\omega)$, $\omega \in \Omega(Y)$, reads

$$\eta^*(\omega)(t) = \varphi(\omega)(t) + dt \rho(\omega)(t) . \quad (8)$$

It is easily checked (see below for a similar computation) that, since η^* is a cochain map, we have

$$d_t \varphi = d_X \rho(t) + \rho(t) d_Y ,$$

where d_X, d_Y are the de Rham differentials. When integrating over I , we thus obtain

$$q^* - p^* = d_X h + h d_Y ,$$

where $h = \int_I \rho(t) dt$ has degree -1 .

Before developing a similar approach to homotopies between morphisms of quasi-free DGZA-s, let us recall that tensoring an ‘algebra’ (resp., ‘coalgebra’) with a DGCA (resp., DGCC) does not change the considered type of algebra (resp., coalgebra); let us also introduce the ‘evaluation’ maps

$$\varepsilon_i^j : \Omega(I) = C^\infty(I) \oplus dt C^\infty(I) \ni f(t) + dt g(t) \mapsto f(i) \in \mathbb{K}, \quad i \in \{0, 1\} .$$

In the following – in contrast with our above notation – we omit stars. Moreover – although the ‘algebraic’ counterpart of a Leibniz infinity algebra over V is $(\text{Zin}(s^{-1}V^*), d_V)$ – we consider Zinbiel algebras of the type $(\text{Zin}(V), d_V)$.

Definition 6. *If $p, q : \text{Zin}(W) \rightarrow \text{Zin}(V)$ are two DGA morphisms, a homotopy or concordance $\eta : p \Rightarrow q$ from p to q is a DGA morphism $\eta : \text{Zin}(W) \rightarrow \Omega(I) \otimes \text{Zin}(V)$, such that*

$$\varepsilon_1^0 \eta = p \quad \text{and} \quad \varepsilon_1^1 \eta = q .$$

The following proposition is basic.

Proposition 2. *Concordances*

$$\eta : \text{Zin}(W) \rightarrow \Omega(I) \otimes \text{Zin}(V)$$

between DGA morphisms p, q can be identified with 1-parameter families

$$\varphi : I \rightarrow \text{Hom}_{\text{DGA}}(\text{Zin}(W), \text{Zin}(V))$$

and

$$\rho : I \rightarrow \varphi \text{Der}(\text{Zin}(W), \text{Zin}(V))$$

of (degree 0) DGA morphisms and of degree 1 φ -Leibniz morphisms, respectively, such that

$$d_t \varphi = [d, \rho(t)] \quad (9)$$

and $\varphi(0) = p$, $\varphi(1) = q$. The RHS of the differential equation (9) is defined by

$$[d, \rho(t)] := d_V \rho(t) + \rho(t) d_W ,$$

where d_V, d_W are the differentials of the quasi-free DGZA-s $\text{Zin}(V), \text{Zin}(W)$.

The notion of φ -derivation or φ -Leibniz morphism appeared for instance in [BKS04]: for $w, w' \in \text{Zin}(W)$, w homogeneous,

$$\rho(w \cdot w') = \rho(w) \cdot \varphi(w') + (-1)^w \varphi(w) \cdot \rho(w') ,$$

where we omitted the dependence of ρ on t .

Proof. As already mentioned in Equation (8), $\eta(w)$, $w \in \text{Zin}(W)$, reads

$$\eta(w)(t) = \varphi(w)(t) + dt \rho(w)(t) ,$$

where $\varphi(t) : \text{Zin}(W) \rightarrow \text{Zin}(V)$ and $\rho(t) : \text{Zin}(W) \rightarrow \text{Zin}(V)$ have degrees 0 and 1, respectively (the grading of $\text{Zin}(V)$ is induced by that of V and the grading of $\Omega(I)$ is the homological one). Let us now translate the remaining properties of η into properties of φ and ρ . We denote by $d_I = dt d_t$ the de Rham differential of I . Since η is a chain map,

$$dt d_t \varphi + d_V \varphi - dt d_V \rho = (d_I \otimes \text{id} + \text{id} \otimes d_V) \eta = \eta d_W = \varphi d_W + dt \rho d_W ,$$

so that

$$d_V \varphi = \varphi d_W \quad \text{and} \quad d_t \varphi = d_V \rho + \rho d_W = [d, \rho] .$$

As η is also an algebra morphism, we have, for $w, w' \in \text{Zin}(W)$,

$$\begin{aligned} \varphi(w \cdot w') + dt \rho(w \cdot w') &= (\varphi(w) + dt \rho(w)) \cdot (\varphi(w') + dt \rho(w')) \\ &= \varphi(w) \cdot \varphi(w') + (-1)^w dt (\varphi(w) \cdot \rho(w')) + dt (\rho(w) \cdot \varphi(w')) , \end{aligned}$$

and φ (resp., ρ) is a family of DGA morphisms (resp., of degree 1 φ -Leibniz maps) from $\text{Zin}(W)$ to $\text{Zin}(V)$. Eventually,

$$p = \varepsilon_1^0 \eta = \varphi(0) \quad \text{and} \quad q = \varepsilon_1^1 \eta = \varphi(1) .$$

□

3.1.2 Horizontal and vertical compositions

In literature, the ‘categories’ of Leibniz (resp., Lie) infinity algebras over V (finite-dimensional) and of quasi-free DGZA-s (resp., quasi-free DGCA-s) over $s^{-1}V^*$ are (implicitly or explicitly) considered equivalent. This conjecture is so far corroborated by the results of this paper. Hence, let us briefly report on compositions of concordances.

Let $\eta : p \Rightarrow q$ and $\eta' : p' \Rightarrow q'$,

$$\begin{array}{ccccc}
 & p & & p' & \\
 & \curvearrowright & & \curvearrowright & \\
 (\text{Zin}(W), d_W) & & (\text{Zin}(V), d_V) & & (\text{Zin}(U), d_U) , \\
 & \downarrow \eta & & \downarrow \eta' & \\
 & q & & q' & \\
 & \curvearrowleft & & \curvearrowleft &
 \end{array} \tag{10}$$

be concordances between DGA morphisms. Their horizontal composite $\eta' \circ_0 \eta : p' \circ p \Rightarrow q' \circ q$,

$$\begin{array}{ccc}
 & p' \circ p & \\
 & \curvearrowright & \\
 (\text{Zin}(W), d_W) & & (\text{Zin}(U), d_U) , \\
 & \downarrow \eta' \circ_0 \eta & \\
 & q' \circ q & \\
 & \curvearrowleft &
 \end{array}$$

is defined by

$$(\eta' \circ_0 \eta)(t) = (\varphi'(t) \circ \varphi(t)) + dt (\varphi'(t) \circ \rho(t) + \rho'(t) \circ \varphi(t)) , \tag{11}$$

with self-explaining notation. It is easily checked that the first term of the RHS and the coefficient of dt in the second term have the properties needed to make $\eta' \circ_0 \eta$ a concordance between $p' \circ p$ and $q' \circ q$.

As for the vertical composite $\eta' \circ_1 \eta : p \Rightarrow r$ of concordances $\eta : p \Rightarrow q$ and $\eta' : q \Rightarrow r$,

$$\begin{array}{ccc}
 & p & \\
 & \downarrow \eta & \\
 (\text{Zin}(W), d_W) & \xrightarrow{q} & (\text{Zin}(V), d_V) , \\
 & \downarrow \eta' & \\
 & r &
 \end{array}$$

note that the composability condition $\varphi(1) = q = t(\eta) = s(\eta') = q = \varphi'(0)$, where s, t denote the source and target maps, does not encode any information about $\rho(1), \rho'(0)$. Hence, the usual ‘half-time’ composition cannot be applied.

Remark 1. The preceding observation is actually the shadow of the fact that the ‘category’ of Leibniz infinity algebras is an infinity category.

3.2 Infinity homotopies

Some authors addressed directly or indirectly the concept of homotopy of Lie infinity algebras (L_∞ -algebras). As mentioned above, in the (equivalent) ‘category’ of quasi-free DGCA-s, the classical picture of homotopy leads to concordances. In the ‘category’ of L_∞ -algebras itself, morphisms can be viewed as Maurer-Cartan (MC) elements of a specific L_∞ -algebra [Dol07],[Sho08], which yields the notion of ‘gauge homotopy’ between L_∞ -morphisms. Additional notions of homotopy between MC elements do exist: Quillen and cylinder homotopies. On the other hand, Markl [Mar02] uses colored operads to construct homotopies for ∞ -morphisms in a systematic way. The concepts of concordance, operadic homotopy, as well as Quillen, gauge, and cylinder homotopies are studied in detail in [DP12], for homotopy algebras over any Koszul operad, and they are shown to be equivalent, essentially due to homotopy transfer.

In this subsection, we focus on the Leibniz infinity case and provide a brief account on the relationship between concordances, gauge homotopies, and Quillen homotopies (in the next section, we explain why the latter concept is the bridge to Getzler’s [Get09] (and Henriques’ [Hen08]) work, as well as to the infinity category structure on the set of Leibniz infinity algebras).

Let us stress that all series in this section converge under some local finiteness or nilpotency conditions (for instance pronilpotency or completeness).

3.2.1 Gauge homotopic Maurer-Cartan elements

Lie infinity algebras over \mathfrak{g} are in bijective correspondence with quasi-free DGCC-s $(\text{Com}^c(\mathfrak{sg}), D)$, see Equation (1). Depending on the definition of the i -ary brackets ℓ_i , $i \geq 1$, from the corestrictions $D_i : (\mathfrak{sg})^{\odot i} \rightarrow \mathfrak{sg}$, where \odot denotes the graded symmetric tensor product, one obtains various sign conventions in the defining relations of a Lie infinity algebra. When setting $\ell_i := D_i$ (resp., $\ell_i := s^{-1}D_i s^i$ (our choice in this paper), $\ell_i := (-1)^{i(i-1)/2} s^{-1}D_i s^i$), we get a Voronov L_∞ -antialgebra [Vor05] (resp., a Lada-Stasheff L_∞ -algebra [LS93], a Getzler L_∞ -algebra [Get09]) made up of graded symmetric multilinear maps $\ell_i : (\mathfrak{sg})^{\times i} \rightarrow \mathfrak{sg}$ of degree -1 (resp., by graded antisymmetric multilinear maps $\ell_i : \mathfrak{g}^{\times i} \rightarrow \mathfrak{g}$ of degree $i - 2$, idem), which verify the conditions

$$\sum_{i+j=r+1} \sum_{\sigma \in \text{Sh}(i,j-1)} \varepsilon(\sigma) \ell_j(\ell_i(sv_{\sigma_1}, \dots, sv_{\sigma_i}), sv_{\sigma_{i+1}}, \dots, sv_{\sigma_r}) = 0, \quad (12)$$

for all homogeneous $sv_k \in \mathfrak{sg}$ and all $r \geq 1$ (resp., the same higher Jacobi identities (12), except that the sign $\varepsilon(\sigma)$ is replaced by $(-1)^{i(j-1)} \varepsilon(\sigma) \text{sign}(\sigma)$, by $(-1)^i \varepsilon(\sigma) \text{sign}(\sigma)$).

As the MC equation of a Lie infinity algebra (\mathfrak{g}, ℓ_i) must correspond to the MC equation given by the D_i , it depends on the definition of the operations ℓ_i . For a Lada-Stasheff L_∞ -algebra (resp., a Getzler L_∞ -algebra), we obtain that the set $\text{MC}(\mathfrak{g})$ of MC elements of \mathfrak{g} is the set of solutions $\alpha \in \mathfrak{g}_{-1}$ of the MC equation

$$\sum_{i=1}^{\infty} \frac{1}{i!} (-1)^{i(i-1)/2} \ell_i(\alpha, \dots, \alpha) = 0 \quad (\text{resp.}, \sum_{i=1}^{\infty} \frac{1}{i!} \ell_i(\alpha, \dots, \alpha) = 0). \quad (13)$$

Remark 2. *Since we prefer the original definition of homotopy Lie [LS93] and homotopy Leibniz [AP10] algebras, but use the results of [Get09], we adopt – to facilitate the comparison with [Get09] – Getzler’s sign convention whenever his work is involved, and change signs appropriately later, when applying the results in our context.*

Hence, we now consider the second MC equation (13). Further, for any $\alpha \in \mathfrak{g}_{-1}$, the twisted brackets

$$\ell_i^\alpha(v_1, \dots, v_i) = \sum_{k=0}^{\infty} \frac{1}{k!} \ell_{k+i}(\alpha^{\otimes k}, v_1, \dots, v_i),$$

$v_1, \dots, v_i \in \mathfrak{g}$, are a sequence of graded antisymmetric multilinear maps of degree $i - 2$. It is well-known that the ℓ_i^α endow \mathfrak{g} with a new Lie infinity structure, if $\alpha \in \text{MC}(\mathfrak{g})$. Finally, any vector $r \in \mathfrak{g}_0$ gives rise to a vector field

$$V_r : \mathfrak{g}_{-1} \ni \alpha \mapsto V_r(\alpha) := -\ell_1^\alpha(r) = -\sum_{k=0}^{\infty} \frac{1}{k!} \ell_{k+1}(\alpha^{\otimes k}, r) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} \ell_k(r, \alpha^{\otimes(k-1)}) \in \mathfrak{g}_{-1}. \quad (14)$$

This field restricts to a vector field of the Maurer-Cartan quadric $\text{MC}(\mathfrak{g})$ [DP12]. It follows that the integral curves

$$d_t \alpha = V_r(\alpha(t)), \quad (15)$$

starting from points in the quadric, are located inside $\text{MC}(\mathfrak{g})$. Hence, the

Definition 7. ([Dol07], [Sho08]) *Two MC elements $\alpha, \beta \in \text{MC}(\mathfrak{g})$ of a Lie infinity algebra \mathfrak{g} are gauge homotopic if there exists $r \in \mathfrak{g}_0$ and an integral curve $\alpha(t)$ of V_r , such that $\alpha(0) = \alpha$ and $\alpha(1) = \beta$.*

This gauge action is used to define the deformation functor $\text{Def} : \text{L}_\infty \rightarrow \text{Set}$ from the category of Lie infinity algebras to the category of sets. Moreover, it will provide a concept of homotopy between Leibniz infinity morphisms.

