ON NONSTRICT MEANS

by

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AGGREGATION OPERATORS

We consider $m$ real numbers $x_1, \ldots, x_m, x_i \in [a, b]$, and we are willing to substitute to the vector $(x_1, \ldots, x_m)$ one simple number $x$ using the aggregation operator $M$:

$$M : \Lambda = \bigcup_{m \in \mathbb{N}_0} [a, b]^m \to R$$

$$(x_1, \ldots, x_m) \to x = M(x_1, \ldots, x_m).$$

Synthesizing judgments is an important part of MCDM methods. The typical situation concerns individuals which form quantitative judgments about a measure. In order to obtain a consensus of these judgments, classical operators are proposed.

Some examples:

$$M(x_1, \ldots, x_m) = \frac{1}{m} \sum_{i} x_i \quad \text{(arithmetic mean)}$$

$$M(x_1, \ldots, x_m) = (\prod_{i} x_i)^{\frac{1}{m}} \quad \text{(geometric mean)}$$

$$M(x_1, \ldots, x_m) = \min_i x_i \quad \text{(minimum)}$$

$$M(x_1, \ldots, x_m) = \max_i x_i \quad \text{(maximum)}$$

$$M(x_1, \ldots, x_m) = \sum_{i} \omega_i^{(m)} x_i, \quad \omega_i^{(m)} \geq 0, \quad \sum_{i} \omega_i^{(m)} = 1$$

(weighted arithmetic mean)

$$M(x_1, \ldots, x_m) = \max_i \{\min(\omega_i^{(m)}, x_i)\}, \quad \omega_i^{(m)} \geq 0, \quad \max_i \omega_i^{(m)} = 1$$

(weighted maximum)

etc.
An aggregation operator $M$ can be

- **Continuous (Co):**
  \[ \forall m \in N_0, \ M^{(m)}(x_1, \ldots, x_m) \text{ is a continuous function}; \]

- **Symmetric (Sy):**
  \[ \forall m \in N_0, \ M^{(m)}(x_1, \ldots, x_m) \text{ is a symmetric function}; \]

- **Increasing (In):**
  \[ \forall m \in N_0, \ M^{(m)}(x_1, \ldots, x_m) \text{ is increasing on each argument}; \]

- **Strictly increasing (SIn):**
  \[ \forall m \in N_0, \ M^{(m)}(x_1, \ldots, x_m) \text{ is strictly increasing on each argument}; \]

- **Idempotent (Id):**
  \[ \forall m \in N_0, \ M^{(m)}(x, \ldots, x) = x. \]

**Proposition 1** If $M$ is In then

\[ M \text{ is Id } \iff \min_i x_i \leq M(x_1, \ldots, x_m) \leq \max_i x_i. \]
AGGREGATION PROPERTIES
ITERATIVE PROPERTIES

An aggregation operator $M$ can be

- **Associative (As):**

  $$M^{(3)}(x_1, x_2, x_3) = M^{(2)}(x_1, M^{(2)}(x_2, x_3)) = M^{(2)}(M^{(2)}(x_1, x_2), x_3);$$

  $$\forall m \in N_0, M^{(m)}(x_1, \ldots, x_m) = M^{(2)}(M^{(m-1)}(x_1, \ldots, x_{m-1}), x_m)$$

  Examples: $M(x_1, \ldots, x_m) = \sum_i x_i \vee \min_i x_i \vee \max_i x_i$.

- **Decomposable (De):** $\forall 1 \leq k \leq m,$

  $$M^{(m)}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) = M^{(m)}(M_k, \ldots, M_k, x_{k+1}, \ldots, x_m)$$

  where $M_k = M^{(k)}(x_1, \ldots, x_k)$.

  Examples: $M(x_1, \ldots, x_m) = \frac{1}{m} \sum_i x_i \vee \min_i x_i \vee \max_i x_i$.

Proposition 2  \hspace{1cm} As & Id $\Rightarrow$ De.
THE GENERALIZED MEAN

Theorem 1  
M is defined on  A (A = ∪m∈N₀[a, b]ᵐ) and fulfills Co, Sy, Sin, Id, De ⇔ ∀m ∈ N₀,

\[ M(x₁, \ldots, xₘ) = f^{-1}\left[ \frac{1}{m} \sum_{i} f(x_i) \right] \]

where f is any continuous strictly monotonic function on [a, b].


Examples:

<table>
<thead>
<tr>
<th>Generator</th>
<th>Mean</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>M(x₁, ..., xₘ)</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>( \frac{1}{m} \sum x_i )</td>
<td>arithmetic</td>
</tr>
<tr>
<td>x²</td>
<td>( \sqrt{\frac{1}{m} \sum x_i^²} )</td>
<td>quadratic</td>
</tr>
<tr>
<td>x⁻¹</td>
<td>( \frac{1}{m} \sum \frac{1}{x_i} )</td>
<td>harmonic</td>
</tr>
<tr>
<td>xα (α ≠ 0)</td>
<td>( (\frac{1}{m} \sum x_i^α)^\frac{1}{α} )</td>
<td>root-power</td>
</tr>
<tr>
<td>log x</td>
<td>( \sqrt[α]{\Pi x_i} )</td>
<td>geometric</td>
</tr>
</tbody>
</table>

THE MAIN PURPOSE

To describe the class D of operators M defined on A and fulfilling Co, Sy, In, Id, De.
Nonstrict Means

