Rapid testing of stabilised finite element formulations for the Reissner-Mindlin plate problem using the FEniCS project

J. S. Hale*, P. M. Baiz

18th June 2012
Outline

- Main research goal
- Plate theories
- Numerical demonstration of locking using FEniCS
- Mixed formulation
- Stabilisation
- A few results
- Summary
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A meshless method for the Reissner-Mindlin plate problem that:

- is based on a sound variational principle
- is free from shear-locking
- avoids problems of previous approaches
- can be extended to the more complicated shell problem

“A locking-free meshfree method for the simulation of shear-deformable plates based on a mixed variational formulation”, Accepted in Computer Methods in Applied Mechanics and Engineering
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Finite Elements

Of course, there are many successful approaches to the Reissner-Mindlin problem in the Finite Element literature.

*How can FE approaches inform the design of a new meshless method?*

**Unifying themes with FEniCS**

- Mixed variational form (robust, general)*
- Stabilisation (bubbles, parameters)*
- Reduction and projection operators to eliminate extra unknowns (‘tricks’ become rigorous)**

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The Reissner-Mindlin Plate Problem
The Kirchhoff Limit

\[ \gamma = \lambda \bar{t}^{-2} (\nabla z_3 - \theta) = 0 \]
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Reissner-Mindlin Equations

**Discrete Displacement Weak Form**

Find \((z_{3h}, \theta_h) \in (V_{3h} \times R_h)\) such that for all \((y_3, \eta) \in (V_{3h} \times R_h)\):

\[
\int_{\Omega_0} L \epsilon(\theta_h) : \epsilon(\eta) \, d\Omega + \lambda \bar{t}^{-2} \int_{\Omega_0} (\nabla z_3 - \theta_h) \cdot (\nabla y_3 - \eta) \, d\Omega = \int_{\Omega_0} g y_3 \, d\Omega
\]

or:

\[
a_b(\theta_h; \eta) + \lambda \bar{t}^{-2} a_s(\theta_h, z_3; \eta, y_3) = f(y_3)
\]
Reissner-Mindlin Equations

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Find \((z_{3h}, \theta_h) \in (\mathcal{V}_{3h} \times \mathcal{R}_h)\) such that for all \((y_3, \eta) \in (\mathcal{V}_{3h} \times \mathcal{R}_h)\):

```python
... 
degree = 1 
V_3 = FunctionSpace(mesh, "Lagrange", degree) 
R = VectorFunctionSpace(mesh, "Lagrange", 
    degree, dim=2) 
U = MixedFunctionSpace([V_3, R])

z_3, theta = TrialFunctions(U) 
y_3, eta = TestFunctions(U) 
... 
```
Find \((z_{3h}, \theta_h) \in (V_{3h} \times \mathcal{R}_h)\) such that for all \((y_3, \eta) \in (V_{3h} \times \mathcal{R}_h)\):

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```
\[ \varepsilon_{ij}(u) := \frac{1}{2}(\nabla u + (\nabla u)^T) \]
\[ L(\varepsilon_{ij}) := D((1 - \nu)\varepsilon + \nu \text{tr} \varepsilon I) \]
\[ a_b(\theta_h, \eta) := \int_{\Omega_0} L\varepsilon(\theta_h) : \varepsilon(\eta) \, d\Omega \]

... 
```
e = lambda theta: 0.5*(grad(theta) +
    grad(theta).T)
L = lambda e: D*((1 - nu)*e +
    nu*tr(e)*Identity(2))

a_b = inner(L(e(theta)), e(eta))*dx
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\]
\[
a_b = \text{inner(L(e(theta)), e(eta))*dx}
\]
...
\[ a_s(\theta_h, z_3; \eta, y_3) = \int_{\Omega_0} (\nabla z_3 - \theta_h) \cdot (\nabla y_3 - \eta) \, d\Omega \]
\[ a_s(\theta_h, z_3; \eta, y_3) = \int_{\Omega_0} (\nabla z_3 - \theta_h) \cdot (\nabla y_3 - \eta) \, d\Omega \]

\[ \ldots \]

\[ a_s = \text{inner}(\text{grad}(z_3) - \theta, \text{grad}(y_3) - \eta) \ast dx \]

\[ \ldots \]
\[ a_b(\theta_h; \eta) + \lambda t^{-2} a_s(\theta_h, z_{3h}; \eta, y_3) = f(y_3) \]

... 

```python
a = a_b + lambda*t**-2*a_s
f = force*y_3*dx
u_h = solve(a == f, bcs=[bc1, bc2])
z_3h, theta_h = u_h.split()
```

Done!
\[ a_b(\theta_h; \eta) + \lambda \bar{t}^{-2} a_s(\theta_h, z_{3h}; \eta, y_3) = f(y_3) \]

... 