Let us first observe that Equation (15) is a 1-variable ordinary differential equation (ODE) and can be solved via an iteration procedure. The integral curve with initial point $\alpha \in \text{MC}(\mathfrak{g})$ is computed in [Get09]. When using our sign convention in the defining relations of a Lie infinity algebra, we get an ODE that contains different signs and the solution of the corresponding Cauchy problem reads

$$\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} e_\alpha^k(r), \quad (16)$$

where the $e_\alpha^k(r)$ admit a nice combinatorial description in terms of rooted trees. Moreover, they can be obtained inductively:

$$\begin{cases} e_\alpha^{i+1}(r) = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^{\frac{n(n+1)}{2}} \sum_{k_1 + \dots + k_n = i} \frac{i!}{k_1! \dots k_n!} \ell_{n+1}(e_\alpha^{k_1}(r), \dots, e_\alpha^{k_n}(r), r), \\ e_\alpha^0(r) = \alpha. \end{cases} \quad (17)$$

It follows that $\alpha, \beta \in \text{MC}(\mathfrak{g})$ are gauge homotopic if

$$\beta - \alpha = \sum_{k=1}^{\infty} \frac{1}{k!} e_\alpha^k(r), \quad (18)$$

for some $r \in \mathfrak{g}_0$.

3.2.2 Simplicial de Rham algebra

We first fix the notation.

Let Δ be the *simplicial category* with objects the nonempty finite ordinals $[n] = \{0, \dots, n\}$, $n \geq 0$, and morphisms the order-respecting functions $f : [m] \rightarrow [n]$. Denote by $\delta_n^i : [n-1] \rightarrow [n]$ the injection that omits the image i and by $\sigma_n^i : [n+1] \rightarrow [n]$ the surjection that assigns the same image to i and $i+1$, $i \in \{0, \dots, n\}$.

A *simplicial object* in a category \mathcal{C} is a functor $X \in [\Delta^{\text{op}}, \mathcal{C}]$. It is completely determined by the simplicial data (X_n, d_i^n, s_i^n) , $n \geq 0$, $i \in \{0, \dots, n\}$, where $X_n = X[n]$ (n -simplices), $d_i^n = X(\delta_n^i)$ (face maps), and $s_i^n = X(\sigma_n^i)$ (degeneracy maps). We denote by SC the functor category $[\Delta^{\text{op}}, \mathcal{C}]$ of simplicial objects in \mathcal{C} .

The simplicial category is embedded in its Yoneda dual category:

$$h_* : \Delta \ni [n] \mapsto \text{Hom}_{\Delta}(-, [n]) \in [\Delta^{\text{op}}, \text{Set}] = \text{SSet} .$$

We refer to the functor of points of $[n]$, i.e. to the simplicial set $\Delta[n] := \text{Hom}_{\Delta}(-, [n])$, as the *standard simplicial n -simplex*. Moreover, the Yoneda lemma states that

$$\text{Hom}_{\Delta}([n], [m]) \simeq \text{Hom}(\text{Hom}_{\Delta}(-, [n]), \text{Hom}_{\Delta}(-, [m])) = \text{Hom}(\Delta[n], \Delta[m]) .$$

This bijection sends $f : [n] \rightarrow [m]$ to φ defined by $\varphi_{[k]}(\bullet) = f \circ \bullet$ and φ to $\varphi_{[n]}(\text{id}_{[n]})$. In the following we identify $[n]$ (resp., f) with $\Delta[n]$ (resp., φ).

The set S_n of n -simplices of a simplicial set S is obviously given by $S_n \simeq \text{Hom}(\text{Hom}_{\Delta}(-, [n]), S) = \text{Hom}(\Delta[n], S)$.

Let us also recall the adjunction

$$|-| : \text{SSet} \rightleftarrows \text{Top} : \text{Sing}$$

given by the ‘geometric realization functor’ $|-|$ and the ‘singular complex functor’ Sing . To define $|-|$, we first define the realization $|\Delta[n]|$ of the standard simplicial n -simplex to be the *standard topological n -simplex*

$$\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0, \sum_i x_i = 1\} .$$

We can view $|-|$ as a functor $|-| \in [\Delta, \text{Top}]$. Indeed, if $f : [n] \rightarrow [m]$ is an order-respecting map, we can define a continuous map $|f| : \Delta^n \rightarrow \Delta^m$ by

$$|f|[x_0, \dots, x_n] = [y_0, \dots, y_m] ,$$

where $y_i = \sum_{j \in f^{-1}\{i\}} x_j$.

Let $\wedge_{\mathbb{K}}^*(x_0, \dots, x_n, dx_0, \dots, dx_n)$ be the free graded commutative algebra generated over \mathbb{K} by the degree 0 (resp., degree 1) generators x_i (resp., dx_i). If we divide out the relations $\sum_i x_i = 1$ and $\sum_i dx_i = 0$ and set $d(x_i) = dx_i$ and $d(dx_i) = 0$, we obtain a quotient DGCA

$$\Omega_n^* = \wedge_{\mathbb{K}}^*(x_0, \dots, x_n, dx_0, \dots, dx_n) / (\sum_i x_i - 1, \sum_i dx_i)$$

that can be identified, for $\mathbb{K} = \mathbb{R}$, with the algebra of polynomial differential forms $\Omega^*(\Delta^n)$ of the standard topological n -simplex Δ^n . When defining $\Omega^* : \Delta^{\text{op}} \rightarrow \text{DGCA}$ by $\Omega^*[n] := \Omega_n^*$ and, for $f : [n] \rightarrow [m]$, by $\Omega^*(f) := |f|^* : \Omega_m^* \rightarrow \Omega_n^*$ (use the standard pullback formula for differential forms given by $y_i = \sum_{j \in f^{-1}\{i\}} x_j$), we obtain a simplicial differential graded commutative algebra $\Omega^* \in \text{SDGCA}$. Hence, the face maps $d_i^n : \Omega_n^* \rightarrow \Omega_{n-1}^*$ are the pullbacks by the $|\delta_n^i| : \Delta^{n-1} \rightarrow \Delta^n$, and similarly for the degeneracy maps. In particular, $d_0^2 = |\delta_2^0|^* : \Omega_2^* \rightarrow \Omega_1^*$ is induced by $y_0 = 0, y_1 = x_0, y_2 = x_1$. Let us eventually introduce, for $0 \leq i \leq n$, the vertex e_i of Δ^n whose $(i+1)$ -th component is equal to 1, as well as the evaluation map $\varepsilon_n^i : \Omega_n^* \rightarrow \mathbb{K}$ at e_i ('pullback' induced by $(y_0, \dots, y_n) = e_i$).

3.2.3 Quillen homotopic Maurer-Cartan elements

We already mentioned that, if (\mathfrak{g}, ℓ_i) is an L_∞ -algebra and (A, \cdot, d) a DGCA, their tensor product $\mathfrak{g} \otimes A$ has a canonical L_∞ -structure $\bar{\ell}_i$. It is given by

$$\bar{\ell}_1(v \otimes a) = (\ell_1 \otimes \text{id} + \text{id} \otimes d)(v \otimes a) = \ell_1(v) \otimes a + (-1)^v v \otimes d(a)$$

and, for $i \geq 2$, by

$$\bar{\ell}_i(v_1 \otimes a_1, \dots, v_i \otimes a_i) = \pm \ell_i(v_1, \dots, v_i) \otimes (a_1 \cdots a_i),$$

where \pm is the Koszul sign generated by the commutation of the variables.

The following concept originates in Rational Homotopy Theory.

Definition 8. *Two MC elements $\alpha, \beta \in \text{MC}(\mathfrak{g})$ of a Lie infinity algebra \mathfrak{g} are Quillen homotopic if there exists a MC element $\bar{\gamma} \in \text{MC}(\bar{\mathfrak{g}}_1)$ of the Lie infinity algebra $\bar{\mathfrak{g}}_1 := \mathfrak{g} \otimes \Omega_1^*$, such that $\varepsilon_1^0 \bar{\gamma} = \alpha$ and $\varepsilon_1^1 \bar{\gamma} = \beta$ (where the ε_1^i are the natural extensions of the evaluation maps).*

From now on, we accept, in the definition of gauge equivalent MC elements, vector fields $V_{r(t)}$ induced by time-dependent $r = r(t) \in \mathfrak{g}_0$. The next result is proved in [Can99] (see also [Man99]). A sketch of the proof will be given later.

Proposition 3. *Two MC elements of a Lie infinity algebra are Quillen homotopic if and only if they are gauge homotopic.*

3.2.4 Infinity morphisms as Maurer-Cartan elements and infinity homotopies

The possibility to view morphisms in $\text{Hom}_{\text{DGPiC}}(C, \mathcal{F}_{P_i}^{\text{gr,c}}(sW))$ as MC elements is known from the theory of the bar and cobar constructions of algebras over an operad. In [DP12], the authors showed that the fact that infinity morphisms between P_∞ -algebras V and W , i.e. morphisms in

$$\text{Hom}_{\text{DGPiC}}(\mathcal{F}_{P_i}^{\text{gr,c}}(sV), \mathcal{F}_{P_i}^{\text{gr,c}}(sW)),$$

are 1:1 with Maurer-Cartan elements of an L_∞ -structure on

$$\text{Hom}_{\mathbb{K}}(\mathcal{F}_{P_i}^{\text{gr,c}}(sV), W),$$

is actually a consequence of a more general result based on the encoding of two P_∞ -algebras and an infinity morphism between them in a DG colored free operad. In the case $P = \text{Lie}$, one recovers the fact [Sho08] that

$$\text{Hom}_{\text{DGCC}}(C, \text{Com}^c(sW)) \simeq \text{MC}(\text{Hom}_{\mathbb{K}}(C, W)), \quad (19)$$

where C is any locally conilpotent DGCC, where W is an L_∞ -algebra, and where the RHS is the set of MC elements of some convolution L_∞ -structure on $\text{Hom}_{\mathbb{K}}(C, W)$.

In the sequel we detail the case $P = \text{Lei}$. Indeed, when interpreting infinity morphisms of Leibniz infinity algebras as MC elements of a Lie infinity algebra, the equivalent notions of gauge and Quillen homotopies provide a concept of homotopy between Leibniz infinity morphisms.

Proposition 4. *Let (V, ℓ_i) and (W, m_i) be two Leibniz infinity algebras and let $(\text{Zin}^c(sV), D)$ be the quasi-free DGZC that corresponds to (V, ℓ_i) . The graded vector space*

$$L(V, W) := \text{Hom}_{\mathbb{K}}(\text{Zin}^c(sV), W)$$

carries a convolution Lie infinity structure given by

$$\mathcal{L}_1 f = m_1 \circ f + (-1)^f f \circ D \quad (20)$$

and, as for $\mathcal{L}_p(f_1, \dots, f_p)$, $p \geq 2$, by

$$\text{Zin}^c(sV) \xrightarrow{\Delta^{p-1}} (\text{Zin}^c(sV))^{\otimes p} \xrightarrow{\sum_{\sigma \in S(p)} \varepsilon(\sigma) \text{sign}(\sigma) f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(p)}} W^{\otimes p} \xrightarrow{m_p} W, \quad (21)$$

where $f, f_1, \dots, f_p \in L(V, W)$, $\Delta^{p-1} = (\Delta \otimes \text{id}^{\otimes (p-2)}) \dots (\Delta \otimes \text{id}) \Delta$, where $S(p)$ denotes the symmetric group on p symbols, and where the central arrow is the graded antisymmetrization operator.

Proof. See [Laa02] or [DP12]. A direct verification is possible as well. \square

Proposition 5. *Let (V, ℓ_i) and (W, m_i) be two Leibniz infinity algebras. There exists a 1:1 correspondence between the set of infinity morphisms from (V, ℓ_i) to (W, m_i) and the set of MC elements of the convolution Lie infinity algebra structure \mathcal{L}_i on $L(V, W)$ defined in Proposition 4.*

Observe that the considered MC series converges pointwise. Indeed, the evaluation of $\mathcal{L}_p(f_1, \dots, f_p)$ on a tensor in $\text{Zin}^c(sV)$ vanishes for $p \gg$, in view of the local conilpotency of $\text{Zin}^c(sV)$. Moreover, convolution L_∞ -algebras are complete, so that their MC equation converges in the topology induced by the filtration (a descending filtration $F^i L$ of the space L of an L_∞ -algebra (L, \mathcal{L}_k) is compatible with the L_∞ -structure \mathcal{L}_k , if $\mathcal{L}_k(F^{i_1} L, \dots, F^{i_k} L) \subset F^{i_1 + \dots + i_k} L$, and it is complete, if, in addition, the ‘universal’ map $L \rightarrow \varprojlim L/F^i L$ from L to the (projective) limit of the inverse system $L/F^i L$ is an isomorphism) [DP12].

Note also that Proposition 5 is a specification, in the case $P = \text{Lei}$, of the abovementioned 1:1 correspondence between infinity morphisms of P_∞ -algebras and MC elements of a convolution L_∞ -algebra. To increase the readability of this text, we give nevertheless a sketchy proof.