Toward a description of \( \mathcal{D} : \text{Co, Sy, In, Id, De.} \)

Two families:

1. \( \mathcal{D}_{a,b,a} \subset \mathcal{D} \) with \( M(a, b) = a \) \((\min \in \mathcal{D}_{a,b,a})\)

2. \( \mathcal{D}_{a,b,b} \subset \mathcal{D} \) with \( M(a, b) = b \) \((\max \in \mathcal{D}_{a,b,b})\)

**Theorem 2** \( M \in \mathcal{D} \iff \exists a \leq \alpha \leq \beta \leq b \) such that \( \forall m \in \mathbb{N}_0 \),

- \( M \in \mathcal{D}_{a,\alpha,\alpha} \) on \([a, \alpha]^m\) \((M(a, \alpha) = \alpha)\)
- \( M \in \mathcal{D}_{\beta,\beta,b} \) on \([\beta, b]^m\) \((M(\beta, b) = \beta)\)
- \( M(x_1, \ldots, x_m) = f^{-1}\left[\frac{1}{m} \sum f[\text{median}(\alpha, x_i, \beta)]\right] \) everywhere else,
  where \( f \) is any continuous strictly monotonic function on \([\alpha, \beta]\).
Three observations

1. \([\alpha, \beta] = [a, b] \iff M \text{ is SIn} \) (generalized mean of Kolmogoroff)

2. If \(\alpha = \beta\), we have

\[
\begin{align*}
X_2 & \\
\alpha & \quad \beta, b, b \\
\alpha & \quad \beta, b, b \\
a & \quad \alpha, a, a \\
\end{align*}
\]

close to:

**Theorem 3**  
\(M\) is defined on \(\Lambda\) and fulfills Co, Sy, In, Id, As \(\Rightarrow\)

\[\exists a \leq \alpha \leq b \text{ such that } \forall m \in N_0, \]

\[M(x_1, \ldots, x_m) = median(\max_i x_i, \alpha, \min_i x_i).\]

Fung and Fu (1975)

3. \(D_{a,\alpha,\alpha}\) and \(D_{\beta,\beta,\beta}\), or equivalently, \(D_{a,\beta,\beta}\) and \(D_{a,\beta,\beta}\) are yet to be described.
**DESCRIPTION OF $D_{a,b,a}$ ($M(a,b) = a$)**

**Theorem 4** $M \in D_{a,b,a} \iff \forall m \in N_0,$

- either $M(x_1, \ldots, x_m) = \min_i x_i,$
- or $M(x_1, \ldots, x_m) = g^{-1} \prod_{i} g(x_i)$

where $g$ is any continuous strictly increasing function on $[a, b]$ with $g(a) = 0,$

- or there exists a countable index set $K$ and a family of disjoint subintervals $\{(a_k, b_k) : k \in K\}$ of $[a, b]$ such that

\[
M(x_1, \ldots, x_m) = \begin{cases} 
g_k^{-1} \prod_i g_k(\min(x_i, b_k)) & \text{if } \exists k \in K \text{ such that } \min_i x_i \in (a_k, b_k) \\
\min_i x_i & \text{otherwise,}
\end{cases}
\]

where $g_k$ is any continuous strictly increasing function on $[a_k, b_k], \text{ with } g_k(a_k) = 0.$

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![Diagram](image.png)
**Description of** $D_{a,b,b}$ \((M(a,b) = b)\)

**Theorem 5** \(M \in D_{a,b,b} \iff \forall m \in \mathbb{N}_0,\)

- either \(M(x_1, \ldots, x_m) = \max_i x_i,\)
- or \(M(x_1, \ldots, x_m) = g^{-1} \prod g(x_i)\)
  where \(g\) is any continuous strictly decreasing function on \([a, b]\) with \(g(b) = 0,\)
- or there exists a countable index set \(K\) and a family of disjoint subintervals \(\{(a_k, b_k)\mid k \in K\}\) of \([a, b]\) such that

\[
M(x_1, \ldots, x_m) = \begin{cases} 
  g_k^{-1} \prod g_k[\max(x_i, b_k)] & \text{if } \exists k \in K \text{ such that } \max_i x_i \in (a_k, b_k) \\
  \max_i x_i & \text{otherwise,}
\end{cases}
\]

where \(g_k\) is any continuous strictly decreasing function on \([a_k, b_k],\)
with \(g_k(b_k) = 0.\)
BISYMMETRY EQUATION

A function \( M(x_1, x_2) \) of two variables is said to be bisymmetric (Bi) if it satisfies the following equation

\[
M[M(x_{11}, x_{12}), M(x_{21}, x_{22})] = M[M(x_{11}, x_{21}), M(x_{12}, x_{22})].
\]

**Theorem 6** \( M \) is defined on \([a, b]^2\) and fulfills Co, Sy, SIn, Id, Bi

\[
M(x_1, x_2) = f^{-1} \left[ \frac{f(x_1) + f(x_2)}{2} \right]
\]

where \( f \) is any continuous monotonic function on \([a, b]\).


**Theorem 7** Let \( B \) be the class of functions \( M \) defined on \([a, b]^2\) and fulfilling Co, Sy, In, Id, Bi.

To obtain a description of \( B \), it suffices to consider the case \( m = 2 \) in the description of \( D \) presented before.