```python
a = a_b + lmbda*t**-2*a_s
f = force*y_3*dx
u_h = solve(a == f, bcs=[bc1, bc2])
z_3h, theta_h = u_h.split()
```

Done!
Figure: Locking; Fix discretisation, decrease $\bar{t}$
Inability of the basis functions to represent the limiting Kirchhoff mode.

\[ \mathcal{V}_b = \{(y_3, \eta) \in (\mathcal{V}_3 \times \mathcal{R}) \mid \nabla y_3 - \eta = 0\} \]

\[ \mathcal{V}_b \cap \mathcal{V}_h = \{0\} \]
Locking

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\[ \mathcal{V}_b \cap \mathcal{V}_h = \{0\} \]
for degree in range(0,6):
    V_3 = FunctionSpace(mesh, "Lagrange", degree)
    R = VectorFunctionSpace(mesh, "Lagrange", degree, dim=2)
    ...

Figure: Fix $\bar{t}$, increase polynomial order $p$
Figure: Fix $\bar{t}$, refine mesh by decreasing $h$
Error estimate

\[ \|u - u_h\| \leq C(\Omega_0, \kappa, E, \nu) h^p \frac{1}{\bar{t}} |u| \]

Conclusion

We can never fully eliminate locking with these approaches. It would be better to remove the dependence on $\bar{t}$ entirely.
Error estimate

\[ \|u - u_h\| \leq C(\Omega_0, \kappa, E, \nu) \frac{h^p}{t} |u| \]

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Conclusion

We can never fully eliminate locking with these approaches. It would be better to remove the dependence on \( \bar{t} \) entirely.
Mixed Form

Treat the shear stress as an *independent* variational quantity:

$$\gamma_h = \lambda \bar{t}^{-2}(\nabla z_3 h - \theta_h) \in S_h$$

**Discrete Mixed Weak Form**

Find \((z_3 h, \theta_h, \gamma_h) \in (V_3 h, R_h, S_h)\) such that for all 
\((y_3 h, \eta, \psi) \in (V_3 h, R_h, S_h)\):

$$a_b(\theta_h; \eta) + (\gamma_h; \nabla y_3 - \eta)_{L^2} = f(y_3)$$

$$\left(\nabla z_3 h - \theta_h; \psi\right)_{L^2} - \frac{\bar{t}^2}{\lambda} (\gamma_h; \psi)_{L^2} = 0$$
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\[
a_b(\theta_h; \eta) + (\gamma_h; \nabla y_3 - \eta)_{L^2} = f(y_3) \\
(\nabla z_3h - \theta_h; \psi)_{L^2} - \frac{\bar{t}^2}{\lambda} (\gamma_h; \psi)_{L^2} = 0
\]
Stability

Theorem (Brezzi 1974)

The classical saddle point problem (\( t = 0 \)) is stable, if and only if, the following conditions hold:

1. **(\( Z \)-Ellipticity of \( a \))** There exists a constant \( \alpha \geq 0 \) such that:
   \[
a(v, v) \geq \alpha \|v\|^2_X \quad \forall v \in Z
   \]
   where \( Z \) is the kernel of the bilinear form \( b \):
   \[
   Z := \{v \in X \mid b(v, q) = 0 \quad \forall q \in M\}
   \]

2. **(inf-sup condition on \( b \))** The bilinear form \( b \) satisfies an inf-sup condition:
   \[
   \inf_{q \in M} \sup_{v \in X} \frac{b(v, q)}{\|v\|_X \|q\|_M} = \beta > 0
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Displacement Formulation
Locking as $\bar{t} \to 0$

Mixed Formulation
Not necessarily stable

Solution
Combine the displacement and mixed formulation to retain the advantageous properties of both
Displacement Formulation
Locking as $\bar{t} \to 0$

Mixed Formulation
Not necessarily stable

Solution
Combine the displacement and mixed formulation to retain the advantageous properties of both
\[ a_s = \alpha a^{\text{displacement}} + (\bar{t}^{-2} - \alpha) a^{\text{mixed}} \]
Stabilised Mixed Weak Form

Discrete Mixed Weak Form

\[ a_b(\theta_h; \eta) + (\gamma_h; \nabla y_3 - \eta)_{L^2} = f(y_3) \]
\[ (\nabla z_{3h} - \theta_h; \psi)_{L^2} - \frac{\bar{t}^2}{\lambda} (\gamma_h; \psi)_{L^2} = 0 \]

Stabilised Mixed Weak Form (Brezzi and Arnold 1993, Boffi and Lovadina 1997)

\[ a_b(\theta; \eta) + \lambda \alpha a_s(\theta, z_3; \eta, y_3) + (\gamma, \nabla y_3 - \eta)_{L^2} = f(y_3) \]
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**Theorem (Brezzi 1974)**

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   *where $\mathcal{Z}$ is the kernel of the bilinear form $b$:*

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   \[ \inf_{q \in \mathcal{M}} \sup_{v \in X} \frac{b(v, q)}{\|v\|_X \|q\|_M} = \beta > 0 \]
...  
S = VectorFunctionSpace(mesh, "DG", 0, dim=2)  
...  
gamma = TrialFunction(S)  
psi = TestFunction(S)  
...  
a = a_b + alpha*lmbda*a_s + inner(gamma, 
    grad(y_3) - eta) + inner(grad(z_3) - 
    theta, psi) - t**2/(lmbda*(1.0 - 
    alpha*t**2))*inner(gamma, psi)  
...
Figure: Various elements, Fix $h$, vary $\alpha$. 
Figure: TRIA0220 element. Fix $h$, vary $\alpha$.
\[ h = \text{CellSize(mesh)} \]
\[ \alpha = h^{(-2.0)} \times \text{constant} \]
Figure: Trying different $\alpha$ recipes; convergence can be improved (Lovadina)
Conclusions

- The FEniCS project allows for rapid testing of different Finite Element strategies.
- ~ 400 lines of code; Displacement, Mixed, Projections, Errors, Command Line Interface, Output Results etc.
- The stabilisation parameter $\alpha$ should be chosen based on some local discretisation dependent length measure.
- Convergence rates can even be improved by a ‘good’ choice of $\alpha$.
- Based on these results, we have designed a novel meshfree method based on a stabilised weak form with a Local Patch Projection technique to eliminate the shear-stress unknowns \textit{a priori}.
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Thanks for listening.