Proof. An MC element is an $\alpha \in \text{Hom}_{\mathbb{K}}(\text{Zin}^c(sV), W)$ of degree -1 that verifies the MC equation. Hence, $s\alpha : \text{Zin}^c(sV) \rightarrow sW$ has degree 0 and, since $\text{Zin}^c(sW)$ is free as GZC, $s\alpha$ coextends uniquely to $\widehat{s\alpha} \in \text{Hom}_{\text{GZC}}(\text{Zin}^c(sV), \text{Zin}^c(sW))$. The fact that α is a solution of the MC equation exactly means that $\widehat{s\alpha}$ is a DGZC-morphism, i.e. an infinity morphism between the Leibniz infinity algebras V and W . Indeed, when using e.g. the relations $\ell_i = s^{-1}D_i s^i$ and $m_i = s^{-1}\mathfrak{D}_i s^i$, and the corresponding version of the MC equation, we get

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{1}{p!} (-1)^{\frac{p(p-1)}{2}} \mathcal{L}_p(\alpha, \dots, \alpha) &= 0 \Leftrightarrow \\ \sum_{p=1}^{\infty} (-1)^{\frac{p(p-1)}{2}} m_p(\alpha \otimes \dots \otimes \alpha) \Delta^{p-1} + (-1)^\alpha \alpha D &= 0 \Leftrightarrow \\ \sum_{p=1}^{\infty} (-1)^{\frac{p(p-1)}{2}} s^{-1} \mathfrak{D}_p s^p (\alpha \otimes \dots \otimes \alpha) \Delta^{p-1} + (-1)^\alpha \alpha D &= 0 \Leftrightarrow \\ \sum_{p=1}^{\infty} s^{-1} \mathfrak{D}_p (s\alpha \otimes \dots \otimes s\alpha) \Delta^{p-1} - s^{-1} s\alpha D &= 0 \Leftrightarrow \\ \mathfrak{D}(\widehat{s\alpha}) &= \widehat{(s\alpha)} D. \end{aligned}$$

□

Hence, the

Definition 9. Two infinity morphisms f, g between Leibniz infinity algebras $(V, \ell_i), (W, m_i)$ are infinity homotopic, if the corresponding MC elements $\alpha = \alpha(f)$ and $\beta = \beta(g)$ of the convolution Lie infinity structure \mathcal{L}_i on $L = L(V, W)$ are Quillen (or gauge) homotopic. In other words, f and g are infinity homotopic, if there exists $\bar{\gamma} \in \text{MC}_1(\bar{L})$, i.e. an MC element $\bar{\gamma}$ of the Lie infinity structure $\bar{\mathcal{L}}_i$ on $\bar{L} = L \otimes \Omega_1^*$ – obtained from the convolution structure \mathcal{L}_i on $L = \text{Hom}_{\mathbb{K}}(\text{Zin}^c(sV), W)$ via extension of scalars –, such that $\varepsilon_1^0 \bar{\gamma} = \alpha$ and $\varepsilon_1^1 \bar{\gamma} = \beta$.

3.2.5 Comparison of concordances and infinity homotopies

Since, according to the prevalent philosophy, the ‘categories’ $\text{qfDGP}^i \text{CoAlg}$ and $P_\infty\text{-Alg}$ are ‘equivalent’, appropriate concepts of homotopy in both categories should be in 1:1 correspondence. It can be shown [DP12] that, for any type of algebras, the concepts of concordance and of Quillen homotopy are equivalent (at least if one defines concordances in an appropriate way); and as Quillen homotopies are already known to be equivalent to gauge homotopies, the desired result follows in whole generality.

To accommodate the reader who is not interested in (nice) technicalities, we provide now a sketchy explanation of both relationships, defining concordances dually and assuming for simplicity that $\mathbb{K} = \mathbb{R}$.

Remember first that we defined concordances, in conformity with the classical picture, in a contravariant way: two infinity morphisms $f, g : V \rightarrow W$ between homotopy Leibniz algebras, i.e. two DGA morphisms $f^*, g^* : \text{Zin}(s^{-1}W^*) \rightarrow \text{Zin}(s^{-1}V^*)$, are concordant if there is a morphism

$$\eta \in \text{Hom}_{\text{DGA}}(\text{Zin}(s^{-1}W^*), \text{Zin}(s^{-1}V^*) \otimes \Omega_1^*),$$

whose values at 0 and 1 are equal to f^* and g^* , respectively. Although we will use this definition in the sequel (observe that we slightly adapted it to future purposes), we temporarily prefer a dual, covariant definition (which has the advantage that the spaces V, W need not be finite-dimensional).

The problem that the linear dual of the infinite-dimensional DGCA Ω_1^* (let us recall that \star stands for the (co)homological degree) is not a coalgebra, has already been addressed in [BM12]. The authors consider a space $(\Omega_1^*)^\vee$ made up by the formal series, with coefficients in \mathbb{K} , of the elements $\alpha_i = (t^i)^\vee$, $i \geq 0$, and $\beta_i = (t^i dt)^\vee$, $i \geq 0$. For instance, $\sum_{i \in \mathbb{N}} \mathfrak{R}^i \alpha_i$ represents the map $\{t^i\}_{i \in \mathbb{N}} \rightarrow \mathbb{K}$ and assigns to each t^i the coefficient \mathfrak{R}^i . The differential ∂ of $(\Omega_1^*)^\vee$ is (dual to the de Rham differential d and is) defined by $\partial(\alpha_i) = 0$ and $\partial(\beta_i) = (i+1)\alpha_{i+1}$. As for the coalgebra structure δ , we set

$$\begin{aligned} \delta(\alpha_i) &= \sum_{a+b=i} \alpha_a \otimes \alpha_b, \\ \delta(\beta_i) &= \sum_{a+b=i} (\beta_a \otimes \alpha_b + \alpha_a \otimes \beta_b). \end{aligned}$$

When extending to all formal series, we obtain a map $\delta : (\Omega_1^*)^\vee \rightarrow (\Omega_1^*)^\vee \widehat{\otimes} (\Omega_1^*)^\vee$, whose target is the completed tensor product. To fix this difficulty, one considers the decreasing sequence of vector spaces $(\Omega_1^*)^\vee =: \Lambda^0 \supset \Lambda^1 \supset \Lambda^2 \supset \dots$, where $\Lambda^i = \delta^{-1}(\Lambda^{i-1} \otimes \Lambda^{i-1})$, $i \geq 1$, and defines the universal DGCC $\Lambda := \bigcap_{i \geq 1} \Lambda^i$.

A concordance can then be defined as a map

$$\eta \in \text{Hom}_{\text{DGCC}}(\text{Zin}^c(sV) \otimes \Lambda, \text{Zin}^c(sW))$$

(with the appropriate boundary values). It is then easily seen that any Quillen homotopy, i.e. any element in $\text{MC}(L \otimes \Omega_1^*)$, gives rise to a concordance. Indeed, set $\mathcal{V} := sV$ and note that

$$\begin{aligned} L \otimes \Omega_1^* &= \text{Hom}_{\mathbb{K}}(\text{Zin}^c(\mathcal{V}), W) \otimes \Omega_1^* = \text{Hom}_{\mathbb{K}}\left(\bigoplus_{i \geq 1} \mathcal{V}^{\otimes i}, W\right) \otimes \Omega_1^* = \left(\prod_{i \geq 1} \text{Hom}_{\mathbb{K}}(\mathcal{V}^{\otimes i}, W)\right) \otimes \Omega_1^* \\ &\longrightarrow \prod_{i \geq 1} (\text{Hom}_{\mathbb{K}}(\mathcal{V}^{\otimes i}, W) \otimes \Omega_1^*) \longrightarrow \prod_{i \geq 1} \text{Hom}_{\mathbb{K}}(\mathcal{V}^{\otimes i} \otimes \Lambda, W) = \text{Hom}_{\mathbb{K}}(\text{Zin}^c(\mathcal{V}) \otimes \Lambda, W). \end{aligned}$$

Only the second arrow is not entirely obvious. Let $(f, \omega) \in \text{Hom}_{\mathbb{K}}(\mathcal{V}^{\otimes i}, W) \times \Omega_1^*$ and let $(v, \lambda) \in \mathcal{V}^{\otimes i} \times \Lambda$. Set $\omega = \sum_{a=0}^N k_a t^a + \sum_{b=0}^N \kappa_b (t^b dt)$ and, for instance, $\lambda = \sum_{i \in \mathbb{N}} \mathfrak{R}^i \alpha_i$. Then

$$g(f, \omega)(v, \lambda) := \left(\sum_{i \in \mathbb{N}} \mathfrak{R}^i \alpha_i\right)(\omega) f(v) := \left(\sum_{a=0}^N \mathfrak{R}^a k_a\right) f(v) \in W$$

defines a map between the mentioned spaces. Eventually, a degree -1 element of $L \otimes \Omega_1^*$ (resp., an MC element of $L \otimes \Omega_1^*$, a Quillen homotopy) is sent to an element of $\text{Hom}_{\text{GC}}(\text{Zin}^c(\mathcal{V}) \otimes \Lambda, \text{Zin}^c \mathcal{W})$ (resp., an element of $\text{Hom}_{\text{DGCC}}(\text{Zin}^c(\mathcal{V}) \otimes \Lambda, \text{Zin}^c \mathcal{W})$, a concordance).

The relationship between Quillen and gauge homotopy is (at least on the chosen level of rigor) much clearer. Indeed, an element $\bar{\gamma} \in \text{MC}_1(\bar{L}) = \text{MC}(L \otimes \Omega_1^*)$ can be decomposed as

$$\bar{\gamma} = \gamma(t) \otimes 1 + r(t) \otimes dt,$$

where $t \in [0, 1]$ is the coordinate of Δ^1 . When unraveling the MC equation of the \mathcal{L}_i according to the powers of dt , one gets

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{1}{p!} \mathcal{L}_p(\gamma(t), \dots, \gamma(t)) &= 0, \\ \frac{d\gamma}{dt} &= - \sum_{p=0}^{\infty} \frac{1}{p!} \mathcal{L}_{p+1}(\gamma(t), \dots, \gamma(t), r(t)). \end{aligned} \tag{22}$$

A (nonobvious) direct computation allows to see that the latter ODE, see Definition 7 of gauge homotopies and Equations (15) and (14), is dual (up to dimensional issues) to the ODE (9), see Proposition 2 that characterizes concordances.

4 Infinity category of Leibniz infinity algebras

We already observed that vertical composition of concordances is not well-defined and that Leibniz infinity algebras should form an infinity category. It is instructive to first briefly look at infinity homotopies between infinity morphisms of DG algebras.

4.1 DG case

Remember that infinity homotopies can be viewed as integral curves of specific vector fields V_r of the MC quadric (with obvious endpoints). In the DG case, we have, for any $r \in L_0$,

$$V_r : L_{-1} \ni \alpha \mapsto V_r(\alpha) = -\mathcal{L}_1(r) - \mathcal{L}_2(\alpha, r) \in L_{-1}.$$

In view of the Campbell-Baker-Hausdorff formula,

$$\exp(tV_r) \circ \exp(tV_s) = \exp(tV_r + tV_s + 1/2 t^2 [V_r, V_s] + \dots).$$

The point is that

$$V : L_0 \rightarrow \text{Vect}(L_{-1})$$

is a Lie algebra morphism – also after restriction to the MC quadric; we will not detail this nonobvious fact. It follows that

$$\exp(tV_r) \circ \exp(tV_s) = \exp(tV_{r+s+1/2 t[r,s]+\dots}).$$

If we accept, as mentioned previously, time-dependent r -s, the problem of the vertical composition of homotopies is solved in the considered DG situation: the integral curve of the composed homotopy of two homotopies $\exp(tV_s)$ (resp., $\exp(tV_r)$) between morphisms f, g (resp., g, h) is given by

$$c(t) = (\exp(tV_r) \circ \exp(tV_s))(f) = \exp(tV_{r+s+1/2 t[r,s]+\dots})(f).$$

Note that this vertical composition is not associative. Moreover, the preceding approach does not go through in the infinity situation (note e.g. that in this case L_0 is no longer a Lie algebra). This again points out that homotopy algebras form infinity categories.

4.2 Shortcut to infinity categories

This subsection is a short digression that should allow us to grasp the spirit of infinity categories. For additional information, we refer the reader to [Gro10], [Fin11], and [Nog12].

Strict n -categories or *strict ω -categories* (in the sense of strict infinity categories) are well understood, see e.g. [KMP11]. Roughly, they are made up by 0-morphisms (objects), 1-morphisms (morphisms between objects), 2-morphisms (homotopies between morphisms)..., up to n -morphisms, except in the ω -case, where this upper bound does not exist. All these morphisms can be composed in various ways, the compositions being associative, admitting identities, etc. However, in most cases of higher categories these defining relations do not hold strictly. A number of concepts of weak infinity category, e.g. infinity categories in which the structural relations hold up to coherent higher homotopy, are developed in literature. Moreover, an (∞, r) -category is an infinity category, with the additional requirement that all j -morphisms, $j > r$, be invertible. In this subsection, we actually confine ourselves to $(\infty, 1)$ -categories, which we simply call ∞ -categories.

4.2.1 First examples

Example 2. ∞ -categories should include ordinary categories.

There is another natural example of infinity category. When considering all the paths in $T \in \text{Top}$, up to homotopy (for fixed initial and final points), we obtain the *fundamental groupoid* $\Pi_1(T)$ of T . Remember that the usual ‘half-time’ composition of paths is not associative, whereas the induced composition of homotopy classes is. Hence, $\Pi_1(T)$, with the points of T as objects and the homotopy classes of paths as morphisms, is (really) a category in which all morphisms are invertible. To encode more information about T , we can use a 2-category, the *fundamental 2-groupoid* $\Pi_2(T)$, whose 0-morphisms (resp., 1-morphisms, 2-morphisms) are the points of T (resp., the paths between points, the homotopy classes of homotopies between paths), the composition of 1-morphisms being associative only up to a 2-isomorphism. More generally, we define the *fundamental k -groupoid* $\Pi_k(T)$, in which associativity of order $j \leq k - 1$ holds up to a $(j + 1)$ -isomorphism. Of course, if we increase k , we grasp more and more information about T . The *homotopy principle* says that the weak fundamental infinity groupoid $\Pi_\infty(T)$ recognizes T , or, more precisely, that $(\infty, 0)$ -categories are the same as topological spaces. Hence,

Example 3. ∞ -categories, i.e. $(\infty, 1)$ -categories, should contain topological spaces.

4.2.2 Kan complexes, quasi-categories, nerves of groupoids and of categories

Let us recall that the *nerve functor* $N : \text{Cat} \rightarrow \text{SSet}$, provides a fully faithful embedding of the category Cat of all (small) categories into SSet and remembers not only the objects and morphisms, but also the compositions. It associates to any $\mathcal{C} \in \text{Cat}$ the simplicial set

$$(N\mathcal{C})_n = \{C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_n\},$$

where the sequence in the RHS is a sequence of composable \mathcal{C} -morphisms between objects $C_i \in \mathcal{C}$; the face (resp., the degeneracy) maps are the compositions and insertions of identities. Let us also

recall that the r -horn $\Lambda^r[n]$, $0 \leq r \leq n$, of $\Delta[n]$ is ‘the part of the boundary of $\Delta[n]$ that is obtained by suppressing the interior of the $(n-1)$ -face opposite to r ’. More precisely, the r -horn $\Lambda^r[n]$ is the simplicial set, whose nondegenerate k -simplices are the injective order-respecting maps $[k] \rightarrow [n]$, except the identity and the map $\delta^r : [n-1] \rightarrow [n]$ whose image does not contain r .

We now detail four different situations based on the properties ‘Any (inner) horn admits a (unique) filler’.

Definition 10. A simplicial set $S \in \mathbf{SSet}$ is fibrant and called a Kan complex, if the map $S \rightarrow \star$, where \star denotes the terminal object, is a Kan fibration, i.e. has the right lifting property with respect to all canonical inclusions $\Lambda^r[n] \subset \Delta[n]$, $0 \leq r \leq n$, $n > 0$. In other words, S is a Kan complex, if any horn $\Lambda^r[n] \rightarrow S$ can be extended to an n -simplex $\Delta[n] \rightarrow S$, i.e. if any horn in S admits a filler.

The following result is well-known and explains that a simplicial set is a nerve under a quite similar extension condition.

Proposition 6. A simplicial set S is the nerve $S \simeq NC$ of some category \mathcal{C} , if and only if any inner horn $\Lambda^r[n] \rightarrow S$, $0 < r < n$, has a unique filler $\Delta[n] \rightarrow S$.

Indeed, it is quite obvious that for $S = NC \in \mathbf{SSet}$, an inner horn $\Lambda^1[2] \rightarrow NC$, i.e. two \mathcal{C} -morphisms $f : C_0 \rightarrow C_1$ and $g : C_1 \rightarrow C_2$, has a unique filler $\Delta[2] \rightarrow NC$, given by the edge $h = g \circ f : C_0 \rightarrow C_2$ and the ‘homotopy’ $\text{id} : h \Rightarrow g \circ f$ (1).

As for Kan complexes $S \in \mathbf{SSet}$, the filler property for an outer horn $\Lambda^0[2] \rightarrow S$ (resp., $\Lambda^2[2] \rightarrow S$) implies for instance that a horn $f : s_0 \rightarrow s_1$, $\text{id} : s_0 \rightarrow s_2 = s_0$ (resp., $\text{id} : s'_0 \rightarrow s'_2 = s'_0$, $g : s'_1 \rightarrow s'_2$) has a filler, so that any map has a ‘left (resp., right) inverse’ (2).

It is clear that simplicial sets S_0, S_1, S_2, \dots are candidates for ∞ -categories. In view of the last remark (2), Kan complexes model ∞ -groupoids. Hence, fillers for outer horns should be omitted in the definition of ∞ -categories. On the other hand, ∞ -categories do contain homotopies $\eta : h \Rightarrow g \circ f$, so that, due to (1), uniqueness of fillers is to be omitted as well. Hence, the

Definition 11. A simplicial set $S \in \mathbf{SSet}$ is an ∞ -category if and only if any inner horn $\Lambda^r[n] \rightarrow S$, $0 < r < n$, admits a filler $\Delta[n] \rightarrow S$.

We now also understand the

Proposition 7. A simplicial set S is the nerve $S \simeq NG$ of some groupoid \mathcal{G} , if and only if any horn $\Lambda^r[n] \rightarrow S$, $0 \leq r \leq n$, $n > 0$, has a unique filler $\Delta[n] \rightarrow S$.

Of course, Kan complexes, i.e. ∞ -groupoids, $(\infty, 0)$ -categories, or, still, topological spaces, are ∞ -categories. Moreover, nerves of categories are ∞ -categories. Hence, the requirement that topological spaces and ordinary categories should be ∞ -categories are satisfied. Note further that what we just defined is a model for ∞ -categories called *quasi-categories* or *weak Kan complexes*.

4.2.3 Link with the intuitive picture of an infinity category

In the following, we explain that the preceding model of an ∞ -category actually corresponds to the intuitive picture of an $(\infty, 1)$ -category, i.e. that in an ∞ -category all types of morphisms do exist, that all j -morphisms, $j > 1$, are invertible, and that composition of morphisms is defined and is associative only up to homotopy. This will be illustrated by showing that any ∞ -category has a homotopy category, which is an ordinary category.

We denote simplicial sets by S, S', \dots , categories by C, D, \dots , and ∞ -categories by $\mathcal{S}, \mathcal{S}', \dots$.

Let \mathcal{S} be an ∞ -category. Its 0 -morphisms are the elements of \mathcal{S}_0 and its 1 -morphisms are the elements of \mathcal{S}_1 . The *source* and *target* maps σ, τ are defined, for any 1 -morphism $f \in \mathcal{S}_1$, by $\sigma f = d_1 f \in \mathcal{S}_0$, $\tau f = d_0 f \in \mathcal{S}_0$, and the *identity map* is defined, for any 0 -morphism $s \in \mathcal{S}_0$, by $\text{id}_s = s_0 s \in \mathcal{S}_1$, with self-explaining notation. In the following, we denote a 1 -morphism f with source s and target s' by $f : s \rightarrow s'$. In view of the simplicial relations, we have $\sigma \text{id}_s = d_1 s_0 s = s$ and $\tau \text{id}_s = d_0 s_0 s = s$, so that $\text{id}_s : s \rightarrow s$.

Consider now two morphisms $f : s \rightarrow s'$ and $g : s' \rightarrow s''$. They define an inner horn $\Lambda^1[2] \rightarrow \mathcal{S}$, which, as \mathcal{S} is an ∞ -category, admits a filler $\phi : \Delta[2] \rightarrow \mathcal{S}$, or $\phi \in \mathcal{S}_2$. The face $d_1 \phi \in \mathcal{S}_1$ is of course a candidate for the composite $g \circ f$.

Remark 3. *Since the face $h := d_1 \phi$ of any filler ϕ is a (candidate for the) composite $g \circ f$, composites of morphisms are in ∞ -categories not uniquely defined. We will show that they are determined only up to ‘homotopy’.*

Definition 12. *Let \mathcal{S} be an ∞ -category and let $f, g : s \rightarrow s'$ be two morphisms. A 2 -morphism or homotopy $\phi : f \Rightarrow g$ between f and g is an element $\phi \in \mathcal{S}_2$ such that $d_0 \phi = g, d_1 \phi = f, d_2 \phi = \text{id}_s$.*

Indeed, if there exists such a 2 -simplex ϕ , there are two candidates for the composite $g \circ \text{id}_s$, namely f and, of course, g . If we wish now that all the candidates be homotopic, the existence of ϕ must entail that f and g are homotopic – which is the case in view of Definition 12. If f is homotopic to g , we write $f \simeq g$.

Proposition 8. *The homotopy relation \simeq is an equivalence in \mathcal{S}_1 .*

Proof. Let $f : s \rightarrow s'$ be a morphism and consider $\text{id}_f := s_0 f \in \mathcal{S}_2$. It follows from the simplicial relations that $d_0 \text{id}_f = f, d_1 \text{id}_f = f, d_2 \text{id}_f = s_0 s = \text{id}_s$, so that id_f is a homotopy between f and f . To prove that \simeq is symmetric, let $f, g : s \rightarrow s'$ and assume that ϕ is a homotopy from f to g . We then have an inner horn $\psi : \Lambda^2[3] \rightarrow \mathcal{S}$ such that $d_0 \psi = \phi, d_1 \psi = \text{id}_g$, and $d_3 \psi = \text{id}_{\text{id}_s} =: \text{id}_s^2$. The face $d_2 \psi$ of a filler $\Psi : \Delta[3] \rightarrow \mathcal{S}$ is a homotopy from g to f . Transitivity can be obtained similarly. \square

Definition 13. *The homotopy category $\text{Ho}(\mathcal{S})$ of an ∞ -category \mathcal{S} is the (ordinary) category with objects the objects $s \in \mathcal{S}_0$, with morphisms the homotopy classes $[f]$ of morphisms $f \in \mathcal{S}_1$, with composition $[g] \circ [f] = [g \circ f]$, where $g \circ f$ is any candidate for the composite in \mathcal{S} , and with identities $\text{Id}_s = [\text{id}_s]$.*

To check that this definition makes sense, we must in particular show that all composites $g \circ f$, see Remark 3, are homotopic. Let thus $\phi_1, \phi_2 \in \mathcal{S}_2$ be two 2 -simplices such that $(d_0 \phi_1, d_1 \phi_1, d_2 \phi_1) =$

(g, h_1, f) and $(d_0\phi_2, d_1\phi_2, d_2\phi_2) = (g, h_2, f)$, so that h_1 and h_2 are two candidates. Consider now for instance the inner horn $\psi : \Lambda^2[3] \rightarrow \mathbb{S}$ given by $\psi = (\phi_1, \phi_2, \bullet, \text{id}_f)$. The face $d_2\Psi$ of a filler $\Psi : \Delta[3] \rightarrow \mathbb{S}$ is then a homotopy from h_2 to h_1 . To prove that the composition of morphisms in $\text{Ho}(\mathbb{S})$ is associative, one shows that candidates for $h \circ (g \circ f)$ and for $(h \circ g) \circ f$ are homotopic (we will prove neither this fact, nor the additional requirements for $\text{Ho}(\mathbb{S})$ to be a category).

Remark 4. *It follows that in an ∞ -category composition of morphisms is defined and associative only up to homotopy.*

We now comment on higher morphisms in ∞ -categories, on their composites, as well as on invertibility of j -morphisms, $j > 1$.

Definition 14. *Let $\phi_1 : f \Rightarrow g$ and $\phi_2 : f \Rightarrow g$ be 2-morphisms between morphisms $f, g : s \rightarrow s'$. A 3-morphism $\Phi : \phi_1 \Rightarrow \phi_2$ is an element $\Phi \in \mathbb{S}_3$ such that $d_0\Phi = \text{id}_g$, $d_1\Phi = \phi_2$, $d_2\Phi = \phi_1$, and $d_3\Phi = \text{id}_s^2$.*

Roughly, a 3-morphism is a 3-simplex with faces given by sources and targets, as well as by identities. Higher morphisms are defined similarly [Gro10].

As concerns composition and invertibility, let us come back to transitivity of the homotopy relation. There we are given 2-morphisms $\phi_1 : f \Rightarrow g$ and $\phi_2 : g \Rightarrow h$, and must consider the inner horn $\psi = (\phi_2, \bullet, \phi_1, \text{id}_s^2)$. The face $d_1\Psi$ of a filler Ψ is a homotopy between f and h and is a candidate for the composite $\phi_2 \circ \phi_1$ of the 2-morphisms ϕ_1, ϕ_2 . If we now look again at the proof of symmetry of the homotopy relation and denote the homotopy from g to f by ψ' , we see that $\psi \circ \psi' \simeq \text{id}_g$. We obtain similarly that $\psi' \circ \psi \simeq \text{id}_f$, so that 2-morphisms are ‘invertible’.

Remark 5. *Eventually, all the requirements of the intuitive picture of an ∞ -category are encoded in the existence of fillers of inner horns.*

4.3 Infinity groupoid of infinity morphisms between Leibniz infinity algebras

4.3.1 Quasi-category of homotopy Leibniz algebras

Let Ω_\bullet^* be the SDGCA introduced in Subsection 3.2.2. The ‘Yoneda embedding’ of Ω_\bullet^* viewed as object of SSet and DGCA, respectively, gives rise to an adjunction that is well-known in Rational Homotopy Theory:

$$\Omega^* : \text{SSet} \rightleftarrows \text{DGCA}^{\text{op}} : \text{Spec}_\bullet .$$

The functor $\Omega^* = \text{Hom}_{\text{SSet}}(-, \Omega_\bullet^*) =: \text{SSet}(-, \Omega_\bullet^*)$ associates to any $S_\bullet \in \text{SSet}$ its Sullivan DGCA $\Omega^*(S_\bullet)$ of piecewise polynomial differential forms, whereas the functor $\text{Spec}_\bullet = \text{Hom}_{\text{DGCA}}(-, \Omega_\bullet^*)$ assigns to any $A \in \text{DGCA}$ its simplicial spectrum $\text{Spec}_\bullet(A)$.

Remember now that an ∞ -homotopy between ∞ -morphisms between two Leibniz infinity algebras V, W , is an element in $\text{MC}_1(\bar{L}) = \text{MC}(L \otimes \Omega_1^*)$, where $L = L(V, W)$.

The latter set is well-known from integration of L_∞ -algebras. Indeed, when looking for an integrating topological space or simplicial set of a positively graded L_∞ -algebra L of finite type (degree-wise

of finite dimension), it is natural to consider the simplicial spectrum of the corresponding quasi-free DGCA $\text{Com}(s^{-1}L^*)$. The dual of Equation (19) yields

$$\text{Spec}_\bullet(\text{Com}(s^{-1}L^*)) = \text{Hom}_{\text{DGCA}}(\text{Com}(s^{-1}L^*), \Omega_\bullet^*) \simeq \text{MC}(L \otimes \Omega_\bullet^*).$$

The integrating simplicial set of a nilpotent L_∞ -algebra L is actually homotopy equivalent to $\text{MC}_\bullet(\bar{L}) := \text{MC}(L \otimes \Omega_\bullet^*)$ [Get09]. It is clear that the structure maps of $\text{MC}_\bullet(\bar{L}) \subset L \otimes \Omega_\bullet^*$ are $\tilde{d}_i^n = \text{id} \otimes d_i^n$ and $\tilde{s}_i^n = \text{id} \otimes s_i^n$, where d_i^n and s_i^n were described in Subsection 3.2.2.

Higher homotopies (n -homotopies) are usually defined along the same lines as standard homotopies (1-homotopies), i.e., e.g., as arrows depending on parameters in $I^{\times n}$ (or Δ^n) instead of I (or Δ^1) [Lei03]. Hence,

Definition 15. ∞ - n -homotopies (∞ - $(n+1)$ -morphisms) between given Leibniz infinity algebras V, W are Maurer-Cartan elements in $\text{MC}_n(\bar{L}) = \text{MC}(L \otimes \Omega_n^*)$, where $L = L(V, W)$ and $n \geq 0$.

Indeed, ∞ -1-morphisms are just elements of $\text{MC}(L)$, i.e. standard ∞ -morphisms between V and W .

Note that if \mathbb{S} is an ∞ -category, the set of n -morphisms, with varying $n \geq 1$, between two fixed objects $s, s' \in \mathbb{S}_0$ can be shown to be a Kan complex [Gro10]. The simplicial set $\text{MC}_\bullet(\bar{L})$, whose $(n-1)$ -simplices are the ∞ - n -morphisms between the considered Leibniz infinity algebras V, W , $n \geq 1$, is known to be a Kan complex ($(\infty, 0)$ -category) as well [Get09].

Remark 6. We interpret this result by saying that Leibniz infinity algebras and their infinity higher morphisms form an ∞ -category ($(\infty, 1)$ -category). Further, as mentioned above and detailed below, composition of homotopies is encrypted in the Kan property.

Note that $\text{MC}_\bullet(\bar{L})$ actually corresponds to the ‘d ecalage’, the ‘down-shifting’, of the simplicial set \mathbb{S} .

Let us also *emphasize* that Getzler’s results are valid only for nilpotent L_∞ -algebras, hence in principle not for L , which is only complete (an L_∞ -algebra is *pronilpotent*, if it is complete with respect to its lower central series, i.e. the intersection of all its compatible filtrations, and it is *nilpotent*, if its lower central series eventually vanishes). However, for our remaining concern, namely the explanation of homotopies and their compositions in the 2-term Leibniz infinity algebra case, this difficulty is irrelevant. Indeed, when interpreting the involved series as formal ones and applying the thus obtained results to the 2-term case, where series become finite for degree reasons, we recover the results on homotopies and their compositions conjectured in [BC04]. An entirely rigorous approach to these issues is being examined in a separate paper: it is rather technical and requires applying Henriques’ method or working over an arbitrary local Artinian algebra.

4.3.2 Kan property

Considering our next purposes, we now review and specify the proof of the Kan property of $\text{MC}_\bullet(\bar{L})$ [Get09]. As announced above, to facilitate comparison, we adopt the conventions of the latter paper and apply the results *mutatis mutandis* and formally to our situation. In particular, we

work in 4.3.2 with the cohomological version of Lie infinity algebras (k -ary bracket of degree $2 - k$), together with Getzler's sign convention for the higher Jacobi conditions, see 3.2.1, and assume that $\mathbb{K} = \mathbb{R}$.

Let us first recall that the lower central filtration of (L, \mathcal{L}_i) is given by $F^1 L = L$ and

$$F^i L = \sum_{i_1 + \dots + i_k = i} \mathcal{L}_k(F^{i_1} L, \dots, F^{i_k} L), \quad i > 1.$$

In particular, $F^2 L = \mathcal{L}_2(L, L)$, $F^3 L = \mathcal{L}_2(L, \mathcal{L}_2(L, L)) + \mathcal{L}_3(L, L, L)$, ..., so that $F^k L$ is spanned by all the nested brackets containing k elements of L . Due to nilpotency, $F^i L = \{0\}$, for $i \gg$.

To simplify notation, let δ be the differential \mathcal{L}_1 of L , let d be the de Rham differential of $\Omega_n^* = \Omega^*(\Delta^n)$, and let $\bar{\delta} + \bar{d}$ be the differential $\bar{\mathcal{L}}_1 = \delta \otimes \text{id} + \text{id} \otimes d$ of $L \otimes \Omega^*(\Delta^n)$. Set now, for any $n \geq 0$ and any $0 \leq i \leq n$,

$$\text{mc}_n(\bar{L}) := \{(\bar{\delta} + \bar{d})\beta : \beta \in (L \otimes \Omega^*(\Delta^n))^0\} \text{ and } \text{mc}_n^i(\bar{L}) := \{(\bar{\delta} + \bar{d})\beta : \beta \in (L \otimes \Omega^*(\Delta^n))^0, \bar{\varepsilon}_n^i \beta = 0\},$$

where $\bar{\varepsilon}_n^i := \text{id} \otimes \varepsilon_n^i$ is the canonical extension of the evaluation map $\varepsilon_n^i : \Omega^*(\Delta^n) \rightarrow \mathbb{K}$, see 3.2.2.

Remark 7. *In the following, we use the extension symbol 'bar' only when needed for clarity.*

- There exist fundamental bijections

$$B_n^i : \text{MC}_n(\bar{L}) \xrightarrow{\sim} \text{MC}(L) \times \text{mc}_n^i(\bar{L}) \subset \text{MC}(L) \times \text{mc}_n(\bar{L}). \quad (23)$$

The proof uses the operators

$$h_n^i : \Omega^*(\Delta^n) \rightarrow \Omega^{*-1}(\Delta^n)$$

defined as follows. Let $\vec{t} = [t_0, \dots, t_n]$ be the coordinates of Δ^n (with $\sum_i t_i = 1$) and consider the maps $\phi_n^i : I \times \Delta^n \ni (u, \vec{t}) \mapsto u\vec{t} + (1 - u)\vec{e}_i \in \Delta^n$. They allow to pull back a polynomial differential form on Δ^n to a polynomial differential form on $I \times \Delta^n$. The operators h_n^i are now given by

$$h_n^i \omega = \int_I (\phi_n^i)^* \omega.$$

They satisfy the relations

$$\{d, h_n^i\} = \text{id}_n - \varepsilon_n^i, \quad \{h_n^i, h_n^j\} = 0, \quad \varepsilon_n^i h_n^i = 0, \quad (24)$$

where $\{-, -\}$ is the graded commutator (remember that ε_n^i vanishes in nonzero cohomological degree). The first relation is a higher dimensional analogue of

$$\{d, \int_0^t\} \omega = \{d, \int_0^t\} (f(u) + g(u)du) = d \int_0^t g(u)du + \int_0^t d_u f du = g(t)dt + f(t) - f(0) = \omega - \varepsilon_1^0 \omega,$$

where $\omega \in \Omega^*(I)$.

The natural extensions of d , h_n^i , and ε_n^i to $L \otimes \Omega^*(\Delta^n)$ satisfy the same relations, and, since we obviously have $\delta h_n^i = -h_n^i \delta$, the first relation holds in the extended setting also for d replaced by $\delta + d$.

Define now B_n^i by

$$B_n^i : \text{MC}_n(\bar{L}) \ni \alpha \mapsto B_n^i \alpha := (\varepsilon_n^i \alpha, (\delta + d)h_n^i \alpha) \in \text{MC}(L) \times \text{mc}_n^i(\bar{L}) . \quad (25)$$

Observe that $\alpha \in (L \otimes \Omega^*(\Delta^n))^1$ reads $\alpha = \sum_{k=0}^n \alpha^k$, $\alpha^k \in L_{1-k} \otimes \Omega^k(\Delta^n)$, so that $\varepsilon_n^i \alpha = \varepsilon_n^i \alpha^0 \in L_1$. Moreover, it follows from the definition of the extended L_∞ -maps \mathcal{L}_i that

$$\sum_{i \geq 1} \frac{1}{i!} \mathcal{L}_i(\varepsilon_n^i \alpha, \dots, \varepsilon_n^i \alpha) = \varepsilon_n^i \sum_{i \geq 1} \frac{1}{i!} \mathcal{L}_i(\alpha, \dots, \alpha) = 0 . \quad (26)$$

In view of the last equation (24), the second component of $B_n^i \alpha$ is clearly an element of $\text{mc}_n^i(\bar{L})$.

The construction of the inverse map is based upon a method similar to the iterative approximation procedure that allows us to prove the fundamental theorem of ODE-s. More precisely, consider the Cauchy problem $y'(t) = F(t, y(t))$, $y(0) = Y$, i.e. the integral equation

$$y(s) = Y + \int_0^s F(t, y(t)) dt .$$

Choose now the ‘Ansatz’ $y_0(s) = Y$ and define inductively

$$y_k(s) = Y + \int_0^s F(t, y_{k-1}(t)) dt ,$$

$k \geq 1$. It is well-known that the y_k converge to a function y , which is the unique solution and depends continuously on the initial value Y .

Note now that, if we are given $\mu \in \text{MC}(L)$ and $\nu = (\delta + d)\beta \in \text{mc}_n^i(\bar{L})$, a solution $\alpha \in \text{MC}_n(\bar{L})$ – i.e. an element $\alpha \in (L \otimes \Omega^*(\Delta^n))^1$ that satisfies

$$(\delta + d)\alpha + \sum_{i \geq 2} \frac{1}{i!} \mathcal{L}_i(\alpha, \dots, \alpha) =: (\delta + d)\alpha + \bar{R}(\alpha) = 0 -$$

such that $\varepsilon_n^i \alpha = \mu$ and $(\delta + d)h_n^i \alpha = \nu$, also satisfies the integral equation

$$\alpha = \text{id}_n \alpha = \{ \delta + d, h_n^i \} \alpha + \varepsilon_n^i \alpha = \mu + \nu + h_n^i (\delta + d)\alpha = \mu + \nu - h_n^i \bar{R}(\alpha) . \quad (27)$$

We thus choose the ‘Ansatz’ $\alpha_0 = \mu + \nu$ and set $\alpha_k = \alpha_0 - h_n^i \bar{R}(\alpha_{k-1})$, $k \geq 1$. It is easily seen that, in view of nilpotency, this iteration stabilizes, i.e. $\alpha_{k-1} = \alpha_k = \dots =: \alpha$, for $k \gg$, or, still,

$$\alpha = \alpha_0 - h_n^i \bar{R}(\alpha) . \quad (28)$$

The limit α is actually a solution in $\text{MC}_n(\bar{L})$. Indeed, remember first that the generalized curvature

$$\bar{\mathcal{F}}(\alpha) = (\delta + d)\alpha + \bar{R}(\alpha) = (\delta + d)\alpha + \sum_{i \geq 2} \frac{1}{i!} \mathcal{L}_i(\alpha, \dots, \alpha) ,$$

whose zeros are the MC elements, satisfies, just like the standard curvature, the Bianchi identity

$$(\delta + d)\bar{\mathcal{F}}(\alpha) + \sum_{k \geq 1} \frac{1}{k!} \mathcal{L}_{k+1}(\alpha, \dots, \alpha, \bar{\mathcal{F}}(\alpha)) = 0 . \quad (29)$$

It follows from (28) and (24) that

$$\tilde{\mathcal{F}}(\alpha) = (\delta + d)(\alpha_0 - h_n^i \bar{R}(\alpha)) + \bar{R}(\alpha) = (\delta + d)\mu + h_n^i(\delta + d)\bar{R}(\alpha) + \varepsilon_n^i \bar{R}(\alpha).$$

From Equation (26) we know that $\varepsilon_n^i \bar{R}(\alpha) = R(\varepsilon_n^i \alpha) = R(\mu)$, with self-explaining notation. As for $\varepsilon_n^i \alpha = \mu$, note that $\varepsilon_n^i \mu = \mu$ and that

$$\varepsilon_n^i v = \varepsilon_n^i(\delta + d)\beta = \varepsilon_n^i(\delta + d) \sum_{k=0}^n \beta^k, \beta^k \in L_{-k} \otimes \Omega^k(\Delta^n), \text{ so that } \varepsilon_n^i v = \varepsilon_n^i \delta \beta^0 = \delta \varepsilon_n^i \beta^0 = \delta \varepsilon_n^i \beta = 0.$$

Hence,

$$\begin{aligned} \tilde{\mathcal{F}}(\alpha) &= (\delta + d)\mu + h_n^i(\delta + d)(\tilde{\mathcal{F}}(\alpha) - (\delta + d)\alpha) + R(\mu) \\ &= \mathcal{F}(\mu) + h_n^i(\delta + d)\tilde{\mathcal{F}}(\alpha) = -h_n^i \sum_{k \geq 1} \frac{1}{k!} \tilde{\mathcal{L}}_{k+1}(\alpha, \dots, \alpha, \tilde{\mathcal{F}}(\alpha)), \end{aligned}$$

in view of (29). Therefore, $\tilde{\mathcal{F}}(\alpha) \in F^i \bar{L}$, for arbitrarily large i , and thus $\alpha \in \text{MC}_n(\bar{L})$. This completes the construction of maps

$$\mathcal{B}_n^i : \text{MC}(L) \times \text{mc}_n^i(\bar{L}) \rightarrow \text{MC}_n(\bar{L}). \quad (30)$$

We already observed that $\varepsilon_n^i \mathcal{B}_n^i(\mu, v) = \varepsilon_n^i \alpha = \mu$. In fact, $B_n^i \mathcal{B}_n^i = \text{id}$, so that B_n^i is surjective. Indeed, Equations (28) and (24) imply that

$$(\delta + d)h_n^i \alpha = -h_n^i(\delta + d)\alpha_0 + \alpha_0 - \varepsilon_n^i \alpha_0 = -h_n^i \delta \mu + v = v.$$

As for injectivity, if $B_n^i \alpha = B_n^i \alpha' =: (\mu, v)$, then both, α and α' , satisfy Equation (27). It is now quite easily seen that nilpotency entails that $\alpha = \alpha'$.

- The bijections

$$B_n^i : \text{MC}_n(\bar{L}) \rightarrow \text{MC}(L) \times \text{mc}_n^i(\bar{L})$$

allow us to prove the Kan property for $\text{MC}_\bullet(\bar{L})$. The extension of a horn in $\text{SSet}(\Lambda^i[n], \text{MC}_\bullet(\bar{L}))$ will be performed as sketched by the following diagram:

$$\begin{array}{ccc} \text{SSet}(\Lambda^i[n], \text{MC}_\bullet(\bar{L})) & \dashrightarrow & \text{MC}_n(\bar{L}) \\ \downarrow & & \uparrow \\ \text{SSet}(\Lambda^i[n], \text{MC}(L) \times \text{mc}_\bullet(\bar{L})) & \longrightarrow & \text{MC}(L) \times \text{mc}_n^i(\bar{L}) \end{array} \quad (31)$$

Of course, the right arrow is nothing but \mathcal{B}_n^i .

- ★ As for the left arrow, imagine, for simplicity, that $i = 1$ and $n = 2$, and let

$$\alpha \in \text{SSet}(\Lambda^1[2], \text{MC}_\bullet(\bar{L})).$$

The restrictions $\alpha|_{01}$ and $\alpha|_{12}$ to the 1-faces 01, 12 (compositions of the natural injections with α) are elements in $\text{MC}_1(\bar{L})$, so that the map B_1^1 sends $\alpha|_{01}$ to (μ, v) in $\text{MC}(L) \times \text{mc}_1(\bar{L})$ (and similarly $B_1^0(\alpha|_{12}) = (\mu', v') \in \text{MC}(L) \times \text{mc}_1(\bar{L})$). Of course, $\mu = \varepsilon_1^1(\alpha|_{01}) = \varepsilon_1^0(\alpha|_{12}) = \mu'$. Since $v = (\delta +$

$d\beta$ and $\beta(1) = \varepsilon_1^1 \beta = 0$, we find $v(1) = \varepsilon_1^1 v = 0$ (and similarly $v' = (\delta + d)\beta'$ and $\beta'(1) = v'(1) = 0$). Thus,

$$(\mu; v, v') \in \text{SSet}(\Lambda^1[2], \text{MC}(L) \times \text{mc}_\bullet(\bar{L})),$$

which explains the left arrow.

★ For the bottom arrow, let again $i = 1, n = 2$. Since μ is constant, it can be extended to the whole simplex. To extend (v, v') , it actually suffices to extend (β, β') . Indeed, restriction obviously commutes with δ . As for commutation with d , remember that $\Omega^* : \Delta^{\text{op}} \rightarrow \text{DGCA}$ and that the DGCA-map $d_2^2 = \Omega^*(\delta_2^2)$ sets the component t_2 to 0. Hence, d_2^2 coincides with restriction to 01 and commutes with d . Let now $\bar{\beta}$ be an extension of (β, β') . Since

$$(d\bar{\beta})|_{01} = d_2^2 d\bar{\beta} = d d_2^2 \bar{\beta} = d\beta$$

and similarly $(d\bar{\beta})|_{12} = d\beta'$.

It now remains to explain that an extension $\bar{\beta}$ does always exist. Consider the slightly more general extension problem of three polynomial differential forms β_0, β_1 , and β_2 defined on the 1-faces 12, 02, and 01 of the 2-simplex Δ^2 , respectively (it is assumed that they coincide at the vertices). Let $\pi_2 : \Delta^2 \rightarrow 01$ be the projection defined, for any $\vec{t} = [t_0, t_1, t_2]$, as the intersection of the line $u\vec{t} + (1-u)\vec{e}_2$ with 01. This projection is of course ill-defined at $\vec{t} = \vec{e}_2$. In coordinates, we get

$$\pi_2 : [t_0, t_1, t_2] \mapsto [t_0/(1-t_2), t_1/(1-t_2)].$$

It follows that the pullback $\pi_2^* \beta_2$ is a rational differential form with denominator $(1-t_2)^N$, for some integer N . Hence,

$$\gamma_2 := (1-t_2)^N \pi_2^* \beta_2$$

is a polynomial differential form on Δ^2 that coincides with β_2 on 01. It now suffices to solve the same extension problem as before, but for the forms $\beta_0 - \gamma_2|_{12}$, $\beta_1 - \gamma_2|_{02}$, and 0. When iterating the procedure – due to Renshaw [Sul77] –, the problem reduces to the extension of 0, 0, 0 (since the pullback preserves 0). This completes the description of the bottom arrow, as well as the proof of the Kan property of $\text{MC}_\bullet(\bar{L})$.

5 2-Category of 2-term Leibniz infinity algebras

Categorification replaces sets (resp., maps, equations) by categories (resp., functors, natural isomorphisms). In particular, rather than considering two maps as equal, one details a way of identifying them. Categorification is thus a sharpened viewpoint that turned out to provide deeper insight. This motivates the interest in e.g. categorified algebras (and in truncated infinity algebras – see below).

Categorified Lie algebras were introduced under the name of Lie 2-algebras in [BC04] and further studied in [Roy07], [SL10], and [KMP11]. The main result of [BC04] states that Lie 2-algebras and 2-term Lie infinity algebras form equivalent 2-categories. However, **infinity homotopies of 2-term Lie infinity algebras** (resp., **compositions of such homotopies**) are not explained, but appear as some God-given natural transformations read through this EQUIVALENCE (resp., compositions are addressed only in [SS07] and performed in the ALGEBRAIC OR COALGEBRAIC SETTINGS).

This circumstance is not satisfactory, and **the attempt to improve our understanding of infinity homotopies and their compositions is one of the main concerns of the present paper**. Indeed, in [KMP11] (resp., [BP12]), the authors show that the EQUIVALENCE between n -term Lie infinity algebras and Lie n -algebras is, for $n > 2$, not as straightforward as expected – which is essentially due to the largely ignored fact that the category $\text{Vect } n\text{-Cat}$ of linear n -categories is symmetric monoidal, but that the corresponding map $\boxtimes : L \times L' \rightarrow L \boxtimes L'$ is not an n -functor (resp., that the understanding of a concept in the ALGEBRAIC FRAMEWORK is far from implying its comprehension in the infinity context – a reality that is corroborated e.g. by the comparison of concordances and infinity homotopies).

In this section, we obtain **explicit formulae for infinity homotopies and their compositions**, applying the KAN PROPERTY of $\text{MC}_\bullet(\bar{L})$ to the 2-term case, thus staying inside the INFINITY SETTING.

5.1 Category of 2-term Leibniz infinity algebras

For the sake of completeness, we first describe 2-term Leibniz infinity algebras and their morphisms. Propositions 9 and 10 are specializations to the 2-term case of Definitions 4 and 5; see also [SL10]. The informed reader may skip the present subsection.

Proposition 9. *A 2-term Leibniz infinity algebra is a graded vector space $V = V_0 \oplus V_1$ concentrated in degrees 0 and 1, together with a linear, a bilinear, and a trilinear map l_1, l_2 , and l_3 on V , of degree $|l_1| = -1, |l_2| = 0$, and $|l_3| = 1$, which verify, for any $w, x, y, z \in V_0$ and $h, k \in V_1$,*

$$(a) \quad \begin{aligned} l_1 l_2(x, h) &= l_2(x, l_1 h) , \\ l_1 l_2(h, x) &= l_2(l_1 h, x) , \end{aligned}$$

$$(b) \quad l_2(l_1 h, k) = l_2(h, l_1 k) ,$$

$$(c) \quad l_1 l_3(x, y, z) = l_2(x, l_2(y, z)) - l_2(y, l_2(x, z)) - l_2(l_2(x, y), z) ,$$

$$(d) \quad \begin{aligned} l_3(x, y, l_1 h) &= l_2(x, l_2(y, h)) - l_2(y, l_2(x, h)) - l_2(l_2(x, y), h) , \\ l_3(x, l_1 h, y) &= l_2(x, l_2(h, y)) - l_2(h, l_2(x, y)) - l_2(l_2(x, h), y) , \\ l_3(l_1 h, x, y) &= l_2(h, l_2(x, y)) - l_2(x, l_2(h, y)) - l_2(l_2(h, x), y) , \end{aligned}$$

$$(e) \quad \begin{aligned} l_2(l_3(w, x, y), z) + l_2(w, l_3(x, y, z)) - l_2(x, l_3(w, y, z)) + l_2(y, l_3(w, x, z)) - l_3(l_2(w, x), y, z) \\ + l_3(w, l_2(x, y), z) - l_3(x, l_2(w, y), z) - l_3(w, x, l_2(y, z)) + l_3(w, y, l_2(x, z)) - l_3(x, y, l_2(w, z)) = 0 . \end{aligned}$$

Proposition 10. *An infinity morphism between 2-term Leibniz infinity algebras (V, l_1, l_2, l_3) and (W, m_1, m_2, m_3) is made up by a linear and a bilinear map f_1, f_2 from V to W , of degree $|f_1| = 0, |f_2| = 1$, which verify, for any $x, y, z \in V_0$ and $h \in V_1$,*

$$(a) \quad m_1 f_1 h = f_1 l_1 h ,$$

$$(b) \quad m_2(f_1 x, f_1 y) + m_1 f_2(x, y) = f_1 l_2(x, y) ,$$

$$(c) \quad \begin{aligned} m_2(f_1 x, f_1 h) &= f_1 l_2(x, h) - f_2(x, l_1 h) , \\ m_2(f_1 h, f_1 x) &= f_1 l_2(h, x) - f_2(l_1 h, x) , \end{aligned}$$

$$(d) \quad m_3(f_1x, f_1y, f_1z) - m_2(f_2(x, y), f_1z) + m_2(f_1x, f_2(y, z)) - m_2(f_1y, f_2(x, z)) = \\ f_1l_3(x, y, z) + f_2(l_2(x, y), z) - f_2(x, l_2(y, z)) + f_2(y, l_2(x, z)) .$$

Corollary 1. *The category 2Lei_∞ of 2-term Leibniz infinity algebras and infinity morphisms is a full subcategory of the category $\text{Lei}_\infty\text{-Alg}$ of Leibniz infinity algebras and infinity morphisms.*

5.2 From the Kan property to 2-term infinity homotopies and their compositions

Definition 16. *A 2-term infinity homotopy between infinity morphisms $f = (f_1, f_2)$ and $g = (g_1, g_2)$, which act themselves between 2-term Leibniz infinity algebras (V, l_1, l_2, l_3) and (W, m_1, m_2, m_3) , is a linear map θ_1 from V to W , of degree $|\theta_1| = 1$, which verifies, for any $x, y \in V_0$ and $h \in V_1$,*

- (a) $g_1x - f_1x = m_1\theta_1x$,
- (b) $g_1h - f_1h = \theta_1l_1h$,
- (c) $g_2(x, y) - f_2(x, y) = \theta_1l_2(x, y) - m_2(f_1x, \theta_1y) - m_2(\theta_1x, g_1y)$.

The characterizing relations (a) - (c) of infinity Leibniz homotopies are the correct counterpart of the defining relations of infinity Lie homotopies [BC04]. However, rather than choosing the preceding relations as a mere definition, we deduce them here from the Kan property of $\text{MC}_\bullet(\bar{L})$. More precisely,

Theorem 3. *There exist surjective maps S_1^i , $i \in \{0, 1\}$, from the class \mathcal{S} of ∞ -homotopies for 2-term Leibniz infinity algebras to the class \mathcal{T} of 2-term ∞ -homotopies for 2-term Leibniz infinity algebras.*

Remark 8. *The maps S_1^i preserve the source and the target, i.e. they are surjections from the class $\mathcal{S}(f, g)$ of ∞ -homotopies from f to g , to the class $\mathcal{T}(f, g)$ of 2-term ∞ -homotopies from f to g . In the sequel, we refer to a preimage by S_1^i of an element $\theta_1 \in \mathcal{T}$ as a lift of θ_1 by S_1^i .*

Proof. Henceforth, we use again the homological version of infinity algebras (k -ary bracket of degree $k - 2$), as well as the Lada-Stasheff sign convention for the higher Jacobi conditions and the MC equation.

Due to the choice of the homological variant of homotopy algebras, $\delta = \mathcal{L}_1$ has degree -1 . For consistency, differential forms are then viewed as negatively graded; hence, $d : \Omega^{-k}(\Delta^n) \rightarrow \Omega^{-k-1}(\Delta^n)$, $k \in \{0, \dots, n\}$, and $\bar{\mathcal{L}}_1 = \delta \otimes \text{id} + \text{id} \otimes d$ has degree -1 as well. Similarly, the degree of the operator h_n^i is now $|h_n^i| = 1$. It is moreover easily checked that L cannot contain multilinear maps of nonnegative degree, i.e. that $L = \bigoplus_{k \geq 0} L_{-k}$. It follows that an element $\bar{\alpha} \in (L \otimes \Omega^*(\Delta^n))^{-k}$, $k \geq 0$, reads

$$\bar{\alpha} = \sum \alpha_{-k} \otimes \omega^0 + \sum \alpha_{-k+1} \otimes \omega^{-1} + \dots ,$$

where the RHS is a finite sum. For instance, if $n = 2$, an element $\bar{\alpha}$ of degree -1 can be decomposed as

$$\bar{\alpha} = \alpha(s, t) \otimes 1 + \beta(s, t) \otimes ds + \beta'(s, t) \otimes dt ,$$

where (s, t) are coordinates of Δ^2 and where $\alpha(s, t) \in L_{-1}[s, t]$ and $\beta(s, t), \beta'(s, t) \in L_0[s, t]$ are polynomial functions in s, t with coefficients in L_{-1} and L_0 , respectively.

In the sequel, we evaluate the L_∞ -structure maps $\bar{\mathcal{L}}_i$ of $L \otimes \Omega^*(\Delta^n)$ mainly on elements of degree -1 and 0 , hence we compute the structure maps \mathcal{L}_i of $L = \text{Hom}_{\mathbb{K}}(\text{Zin}^c(sV), W)$ on elements α and β of degree -1 and 0 , respectively. Let

$$\begin{aligned}\alpha &= \sum_{p \geq 1} \alpha^p \in L, \quad |\alpha| = -1, \\ \beta &= \sum_{p \geq 1} \beta^p \in L, \quad |\beta| = 0,\end{aligned}$$

where $\alpha^p, \beta^p : (sV)^{\otimes p} \rightarrow W$. The point is that the concentration of V, W in degrees $0, 1$ entails that almost all components α^p, β^p vanish and that all series converge (which explains why the formal application of Getzler's method to the present situation leads to the correct counterpart of the findings of [BC04]). Indeed, the only nonzero components of α, β are

$$\begin{aligned}\alpha^1 &: sV_0 \rightarrow W_0, \quad sV_1 \rightarrow W_1, \\ \alpha^2 &: (sV_0)^{\otimes 2} \rightarrow W_1, \\ \beta^1 &: sV_0 \rightarrow W_1.\end{aligned}\tag{32}$$

Similarly, the nonzero components of the nonzero evaluations of the maps \mathcal{L}_i on α -s and β -s are

$$\begin{aligned}\mathcal{L}_1(\alpha) &: sV_1 \rightarrow W_0, \quad (sV_0)^{\otimes 2} \rightarrow W_0, \quad sV_0 \otimes sV_1 \rightarrow W_1, \quad (sV_0)^{\otimes 3} \rightarrow W_1, \\ \mathcal{L}_1(\beta) &: sV_0 \rightarrow W_0, \quad sV_1 \rightarrow W_1, \quad (sV_0)^{\otimes 2} \rightarrow W_1, \\ \mathcal{L}_2(\alpha_1, \alpha_2) &: (sV_0)^{\otimes 2} \rightarrow W_0, \quad sV_0 \otimes sV_1 \rightarrow W_1, \quad (sV_0)^{\otimes 3} \rightarrow W_1, \\ \mathcal{L}_2(\alpha, \beta) &: (sV_0)^{\otimes 2} \rightarrow W_1, \\ \mathcal{L}_3(\alpha_1, \alpha_2, \alpha_3) &: (sV_0)^{\otimes 3} \rightarrow W_1,\end{aligned}\tag{33}$$

see Proposition 4.

We are now prepared to concretize the iterative construction of $\mathcal{B}_n^i(\mu, \nu) \in \text{MC}_n(\bar{L})$ from $\mu \in \text{MC}(L)$ and $\nu = (\delta + d)\beta$, $\beta \in (L \otimes \Omega^*(\Delta^n))^0$, $\varepsilon_n^i \beta = 0$ (the explicit forms of $\mathcal{B}_n^i(\mu, \nu)$ for $n = 1$ and $n = 2$ will be the main ingredients of the proofs of Theorems 3 and 4).

- Let $\alpha \in \mathcal{I}(f, g)$, i.e. let

$$\alpha \in \text{MC}_1(\bar{L}) \xrightarrow{\sim} (\mu, (\delta + d)\beta) \in \text{MC}(L) \times \text{mc}_1^0(\bar{L}),$$

such that $\varepsilon_1^0 \alpha = f$ and $\varepsilon_1^1 \alpha = g$. To construct

$$\alpha = \mathcal{B}_1^0 \mathcal{B}_1^0 \alpha = \mathcal{B}_1^0(\varepsilon_1^0 \alpha, (\delta + d)h_1^0 \alpha) =: \mathcal{B}_1^0(f, (\delta + d)\beta) =: \mathcal{B}_1^0(\mu, \nu),$$

we start from

$$\alpha_0 = \mu + (\delta + d)\beta.$$

The iteration unfolds as

$$\alpha_k = \alpha_0 - \sum_{j=2}^{\infty} \frac{1}{j!} h_1^0 \bar{\mathcal{L}}_j(\alpha_{k-1}, \dots, \alpha_{k-1}), \quad k \geq 1.$$

Explicitly,

$$\begin{aligned}
 \alpha_1 &= \mu + (\delta + d)\beta - \frac{1}{2}h_1^0\bar{\mathcal{L}}_2(\alpha_0, \alpha_0) - \frac{1}{3!}h_1^0\bar{\mathcal{L}}_3(\alpha_0, \alpha_0, \alpha_0) \\
 &= \mu + (\delta + d)\beta - h_1^0\bar{\mathcal{L}}_2(\mu + \delta\beta, d\beta) - \frac{1}{2}h_1^0\bar{\mathcal{L}}_3(\mu + \delta\beta, \mu + \delta\beta, d\beta) \\
 &= \mu + (\delta + d)\beta - h_1^0\bar{\mathcal{L}}_2(\mu + \delta\beta, d\beta) .
 \end{aligned}$$

Observe that $\mu + \delta\beta \in L_{-1}[t]$ and $d\beta \in L_0[t] \otimes dt$, that differential forms are concentrated in degrees 0 and -1 , that h_1^0 annihilates 0-forms, and that the term in $\bar{\mathcal{L}}_3$ contains a factor of the type $\mathcal{L}_3(\alpha_1, \alpha_2, \beta)$ (notation of (33)), whose components vanish – see above. Analogously,

$$\begin{aligned}
 \alpha_2 &= \mu + (\delta + d)\beta - h_1^0\bar{\mathcal{L}}_2(\mu + \delta\beta - h_1^0\bar{\mathcal{L}}_2(\mu + \delta\beta, d\beta), d\beta) \\
 &\quad - \frac{1}{2}h_1^0\bar{\mathcal{L}}_3(\mu + \delta\beta - h_1^0\bar{\mathcal{L}}_2(\mu + \delta\beta, d\beta), \mu + \delta\beta - h_1^0\bar{\mathcal{L}}_2(\mu + \delta\beta, d\beta), d\beta) \\
 &= \mu + (\delta + d)\beta - h_1^0\bar{\mathcal{L}}_2(\mu + \delta\beta, d\beta) .
 \end{aligned}$$

Indeed, the term $h_1^0\bar{\mathcal{L}}_2(h_1^0\bar{\mathcal{L}}_2(\mu + \delta\beta, d\beta), d\beta)$ contains a factor of the type $\mathcal{L}_2(\mathcal{L}_2(\alpha, \beta_1), \beta_2)$ (notation of (33)), and the only nonvanishing component of this factor, as well as of its first internal map $\mathcal{L}_2(\alpha, \beta_1)$, is the component $(sV_0)^{\otimes 2} \rightarrow W_1$ – which entails, in view of Proposition 4, that the considered term vanishes. Hence, the iteration stabilizes already at its second stage and

$$\alpha = \mathcal{B}_1^0(\mu, \nu) = \mu + (\delta + d)\beta - h_1^0\bar{\mathcal{L}}_2(\mu + \delta\beta, d\beta) \in \text{MC}_1(\bar{L}) . \quad (34)$$

Remark first that the integral h_1^0 can be evaluated since $\bar{\mathcal{L}}_2(\mu + \delta\beta, d\beta)$ is a total derivative. Indeed, when setting $\beta = \beta_0 \otimes P$ (sum understood), $\beta_0 \in L_0$ and $P \in \Omega^0(\Delta^1)$, we see that

$$\bar{\mathcal{L}}_2(\mu, d\beta) = \mathcal{L}_2(\mu, \beta_0) \otimes dP = -d\bar{\mathcal{L}}_2(\mu, \beta) .$$

As for the term $\bar{\mathcal{L}}_2(\delta\beta, d\beta)$, we have

$$0 = (\delta + d)\bar{\mathcal{L}}_2(\beta, d\beta) = \bar{\mathcal{L}}_2(\delta\beta, d\beta) + \bar{\mathcal{L}}_2(\beta, \delta d\beta) ,$$

since $\bar{\mathcal{L}}_1 = \delta + d$ is a graded derivation of $\bar{\mathcal{L}}_2$ and as $\bar{\mathcal{L}}_2(\beta, d\beta) = \bar{\mathcal{L}}_2(d\beta, d\beta) = 0$. It is now easily checked that

$$\bar{\mathcal{L}}_2(\delta\beta, d\beta) = -\frac{1}{2}d\bar{\mathcal{L}}_2(\delta\beta, \beta) .$$

Eventually,

$$\begin{aligned}
 \alpha &= \mu + (\delta + d)\beta + h_1^0d\bar{\mathcal{L}}_2(\mu, \beta) + \frac{1}{2}h_1^0d\bar{\mathcal{L}}_2(\delta\beta, \beta) \\
 &= \mu + (\delta + d)\beta + \bar{\mathcal{L}}_2(\mu, \beta) + \frac{1}{2}\bar{\mathcal{L}}_2(\delta\beta, \beta) .
 \end{aligned}$$

Indeed, it suffices to observe that, for any $\ell_{-1} \otimes P \in L_{-1} \otimes \Omega^0(\Delta^1)$ which vanishes under the action of ε_1^0 , we have

$$h_1^0d(\ell_{-1} \otimes P) = -dh_1^0(\ell_{-1} \otimes P) + \ell_{-1} \otimes P - \varepsilon_1^0(\ell_{-1} \otimes P) = \ell_{-1} \otimes P .$$

We are now able to write the components of $g = \varepsilon_1^1 \alpha \in L_{-1}$ (see (32)) in terms of $f = \mu$ and β :

$$\begin{aligned} g^1 &= \varepsilon_1^1(\mu + (\delta + d)\beta - \mathcal{L}_2(\mu, \beta) - \frac{1}{2}\mathcal{L}_2(\delta\beta, \beta))^1 = \varepsilon_1^1(f + \delta\beta)^1 = f^1 + \varepsilon_1^1(\delta\beta)^1, \\ g^2 &= \varepsilon_1^1(\mu + (\delta + d)\beta - \mathcal{L}_2(\mu, \beta) - \frac{1}{2}\mathcal{L}_2(\delta\beta, \beta))^2 = \varepsilon_1^1(f + \delta\beta - \mathcal{L}_2(f, \beta) - \frac{1}{2}\mathcal{L}_2(\delta\beta, \beta))^2, \\ g^3 &= 0, \end{aligned} \quad (35)$$

where we changed signs according to our sign conventions and remembered that the first component of a morphism of the type $\mathcal{L}_2(\alpha, \beta)$ (see (33)) vanishes.

To obtain a 2-term ∞ -homotopy $\theta_1 \in \mathcal{T}(f, g)$, it now suffices to further develop the equations (35).

As

$$g_1 := g^1 s, f_1 := f^1 s \in \text{Hom}_{\mathbb{K}}^0(V, W),$$

we evaluate the first equation on $x \in V_0$ and $h \in V_1$. Therefore, we compute $\varepsilon_1^1(\delta\beta)^1 s = \delta\beta(1)^1 s$ on x and h . Since

$$\delta\beta(1) = \mathcal{L}_1\beta(1) = m_1\beta(1) + \beta(1)D_V,$$

where $D_V \in \text{CoDer}^{-1}(\text{Zin}^c(sV))$, we have $D_V : sV_1 \rightarrow sV_0, sV_0 \otimes sV_0 \rightarrow sV_0, \dots$ Hence,

$$\delta\beta(1)^1 s x = m_1 \beta(1) s x = m_1 \theta_1 x, \quad (36)$$

where we defined the *homotopy parameter* θ_1 by

$$\theta_1 := \beta(1)s = \beta(1)s - \beta(0)s. \quad (37)$$

Similarly,

$$\delta\beta(1)^1 s h = \beta(1)D_V s h = \beta(1)s s^{-1}D_V s h = \theta_1 l_1 h. \quad (38)$$

The characterizing equations (a) and (b) follow.

Since

$$g_2 := g^2 s^2, f_2 := f^2 s^2 \in \text{Hom}_{\mathbb{K}}^1(V \otimes V, W),$$

it suffices to evaluate the second equation on $x, y \in V_0$. When computing e.g. $\varepsilon_1^1 \mathcal{L}_2(\delta\beta, \beta)^2 s^2(x, y)$, we get

$$\begin{aligned} \mathcal{L}_2(\delta\beta(1), \beta(1))(s x, s y) &= m_2(\delta\beta(1) s x, \beta(1) s y) + m_2(\beta(1) s x, \delta\beta(1) s y) = \\ &= m_2(m_1 \theta_1 x, \theta_1 y) + m_2(\theta_1 x, m_1 \theta_1 y) = 2m_2(\theta_1 x, m_1 \theta_1 y), \end{aligned} \quad (39)$$

in view of Equation (36) and Relation (b) of Proposition 9. Similarly,

$$\varepsilon_1^1 \mathcal{L}_2(f, \beta)^2 s^2(x, y) = m_2(f_1 x, \theta_1 y) + m_2(\theta_1 x, f_1 y). \quad (40)$$

Further, one easily finds

$$\varepsilon_1^1(\delta\beta)^2 s^2(x, y) = \theta_1 l_2(x, y). \quad (41)$$

When collecting the results (39), (40), and (41), and taking into account Relation (a), we finally obtain the characterizing equation (c).

- Recall that in the preceding step we started from $\alpha \in \mathcal{S}(f, g)$, set $\mu = f$,

$$\beta = h_1^0 \alpha ,$$

$\nu = (\delta + d)\beta$, defined

$$\theta_1 = (\beta(1) - \beta(0))s ,$$

and deduced the characterizing relations $g = f + \mathcal{E}(f, \beta(1)s) = f + \mathcal{E}(f, \theta_1)$ of $\theta_1 \in \mathcal{T}(f, g)$ by computing

$$\alpha = \mathcal{B}_1^0(\mu, \nu) = \mu + (\delta + d)\beta + \mathcal{L}_2(\mu, \beta) + \frac{1}{2}\mathcal{L}_2(\delta\beta, \beta)$$

at 1. Let us mention that instead of defining the map $S_1^0 : \mathcal{S} \ni \alpha \mapsto \theta_1 \in \mathcal{T}$, we can consider the similarly defined map S_1^1 .

To prove surjectivity of S_1^i , let $\theta_1 \in \mathcal{T}(f, g)$ and set $\beta(i) = 0, i \in \{0, 1\}$, and $\beta(1-i) = (-1)^i \theta_1 s^{-1}$. Note that by construction $\theta_1 = (\beta(1) - \beta(0))s$. Use now Renshaw's method [Sul77] to extend $\beta(0)$ and $\beta(1)$ to some $\beta \in L_0 \otimes \Omega^0(\Delta^1)$, set

$$\mu = (1-i)f + ig \quad \text{and} \quad \nu = (\delta + d)\beta ,$$

and construct

$$\alpha = \mathcal{B}_1^i(\mu, \nu) \in \text{MC}_1(\bar{L}) . \tag{42}$$

If $i = 0$, then

$$\alpha(0) = \varepsilon_1^0 \alpha = \mu = f \quad \text{and} \quad \alpha(1) = (\mathcal{B}_1^0(\mu, \nu))(1) = f + \mathcal{E}(f, \theta_1) = g ,$$

in view of the characterizing relations (a)-(c) of θ_1 . If $i = 1$, one has also $\alpha(1) = g$ and $\alpha(0) = g + \mathcal{E}(g, -\theta_1) = f$, but to obtain the latter result, the characterizing equations (a)-(c), as well as Equation (b) of Proposition 9 are needed. To determine the image of $\alpha \in \mathcal{S}(f, g)$ by S_1^i , one first computes $h_1^i \alpha$, which, since h_1^i sends 0-forms to 0, is equal to

$$h_1^i(\delta + d)\beta = -(\delta + d)h_1^i \beta + \beta - \varepsilon_1^i \beta = \beta ,$$

then one gets

$$S_1^i \alpha = (\beta(1) - \beta(0))s = \theta_1 ,$$

which completes the proof. □

Theorem 4 (Definition). *If $\theta_1 : f \Rightarrow g, \tau_1 : g \Rightarrow h$ are 2-term ∞ -homotopies between infinity morphisms $f, g, h : V \rightarrow W$, the vertical composite $\tau_1 \circ_1 \theta_1$ is given by $\tau_1 + \theta_1$.*

We will actually lift $\theta_1, \tau_1 \in \mathcal{T}$ to $\alpha', \alpha'' \in \text{MC}_1(\bar{L})$ (which involves choices), then compose these lifts in the infinity groupoid $\text{MC}_\bullet(\bar{L})$ (which is not a well-defined operation), and finally project the result back to \mathcal{T} (despite all the intermediate choices, the final result will turn out to be well-defined).

Proof. Let now $n = 2$, take $\mu \in \text{MC}(L)$ and $\nu = (\delta + d)\beta \in \text{mc}_2^1(\bar{L})$, then construct $\alpha = \mathcal{B}_2^1(\mu, \nu)$. The computation is similar to that in the 1-dimensional case and gives the same result:

$$\alpha = \mu + (\delta + d)\beta + \mathcal{L}_2(\mu, \beta) + \frac{1}{2}\mathcal{L}_2(\delta\beta, \beta). \quad (43)$$

To obtain $\tau_1 \circ_1 \theta_1$, proceed as in (42) and lift θ_1 (resp., τ_1) to

$$\alpha' := \mathcal{B}_1^1(g, (\delta + d)\beta') \in \mathcal{S}(f, g) \subset \text{MC}_1(\bar{L}) \quad (\text{resp., } \alpha'' := \mathcal{B}_1^0(g, (\delta + d)\beta'') \in \mathcal{S}(g, h) \subset \text{MC}_1(\bar{L})),$$

where

$$\beta'(0) = -\theta_1 s^{-1} \text{ and } \beta'(1) = 0 \quad (\text{resp., } \beta''(0) = 0 \text{ and } \beta''(1) = \tau_1 s^{-1}).$$

As mentioned above, we have by construction

$$\theta_1 = (\beta'(1) - \beta'(0))s \quad (\text{resp., } \tau_1 = (\beta''(1) - \beta''(0))s). \quad (44)$$

If we view α' (resp., α'') as defined on the face 01 (resp., 12) of Δ^2 , the equation $\varepsilon_1^1 \alpha' = \varepsilon_1^0 \alpha'' = g$ reads $\varepsilon_2^1 \alpha' = \varepsilon_2^1 \alpha'' = g =: \mu$. This means that

$$(\alpha', \alpha'') \in \text{SSet}(\Lambda^1[2], \text{MC}_\bullet(\bar{L})).$$

We now follow the extension square (31). The left arrow leads to

$$(\mu; (\delta + d)\beta', (\delta + d)\beta'') \in \text{SSet}(\Lambda^1[2], \text{MC}(L) \times \text{mc}_\bullet(\bar{L})),$$

the bottom arrow to

$$(\mu, (\delta + d)\beta) \in \text{MC}(L) \times \text{mc}_2^1(\bar{L}),$$

where β is any extension of (β', β'') to Δ^2 , and the right arrow provides $\alpha \in \text{MC}_2(\bar{L})$ given by Equation (43). From Subsection 4.2.3, we know that all composites of α', α'' are ∞ -2-homotopic and that a possible composite is obtained by restricting α to 02. This restriction $(-)|_{02}$ is given by the DGCA-map d_1^2 . Hence, we get

$$\alpha|_{02} = \mu + (\delta + d)\beta|_{02} + \mathcal{L}_2(\mu, \beta|_{02}) + \frac{1}{2}\mathcal{L}_2(\delta\beta|_{02}, \beta|_{02}) \in \mathcal{S}(f, h) \subset \text{MC}_1(\bar{L}).$$

We now choose the projection $S_1^0 \alpha|_{02} \in \mathcal{S}(f, h)$ of the composite-candidate of the chosen lifts of θ_1, τ_1 , as composite $\tau_1 \circ_1 \theta_1$. Since

$$h_1^0 \alpha|_{02} = -(\delta + d)h_1^0 \beta|_{02} + \beta|_{02} - \beta(0) = \beta|_{02} - \beta(0),$$

we get

$$\begin{aligned} S_1^0 \alpha|_{02} &= (\beta|_{02}(2) - \beta(0) - \beta|_{02}(0) + \beta(0))s = (\beta(2) - \beta(0))s = \\ &= (\beta''(2) - \beta''(1))s + (\beta'(1) - \beta'(0))s = \tau_1 + \theta_1, \end{aligned}$$

in view of (44). Hence, by definition, the vertical composite of $\theta_1 \in \mathcal{S}(f, g)$ and $\tau_1 \in \mathcal{S}(g, h)$ is given by

$$\tau_1 \circ_1 \theta_1 = \tau_1 + \theta_1 \in \mathcal{S}(f, h). \quad (45)$$

□

Remark 9. *The composition of elements of $\mathcal{S} = \text{MC}_1(\bar{L})$ in the infinity groupoid $\text{MC}_\bullet(\bar{L})$, which is defined and associative only up to higher morphisms, projects to a well-defined and associative vertical composition in \mathcal{T} .*

Just as for concordances, horizontal composition of ∞ -homotopies is without problems. The horizontal composite of $\theta_1 \in \mathcal{T}(f, g)$ and $\tau_1 \in \mathcal{T}(f', g')$, where $f, g : V \rightarrow W$ and $f', g' : W \rightarrow X$ act between 2-term Leibniz infinity algebras, is defined by

$$\tau_1 \circ_0 \theta_1 = g'_1 \theta_1 + \tau_1 f_1 = f'_1 \theta_1 + \tau_1 g_1 . \quad (46)$$

The two definitions coincide, since θ_1, τ_1 are chain homotopies between the chain maps f, g and f', g' , respectively, see Definition 16, Relations (a) and (b). The identity associated to a 2-term ∞ -morphism is just the zero-map. As announced in [BC04] (in the Lie case and without information about composition), we have the

Proposition 11. *There is a strict 2-category $2\text{Lei}_\infty\text{-Alg}$ of 2-term Leibniz infinity algebras.*

6 2-Category of categorified Leibniz algebras

6.1 Category of Leibniz 2-algebras

Leibniz 2-algebras are categorified Leibniz structures on a categorified vector space. More precisely,

Definition 17. *A Leibniz 2-algebra $(L, [-, -], \mathbf{J})$ is a linear category L equipped with*

1. *a bracket $[-, -]$, i.e. a bilinear functor $[-, -] : L \times L \rightarrow L$, and*
2. *a Jacobiator \mathbf{J} , i.e. a trilinear natural transformation*

$$\mathbf{J}_{x,y,z} : [x, [y, z]] \rightarrow [[x, y], z] + [y, [x, z]], \quad x, y, z \in L_0,$$

which verify, for any $w, x, y, z \in L_0$, the Jacobiator identity

$$\begin{array}{ccc}
 & [w, [x, [y, z]]] & \\
 \swarrow \scriptstyle [\mathbf{1}_w, \mathbf{J}_{x,y,z}] & & \searrow \scriptstyle \mathbf{1} \\
 [w, [[x, y], z]] + [w, [y, [x, z]]] & & [w, [x, [y, z]]] \\
 \downarrow \scriptstyle \mathbf{J}_{w,[x,y],z} + \mathbf{J}_{w,y,[x,z]} & & \downarrow \scriptstyle \mathbf{J}_{w,x,[y,z]} \\
 [[w, [x, y]], z] + [[x, y], [w, z]] & & [[w, x], [y, z]] + [x, [w, [y, z]]] \\
 + [[w, y], [x, z]] + [y, [w, [x, z]]] & & \\
 \downarrow \scriptstyle \mathbf{1} + [\mathbf{1}_y, \mathbf{J}_{w,x,z}] & & \downarrow \scriptstyle \mathbf{1} + [\mathbf{1}_x, \mathbf{J}_{w,y,z}] \\
 [[w, [x, y]], z] + [[x, y], [w, z]] & & [[w, x], [y, z]] + [x, [[w, y], z]] \\
 + [[w, y], [x, z]] + [y, [[w, x], z]] & & + [x, [y, [w, z]]] \\
 + [y, [x, [w, z]]] & & \\
 \swarrow \scriptstyle [\mathbf{J}_{w,x,y}, \mathbf{1}_z] & & \swarrow \scriptstyle \mathbf{J}_{[w,x],y,z} + \mathbf{J}_{x,[w,y],z} + \mathbf{J}_{x,y,[w,z]} \\
 & [[w, x], y], z] + [[x, [w, y]], z] & \\
 & + [[x, y], [w, z]] + [[w, y], [x, z]] & \\
 & + [y, [[w, x], z]] + [y, [x, [w, z]]] & \\
 & & (47)
 \end{array}$$

The Jacobiator identity is a coherence law that should be thought of as a higher Jacobi identity for the Jacobiator.

The preceding hierarchy ‘category, functor, natural transformation’ together with the coherence law is entirely similar to the known hierarchy ‘linear, bilinear, trilinear maps l_1, l_2, l_3 ’ with the L_∞ -conditions (a)-(e). More precisely,

Proposition 12. *There is a 1-to-1 correspondence between Leibniz 2-algebras and 2-term Leibniz infinity algebras.*

This proposition was proved in the Lie case in [BC04] and announced for the Leibniz case in [SL10]. A generalization of the latter correspondence to Lie 3-algebras and 3-term Lie infinity algebras can be found in [KMP11]. This paper allows to understand that the correspondence between

higher categorified algebras and truncated infinity algebras is subject to cohomological conditions, and to see how the coherence law corresponds to the last nontrivial L_∞ -condition.

The definition of Leibniz 2-algebra morphisms is God-given: such a morphism must be a functor that respects the bracket up to a natural transformation, which in turn respects the Jacobiator. More precisely,

Definition 18. Let $(L, [-, -], \mathbf{J})$ and $(L', [-, -]', \mathbf{J}')$ be Leibniz 2-algebras (in the following, we write $[-, -], \mathbf{J}$ instead of $[-, -]', \mathbf{J}'$). A morphism (F, \mathbf{F}) of Leibniz 2-algebras from L to L' consists of

1. a linear functor $F : L \rightarrow L'$, and
2. a bilinear natural transformation

$$\mathbf{F}_{x,y} : [Fx, Fy] \rightarrow F[x, y], \quad x, y \in L_0,$$

which make the following diagram commute

$$\begin{array}{ccc}
 [Fx, [Fy, Fz]] & \xrightarrow{\mathbf{J}_{Fx, Fy, Fz}} & [[Fx, Fy], Fz] + [Fy, [Fx, Fz]] \\
 \downarrow [\mathbf{1}_x, \mathbf{F}_{y,z}] & & \downarrow [\mathbf{F}_{x,y}, \mathbf{1}_z] + [\mathbf{1}_y, \mathbf{F}_{x,z}] \\
 [Fx, F[y, z]] & & [F[x, y], Fz] + [Fy, F[x, z]] \\
 \downarrow \mathbf{F}_{x,[y,z]} & & \downarrow \mathbf{F}_{[x,y],z} + \mathbf{F}_{y,[x,z]} \\
 F[x, [y, z]] & \xrightarrow{F\mathbf{J}_{x,y,z}} & F[[x, y], z] + F[y, [x, z]]
 \end{array} \tag{48}$$

Proposition 13. There is a 1-to-1 correspondence between Leibniz 2-algebra morphisms and 2-term Leibniz infinity algebra morphisms.

For a proof, see [BC04] and [SL10].

Composition of Leibniz 2-algebra morphisms (F, \mathbf{F}) is naturally given by composition of functors and whiskering of functors and natural transformations.

Proposition 14. There is a category Lei2 of Leibniz 2-algebras and morphisms.

6.2 2-morphisms and their compositions

The definition of a 2-morphism is canonical:

Definition 19. Let $(F, \mathbf{F}), (G, \mathbf{G})$ be Leibniz 2-algebra morphisms from L to L' . A Leibniz 2-algebra 2-morphism θ from F to G is a linear natural transformation $\theta : F \Rightarrow G$, such that, for any $x, y \in L_0$,

the following diagram commutes

$$\begin{array}{ccc}
 [Fx, Fy] & \xrightarrow{\mathbf{F}_{x,y}} & F[x, y] \\
 \downarrow [\boldsymbol{\theta}_x, \boldsymbol{\theta}_y] & & \downarrow \boldsymbol{\theta}_{[x,y]} \\
 [Gx, Gy] & \xrightarrow{\mathbf{G}_{x,y}} & G[x, y]
 \end{array} \tag{49}$$

Theorem 5. *There is a 1:1 correspondence between Leibniz 2-algebra 2-morphisms and 2-term Leibniz ∞ -homotopies.*

Horizontal and vertical compositions of Leibniz 2-algebra 2-morphisms are those of natural transformations.

Proposition 15. *There is a strict 2-category Lei2Alg of Leibniz 2-algebras.*

Corollary 2. *The 2-categories $2\text{Lei}_\infty\text{-Alg}$ and Lei2Alg are 2-equivalent.*

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